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REPRESENTATIONS OF DIRECT PRODUCTS OF FINITE GROUPS

BURTON FEIN

Let G be a finite group and K an arbitrary field. We denote by $K(G)$ the group algebra of G over K. Let G be the direct product of finite groups G_1 and $G_2, G = G_1 \times G_2$, and let M_i be an irreducible $K(G_i)$-module, $i = 1, 2$. In this paper we study the structure of M_1, M_2, the outer tensor product of M_1 and M_2.

While M_1, M_2 is not necessarily an irreducible $K(G)$-module, we prove below that it is completely reducible and give criteria for it to be irreducible. These results are applied to the question of whether the tensor product of division algebras of a type arising from group representation theory is a division algebra.

We call a division algebra D over K K-derivable if $D \cong \mathrm{Hom}_{K(G)}(M, M)$ for some finite group G and irreducible $K(G)$-module M. If $B(K)$ is the Brauer group of K, the set $B_0(K)$ of classes of central simple K-algebras having division algebra components which are K-derivable forms a subgroup of $B(K)$. We show also that $B_0(K)$ has infinite index in $B(K)$ if K is an algebraic number field which is not an abelian extension of the rationals.

All $K(G)$-modules considered are assumed to be unitary finite dimensional left $K(G)$-modules. If M_i is a $K(G_i)$-module, $i = 1, 2$, the outer tensor product $M_1 \# M_2$ of M_1 and M_2 is the $K(G)$-module whose underlying space is $M_1 \otimes_K M_2$ and where $(g_1, g_2) \in G$ acts on $M_1 \otimes_K M_2$ by

$$(g_1, g_2) \sum m_i \otimes m_i' = \sum g_1 m_i \otimes g_2 m_i', \; m_i \in M_1, \; m_i' \in M_2, \; g_j \in G_j, \; j = 1, 2 .$$

It will be necessary to refer to the theory of the Schur index of absolutely irreducible representations of finite groups. In § 1 we present a treatment of this theory where the relevant theorems are proved for arbitrary fields. This treatment is included in the author's doctoral dissertation supervised by Professor Charles W. Curtis at the University of Oregon. During the preparation of this paper the author held a National Science Foundation Graduate Fellowship.

1. The Schur index. The method used in [3, § 70] to prove the relevant theorems about the Schur index for fields of characteristic zero does not seem to generalize to arbitrary fields. In that treatment attention is focused on the enveloping algebra of the representations rather than on the representations themselves. We work directly with modules over group algebras. After Theorem 1.1 has been proved, the methods of [3, § 70] can be generalized to arbitrary fields. However, this approach seems to be unnecessarily long and complicated and we have chosen to present a unified treatment independent of these methods. For the convenience of the reader we have included several short arguments that are similar to ones appearing in [3].

Before we can state our main results we need to introduce some terminology. We refer the reader to [3] for the relevant theory.

Let G be a finite group. A field E is a splitting field for G if every irreducible $E(G)$-module is absolutely irreducible. Let K be a field. By Theorem 69.11 of [3] there is a finite normal separable extension E of K which is a splitting field for G. For if K has characteristic p, there is a finite field F of characteristic p which is a splitting field for G. Since F is an extension of its prime field by roots of unity, a composite $E = F \cdot K$ of F and K is a splitting field of the desired type. *We shall assume throughout this section that E is a normal separable extension of K which is splitting field for G.* K will be assumed to be an arbitrary field.

We denote the Galois group of E over K by $\mathscr{G}(E \mid K)$. Let N be an $E(G)$-module with basis $m_1, \cdots, m_n$ over E, and let the action of G on N be given by $gm_i = \sum_j a_{ij}(g)m_j$, $g \in G$, $a_{ij}(g) \in E$. Let V be an n-dimensional vector space over K with basis $v_1, \cdots, v_n$ and let $\sigma \in \mathscr{G}(E \mid K)$. Under the action $gv_i = \sum_j \sigma(a_{ij}(g))v_j$, $g \in G$, V becomes an $E(G)$-module which we denote by σN. σN is called a conjugate module of N. If χ is the character of N, then we denote by $\sigma\chi$ the character of σN, where $(\sigma\chi)(g) = \sigma(\chi(g))$, $g \in G$. σ and τ will always denote elements of $\mathscr{G}(E \mid K)$ while χ and ψ will always be characters of modules over group algebras.

Let N be an irreducible $E(G)$-module and let E^* denote an algebraic closure of E. *All fields considered will be assumed to be subfields of E^*.* $N^* = N \otimes_E E^*$ is an irreducible $E^*(G)$-module. N^* is said to be realizable in a subfield J of E^* if there is a $J(G)$-module V such that $V \otimes_J E^* \cong N^*$. Let χ be the character of N, χ^* the character of N^*. Then $\chi^*(g) = \chi(g)$ for all $g \in G$. We denote by $K(\chi)$ the field generated over K by the values $\chi(g)$, $g \in G$. The Schur index $m_K(N)$ of N over K is the minimum value of $(J : K(\chi))$, the degree of J over $K(\chi)$, taken over all fields J in which N^* is realizable, where $K(\chi) \subset J \subset E^*$. In general, there will not exist a subfield J of E in which N is realizable and such that $(J : K(\chi)) = m_K(N)$ [2].

Let M be an irreducible $K(G)$-module. M is isomorphic to a minimal left ideal of a simple component A of $K(G)/\text{rad}\, K(G)$ [3, Th. 25.10]. A is isomorphic to a complete matrix ring $(D)_n$, D a division algebra with center L, $L \supset K$, and $D \cong \text{Hom}_{K(G)}(M, M)$ [3, Th. 26.8]. The index $m(D)$ of D is $(F : L)$ where F is any maximal subfield of

D [3, Th. 68.6]. We shall let rM denote the direct sum of r copies of M, where r is a natural number. We set $M^E = M \otimes_K E$. *N will be assumed to be an irreducible $E(G)$-module which is a composition factor of M^E.* χ will be the character of N. Since A is associative, A may be viewed as an L-algebra. We denote this algebra by ${}_LA$. A will denote ${}_KA$. We shall maintain the above context throughout this entire section.

THEOREM 1.1. *The center L of D is $K(\chi)$. $A \otimes_K K(\chi)$ is isomorphic to a direct sum of t copies of ${}_{K(\chi)}A$, where $t = (K(\chi) : K)$.*

We begin with a lemma which is essentially proved in [3, Th. 70.15].

LEMMA 1.2: *$M^E \cong k(\sigma_1 N \oplus \cdots \oplus \sigma_t N)$ where the $\sigma_i \in \mathscr{G}(E \mid K)$, $\sigma_1 = 1$, k is a natural number, the $\{\sigma_i N\}$ form a complete set of nonisomorphic conjugates of the irreducible $E(G)$-module N, and $t = (K(\chi) : K)$.*

Proof. M^E is a completely reducible and $E \otimes_K (K(G)/\text{rad}\, K(G)) \cong E(G)/\text{rad}\, E(G)$ [3, Ths. 69.9, 69.10]. A^E is a component (not necessarily simple) of $E(G)/\text{rad}\, E(G)$. Since A^E is semi-simple [3, Th. 69.4] we have $A^E = C \cong Ce_1 \oplus \cdots \oplus Ce_t$ where the e_i are primitive orthogonal central idempotents of C. For any $\sigma \in \mathscr{G}(E \mid K)$ we define a K-automorphism of A^E by $\sigma(\sum a_j \otimes f_j) = \sum a_j \otimes \sigma f_j$, $a_j \in A$, $f_j \in E$. $\sigma(f_i)$ is again a primitive central idempotent of C and so coincides with some f_j, $1 \leqq j \leqq t$. If $f_1, \cdots, f_r$ are the different conjugates $\sigma(f_1)$ of f_1 then $f = f_1 + \cdots + f_r$ is a central idempotent of A. Since A is simple, $r = t$. Let $f_1 = \sum f_{1j}$ be the decomposition of f_1 into primitive idempotents. $Cf_{1j} \cong N$ for some irreducible $E(G)$-module N and $C\sigma(f_{1j}) \cong \sigma N$. Since $C \cong Cf_1 \oplus C\sigma_2(f_1) \oplus \cdots \oplus C\sigma_t(f_1)$, we see that $N, \sigma_2 N, \cdots, \sigma_t N$ are the distinct $E(G)$-components of M^E and that the $\{\sigma_i N\}$ form a complete set of nonisomorphic conjugates of N. This proves that $M^E \cong d(1)N \oplus d(2)\sigma_2 N \oplus \cdots \oplus d(t)\sigma_t N$, where the $d(i)$ are natural numbers. Since $\sigma M^E \cong M^E$ for all $\sigma \in \mathscr{G}(E \mid K)$,

$$d(1)N \oplus \cdots \oplus d(t)\sigma_t N \cong d(1)\sigma N \oplus d(2)\sigma\sigma_2 N \oplus \cdots \oplus d(t)\sigma\sigma_t\,.$$

By the Krull-Schmidt Theorem $d(1) = d(2) = \cdots = d(t)$. It only remains to prove that $t = (K(\chi) : K)$. Let $\mathscr{H} = \{\sigma \in \mathscr{G}(E \mid K) \mid \sigma N \cong N\}$. From Galois Theory $t = [\mathscr{G}(E \mid K) : \mathscr{H}]$. But $\sigma N \cong N$ if and only if $\sigma\chi = \chi$, where χ is the character of N. Therefore

$$\mathscr{H} = \{\sigma \in \mathscr{G}(E \mid K) \mid \sigma\chi = \chi\}$$

and so $t = (K(\chi) : K)$.

Let h be the exponent of G. For $g \in G, \chi(g)$ is a sum of h-th roots of unity. Therefore $K(\chi) \subset K(\sqrt[h]{1})$ and since $\mathscr{G}(K(\sqrt[h]{1}) \mid K)$ is abelian $K(\chi)$ is a normal separable extension of K. If $\sigma \in \mathscr{G}(E \mid K)$, then $K(\sigma\chi) = K(\chi)$.

Proof of Theorem 1.1. Let $A \otimes_K K(\chi) \cong B_1 \oplus \cdots \oplus B_s$, the B_i simple $K(\chi)$-algebras. If the irreducible $K(\chi)(G)$-module U is isomorphic to a minimal left ideal of B_1, then $U^E \cong r(\sigma_i N \oplus \cdots)$ by Lemma 1.2. However, since $K(\sigma\chi) = K(\chi)$ for all $\sigma \in \mathscr{G}(E \mid K)$, it follows that $U^E \cong r(\sigma N)$ for some $\sigma \in \mathscr{G}(E \mid K)$. Since A^E has t distinct non-isomorphic simple components we have $s \leqq t$ and $B_i \not\cong B_j$ for all i, j. Therefore $s = t$ and each $B_i \otimes_{K(\chi)} E$ is simple with center E [3, Th. 29.13]. If F_i is the center of B_i, then the centroid of $B_i \otimes_{K(\chi)} E$ is $F_i \otimes_{K(\chi)} E$ [7, Th. 1, p. 114]. Counting dimensions we see that $F_i \otimes_{K(\chi)} E \cong E$ if and only if $F_i = K(\chi)$. Therefore the centroid of $A \otimes_K K(\chi)$ is isomorphic to a direct sum of t copies of $K(\chi)$. The center of D is L. Then the centroid of $A \otimes_K K(\chi)$ is $L \otimes_K K(\chi)$ and so $L \otimes_K K(\chi)$ is a direct sum of t copies of $K(\chi), t = (K(\chi) : K)$. If I is a maximal ideal of $L \otimes_K K(\chi)$, then $(L \otimes_K K(\chi))/I \cong K(\chi)$ and so every composite of $K(\chi)$ and L over K is isomorphic to $K(\chi)$ [8, p. 84, Th. 21]. Therefore $K(\chi) = L$. But then A is an algebra over $K(\chi)$ and we have

$$\begin{aligned} A \otimes_K L &\cong ({}_L A \otimes_L L) \otimes_K L \cong {}_L A \otimes_L (L \otimes_K L) \\ &\cong {}_L A \otimes_L (L \oplus \cdots \oplus L) \cong {}_L A \otimes_L L \oplus \cdots \oplus_L A \otimes_L L \\ &\cong {}_L A \oplus \cdots \oplus {}_L A \,. \end{aligned}$$

Since L is normal over K, $L \otimes_K L$ is a direct sum of t copies of L, $t = (L : K)$ [8, p. 87].

It will always be clear from the context whether we are viewing A as a K-algebra or as an L-algebra. We shall, therefore, not continue to distinguish between these algebras but shall simply write A for both ${}_K A$ and ${}_L A$. We recall that a finite extension F of L is a splitting field for $D(A = (D)_n)$ if $D \otimes_L F = (F)_s$ for some integer s.

LEMMA 1.3. *Let K be a perfect field and F a finite extension of L. Then F is a splitting field for D if and only if N^* is realizable in F, $N^* = N \otimes_E E^*$.*

Proof. Since F is a separable extension of L.

$$F \otimes_L L(G)/\mathrm{rad}\, L(G) \cong F(G)/\mathrm{rad}\, F(G)$$

[3, Th. 69.10]. Let U be an $L(G)$-module so that $U \otimes_L E^* \cong rN^*$, r a natural number, and such that U is isomorphic to a minimal left ideal of A (the existence of such a U was proved in the proof of Theorem 1.2). $A \otimes_L F$ is a simple component of $F(G)/\mathrm{rad}\, F(G)$. Let V be a minimal left ideal of $A \otimes_L F$. Then $\mathrm{Hom}_{F(G)}(V, V) \cong D'$ where $A \otimes_L F = (D')_v$. N^* is realizable in F if and only if $D' = F$ [3, Th. 29.13]. Since F is a splitting field for D if and only if $A \otimes_L F = (F)_v$ we are done.

THEOREM 1.4. (a) *$M^E \cong m_K(N)(N \oplus \sigma_2 N \oplus \cdots \oplus \sigma_t N)$ where the $\sigma_i \in \mathscr{G}(E \mid K)$, the $\{\sigma_i N\}$ form a complete set of nonisomorphic conjugates of the irreducible $E(G)$-module N, and $t = (K(\chi) : K)$.*

(b) $m_K(N) = m(D)$.

(c) *If K has prime characteristic, then $m_K(N) = 1$,* i.e. *$\mathrm{Hom}_{K(G)}(M, M)$ is commutative.*

(d) *$m_K(N)$ divides the dimension $(N : E)$ of N over E.*

(e) *For any finite algebraic extension J of K in which N^* is realizable, $m_K(N)$ divides $(J : K(\chi))$.*

Proof. We have $A \otimes_K E = C \cong Ce_1 \oplus \cdots \otimes C\sigma_t(e_1)$. Since E is a splitting field for G, $Ce_1 \cong C\sigma_i(e_1) \cong (E)_r$

$$(A \otimes_K E : E) = tr^2 = (A : K) = ((D)_n : K) = n^2(D : K)$$
$$= n^2(D : L)(L : K) = n^2 t[m(D)]^2 .$$

Therefore $r = n \cdot m(D)$. M is isomorphic to a minimal left ideal I of A. Since $A = (D)_n$, A is isomorphic to a direct sum of n copies of I. Set $m = m(D)$. Then $A \otimes_K E$ is isomorphic to a direct sum of t copies of $(E)_{mn}$ so $A \otimes_K E$ is a direct sum of tmn minimal left ideals. $\sigma_j N$ is isomorphic to a minimal left ideal I_j of $A \otimes_K E$, $j = 1, \cdots, t$, $\sigma_1 = 1$. Since $M \cong I$, the $\{I_j\}$ appear with equal multiplicity in $I \otimes_K E$. By Lemma 1.2 $k = m(D)$ and $M^E \cong m(D)(N \oplus \sigma_2 N \oplus \cdots \oplus \sigma_t N)$. The rest of the proof is divided into two parts.

Case 1. K is perfect. Let V be a maximal subfield of D. V is a splitting field for D of minimal K-dimension. By Lemma 1.3 N^* is realizable in V. Therefore $m_K(N) \geqq (V : L) = m(D)$. Conversely, if N^* is realizable in a finite extension F of L then F is a splitting field for D. Hence $m_K(N) \leqq m(D)$. This proves (a) and (b) when K is perfect. Let K now have characteristic zero. We have seen that N is isomorphic to a minimal left ideal of $(E)_{nm}$. Then $\mathrm{Hom}_E(N, N) \cong (E)_{mn}$ so $(N : E) = nm = n \cdot m_K(N)$. If N^* is realizable in a finite algebraic extension J of K, then J is a splitting field for D by Lemma 1.3. $m(D) = m_K(N)$ divides $(J : K(\chi))$ by [3, Th. 68.7]. This proves

(d) and (e) for K of characteristic zero.

Case 2. K has characteristic $p, p > 0$. Assume first of all that K is finite. Then D is a finite skewfield and hence a field, $D = K(\chi)$ [3, Th. 68.9]. Since K is perfect and $K(\chi)$ is a splitting field for D, N^* is realizable in $K(\chi)$. Therefore $m_K(N) = 1$ by Case 1. We have $m(D) = 1$ also. We now assume that K is infinite. Let $F = Z_p(\chi, \sigma_2\chi, \cdots, \sigma_t\chi)$ where Z_p is the prime field and the $\{\sigma_i\chi\}$ are the characters of the $\{\sigma_i N\}$. F is a finite field so the $\{\sigma_i N\}$ are all realizable in F, say $V_i \otimes_F E \cong \sigma_i N$, $i = 1, \cdots, t$, $\sigma_1 = 1$. The V_i are irreducible $F(G)$-modules. Let $W = V_1 \oplus \cdots \oplus V_t$. The character of W lies in $K \cap F = R$. Since $F \otimes_R R(G)/\mathrm{rad}\, R(G) \cong F(G)/\mathrm{rad}\, F(G)$, there is an $R(G)$-module T such that

$$W \cong T^F \cong V_1 \oplus \cdots \oplus V_t\,.$$

Therefore $(m(D)T)^K \cong M$; and since M is irreducible, $m(D) = 1$. Since N^* is realizable in $Z_p(\chi)$, it will be realizable in $K(\chi)$; so $\mathrm{m}_K(N) = 1$. (d) and (e) are now immediate.

COROLLARY 1.5. *The characters of the nonisomorphic irreducible $K(G)$-modules are linearly independent over K.*

Proof. The characters of the nonisomorphic $E(G)$-modules are linearly independent over E [3, Th. 30.12]. Since the characters of M and M^E are identical, the desired result is immediate from Theorem 1.4 (a) and (c).

REMARK. We have only stated the results concerning the Schur index that we will need in the following sections. Analogues of the other important theorems found in [3, § 70] can also be easily proved by the methods used here.

It will be useful to have an expression for the relationship between the simple component A of $K(G)/\mathrm{rad}\, K(G)$ and the irreducible $E(G)$-module N.

DEFINITION 1.6. Let K be an arbitrary field, E a finite separable extension of K. The simple component A of $K(G)/\mathrm{rad}\, K(G)$ is *associated* with the irreducible $E(G)$-module N if N is isomorphic to a minimal left ideal of $A \otimes_K E$.

2. Outer tensor products of irreducible modules. Throughout this section K will denote an arbitrary field, G_1 and G_2 will denote finite groups, and G will be the direct product of G_1 and G_2, $G = G_1 \times G_2$.

E will denote a finite normal separable extension of K which is a splitting field for G. M_i will be an irreducible $K(G_i)$-module, $i = 1, 2$, and $M_1 \# M_2$ will denote the outer tensor product of M_1 and M_2. A_i will denote the simple component of $K(G_i)/\text{rad}\,(KG_i)$ corresponding (in the sense of Definition 1.6) to M_i, $i = 1, 2$. Let N_i be an irreducible $E(G_i)$-component of M_i^E. For any $\sigma, \tau \in \mathscr{G}(E \mid K)$, $\sigma N_1 \# \tau N_2$ is an irreducible $E(G)$-module [1, Footnote, p. 587]. $\sigma N_1 \# \tau N_2$ will not, in general, be a conjugate of $N_1 \# N_2$. We shall let ψ_i denote the character of N_i, $i = 1, 2$. All fields considered will be assumed to be subfields of E^*, a fixed algebraic closure of E. Let $L_i = K(\psi_i)$, $\mathscr{H}_i = \mathscr{G}(E \mid L_i)$, $i = 1, 2$. Let $\overline{\mathscr{H}_i}$ be a fixed set of coset representatives of $\mathscr{H}_i$ in $\mathscr{G}(E \mid K)$. Theorem 1.4 (a) implies that $M_i^E \cong m_K(N_i)(\sum \sigma_i N_i)$, the sum being over all $\sigma_i \in \overline{\mathscr{H}_i}$. Since the $\{L_i\}$ are normal over K, there is an unique composite L of L_1 and L_2 over K.

PROPOSITION 2.1. $m_K(N_1 \# N_2) = m_K(\sigma N_1 \# \tau N_2)$ for all $\sigma, \tau \in \mathscr{G}(E \mid K)$.

Proof. Let $\sigma, \tau \in \mathscr{G}(E \mid K)$. We may clearly assume that $\sigma \in \overline{\mathscr{H}_1}$, $\tau \in \overline{\mathscr{H}_2}$. In § 1 we observed that $K(\psi_i) = K(\sigma\psi_i) = L_i$, $i = 1, 2$, for all $\sigma \in \mathscr{G}(E \mid K)$. Let L be the composite of L_1 and L_2 over K. Then

$$m_K(N_1 \# N_2) = m_L(N_1 \# N_2),\ m_K(\sigma N_1 \# \tau N_2) = m_L(\sigma N_1 \# \tau N_2)\ .$$

Because of Theorem 1.4 (c) we may assume that K has characteristic zero. Let C, D be the simple components of $L(G)$ associated with $N_1 \# N_2$ and $\sigma N_1 \# \tau N_2$ respectively. In view of Theorem 1.4 (b) it is sufficient to prove that $C \cong D$. Since $K(G) \cong K(G_1) \otimes_K K(G_2)$, $A_1 \otimes_K A_2$ is a component of $K(G)$. Therefore $A_1^L \otimes_L A_2^L \cong (A_1 \otimes_K A_2)^L$ is a component of $L(G)$. It follows from Theorem 1.2 that $A_1^L \otimes_L A_2^L$ is a direct sum of isomorphic simple algebras. From Theorem 1.4 (a) we have $M_1^E \cong m_K(N_i)(\sum \sigma_i N_i)$, the sum being over all $\sigma_i \in \overline{\mathscr{H}_i}$. Then

$$(M_1 \# M_2)^E \cong M_1^E \# M_2^E \cong m_K(N_1)m_K(N_2) \sum \alpha N_1 \# \beta N_2\ ,$$

where the sum is taken over all $\alpha \in \overline{\mathscr{H}_1}, \beta \in \overline{\mathscr{H}_2}$. From this we see that both $N_1 \# N_2$ and $\sigma N_1 \# \tau N_2$ are associated with $A_1 \otimes_K A_2$, for all $\sigma \in \overline{\mathscr{H}_1}, \tau \in \overline{\mathscr{H}_2}$. This proves that C and D are components of $A_1^L \otimes_L A_2^L$ and so $C \cong D$.

Using this result we can determine the structure of $M_1 \# M_2$ We first recall some properties of principal indecomposable modules [3, § 54]. Let V_1, V_2 be principal indecomposable $K(G)$-modules. Then $V_i \cong K(G)e_i$ from some primitive idempotent e_i, $i = 1, 2$. V_i has an unique maximal submodule isomorphic to rad $K(G)e_i$. We denote this submodule rad V_i. $K(G)e_i/\text{rad}\,K(G)e_i$ is an irreducible $K(G)$-module which we denote by $\bar{V}_i$, $i = 1, 2$. $V_1 \cong V_2$ if and only if $\bar{V}_1 \cong \bar{V}_2$.

THEOREM 2.2. *$M_1 \# M_2$ is completely reducible. $M_1 \# M_2 \cong k(T_1 \oplus \cdots \oplus T_r)$, where the $\{T_i\}$ are nonisomorphic irreducible $K(G)$-modules and $k = m_K(N_1)m_K(N_2)/m_K(N_1 \# N_2)$. The $\{T_i\}$ have common K-dimension s, where $s = m_K(N_1 \# N_2)(L : K)(N_1 \# N_2 : E)$.*

Proof. Let $V_{i1}, V_{i2}, \cdots, V_{in(i)}$ be the set of (isomorphic) principal indecomposable $K(G_i)$-modules such that $\bar{V}_{ij} \cong M_i$, $i = 1, 2$, $j = 1, 2, \cdots, n(i)$. Let $C_i = V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{in(i)}$. C_i is a component of $K(G_i)$, $i = 1, 2$. $\bar{C}_i = \bar{V}_{i1} \oplus \cdots \oplus \bar{V}_{in(i)} \cong M_i \oplus \cdots \oplus M_i$, $i = 1, 2$. $\bar{C}_i$ is the sum of all the minimal left ideals of $K(G_i)/\mathrm{rad}\, K(G_i)$ which are isomorphic to M_i, $i = 1, 2$. Therefore $\bar{C}_i$ is a simple component of $K(G_i)/\mathrm{rad}\, K(G_i)$, $i = 1, 2$ [3, Th. 25.15]. Let N be the radical of $\bar{C}_1 \otimes_K \bar{C}_2$. Then N^L is contained in the radical of $\bar{C}_1^L \otimes_L \bar{C}_2^L$. But $\bar{C}_i^L \cong D_{i1} \oplus \cdots \oplus D_{im(i)}$, $i = 1, 2$, where the $\{D_{ij}\}$ are central simple algebras over L. Therefore $\bar{C}_1^L \otimes_L \bar{C}_2^L$ has zero radical, so $N = 0$, i.e. $\bar{C}_1 \otimes_K \bar{C}_2 \cong C_1 \otimes C_2/\mathrm{rad}\,(C_1 \otimes_K C_2)$. Since $C_1 \otimes_K C_2$ is a component of $K(G)$, we may express it as a direct sum of principal indecomposable $K(G)$-modules, $C_1 \otimes_K C_2 = Y_1 \oplus \cdots \oplus Y_s$. Then $C_1 \otimes_K C_2/\mathrm{rad}\, C_1 \otimes_K C_2 \cong \bar{Y}_1 \oplus \cdots \oplus \bar{Y}_s$ and the $\{\bar{Y}_i\}$ are irreducible $K(G)$-modules. We have

$$\bar{Y}_1 \oplus \cdots \oplus \bar{Y}_s \cong \bar{C}_1 \otimes_K \bar{C}_2 \cong \sum_{i,j} \bar{V}_{1i} \otimes_K \bar{V}_{2j} \cong \sum M_1 \# M_2 .$$

Let $M_1 \# M_2 = X_1 \oplus \cdots \oplus X_r$, the X_i being indecomposable $K(G)$-modules. By the Krull-Schmidt Theorem each X_i is isomorphic to some $\bar{Y}_j$. Therefore $M_1 \# M_2$ is completely reducible [3, Th. 15.3]. Let $M_1 \# M_2 \cong \bar{Y}_1 \oplus \cdots \oplus \bar{Y}_r$. We have previously observed that

$$(M_1 \# M_2)^E \cong m_K(N_1)m_K(N_2)(\sum \sigma N_1 \# \tau N_2),\ \sigma \in \overline{\mathscr{H}_1},\ \tau \in \overline{\mathscr{H}_2} .$$

Let σ_1, σ_2 be elements of $\overline{\mathscr{H}_1}$. If $\sigma_1 N_1 \# \tau_1 N_2 \cong \sigma_2 N_2 \# \tau_2 N_2$ where τ_1 and τ_2 are in $\overline{\mathscr{H}_2}$, then $\sigma_1\psi_1 = \sigma_2\psi_2$ and so $\sigma_1 = \sigma_2$. Since $(M_1 \# M_2)^E \cong Y_1^E \oplus \cdots \oplus Y_r^E$, it is immediate from Proposition 2.1 that $M_1 \# M_2 \cong k(T_1 \oplus \cdots \oplus T_r)$, where the T_i are nonisomorphic irreducible $K(G)$-modules and $k = m_K(N_2)/m_K(N_1 \# N_2)$. The K-dimension of any of the T_i's is $m_K(N_1 \# N_2)(L : K)(N_1 \# N_2 : E)$ in view of Theorem 1.4 (a) and $K(\psi_1, \psi_2) = K(\sigma\psi_1, \tau\psi_2) = L$ for all $\sigma, \tau \in \mathscr{G}(E \mid K)$.

REMARK. Let V_i be a principal indecomposable $K(G_i)$-module $i = 1, 2$. The proof of Theorem 2.2 shows that $V_1 \# V_2$ is a principal indecomposable $K(G)$-module if and only if $\bar{V}_1 \# \bar{V}_2$ is an irreducible $K(G)$-module. If the V_i are indecomposable, but not necessarily principal indecomposable $K(G_i)$-modules, $i = 1, 2$, then $V_1 \# V_2$ is indecomposable if and only if $d_{G_1}(V_1) \otimes_K d_{G_2}(V_2)$ is a division algebra, where

$$d_{G_i}(V_i) = \operatorname*{Hom}_{K(G_i)}(V_i, V_i)/\mathrm{rad} \operatorname*{Hom}_{K(G_i)}(V_1, V_i) , \qquad i = 1, 2$$

[5, p. 438].

We now turn to the question of when $M_1 \# M_2$ is irreducible.

THEOREM 2.3. *$M_1 \# M_2$ is an irreducible $K(G)$-module if and only if the following conditions are satisfied:*

(a) $m_K(N_1)m_K(N_2) = m_K(N_1 \# N_2)$.

(b) $\mathscr{G}(E \mid K) = \mathscr{H}_1\mathscr{H}_2$.

(c) $(K(\psi_1) : K)\cdot(K(\psi_2) : K) = (K(\psi_1, \psi_2) : K)$.

Proof. We begin by showing that $\mathscr{G}(E \mid K) = \mathscr{H}_1\mathscr{H}_2$ if and only if $\sigma\mathscr{H}_1 \cap \tau\mathscr{H}_2$ is nonempty for all $\sigma, \tau \in \mathscr{G}(E \mid K)$. Since $K(\psi_i)$ is a normal extension of K, $\mathscr{H}_1$ is a normal subgroup of $\mathscr{G}(E \mid K)$, $i = 1, 2$. Assume that $\mathscr{G}(E \mid K) = \mathscr{H}_1\mathscr{H}_2$ and let $\sigma, \tau \in \mathscr{G}(E \mid K)$. Then $\sigma\mathscr{H}_1 = h_2\mathscr{H}_1$, $\tau\mathscr{H}_2 = h_1\mathscr{H}_2$ where $h_i \in \mathscr{H}_i$, $i = 1, 2$. Then

$$h_2h_1 \in \sigma\mathscr{H}_1 \cap \tau\mathscr{H}_2 = h_2\mathscr{H}_1 \cap \mathscr{H}_2 h_1 .$$

Conversely, assume that $\sigma\mathscr{H}_1 \cap \tau\mathscr{H}_2$ is nonempty for all $\sigma, \tau \in \mathscr{G}(E \mid K)$ and let $x \in \mathscr{G}(E \mid K)$. Then $x\mathscr{H}_1 \cap \mathscr{H}_2$ is nonempty so $xh_1 = h_2$ for some $h_1 \in \mathscr{H}_1$, $h_2 \in \mathscr{H}_2$. Therefore $x \in \mathscr{H}_2\mathscr{H}_1 = \mathscr{H}_1\mathscr{H}_2$ so $\mathscr{G}(E \mid K) = \mathscr{H}_1\mathscr{H}_2$. We have $(M_1 \# M_2)^E \cong m_K(N_1)m_K(N_2)(\sum \sigma N_1 \# \tau N_2)$, the sum take over all $\sigma \in \overline{\mathscr{H}_1}$, $\tau \in \overline{\mathscr{H}_2}$. Assume that $M_1 \# M_2$ is irreducible. By Theorem 1.4 (a) we see that (a) is necessary. For each $\sigma \in \mathscr{H}_1$, $\tau \in \mathscr{H}_2$ there must exist a $\gamma \in \mathscr{G}(E \mid K)$ such that $\sigma N_1 \# \tau N_2 \cong \gamma N_1 \# \gamma N_2 \cong \gamma(N_1 \# N_2)$. Then $\lambda \in \sigma\mathscr{H}_1 \cap \tau\mathscr{H}_2$ so (b) is necessary. By Theorem 1.4 (a) the total number of composition factors of $(M_1 \# M_2)^E$ must be $m_K(N_1 \# N_2)\cdot(K(\psi_1, \psi_2) : K)$. Therefore

$$(K(\psi_1, \psi_2) : K) = (K(\psi_1) : K)\cdot(K(\psi_2) : K)$$

so (c) is necessary. The same argument shows that (a), (b) and (c) are sufficient since $(M_1 \# M_2)^E \cong W^E$, W an irreducible $K(G)$-module, implies $M_1 \# M_2 \cong W$.

COROLLARY 2.4. *Let $G_1 = G_2$, $G = G_1 \times G_1$. Let M_1 be an irreducible $K(G_1)$-module. Then $M_1 \# M_1$ is irreducible if and only if M_1 is an absolutely irreducible $K(G_1)$-module.*

Proof. The if part of the theorem is immediate from [1, Footnote, p. 587]. Conversely, let $M_1 \# M_1$ be irreducible. In order for condition (b) of Theorem 2.3 hold, $K(\psi_1)$ must equal K. If N_1^* is realizable (in the context of § 1) in a field F then so is N_1. Therefore $m_K(N_1 \# N_1) \leqq m_K(N_1)$. But then condition (a) of Theorem 2.3 holds if and only if $m_K(N_1) = 1$. Therefore N_1^* is realizable in K and so M_1 is absolutely irreducible.

The next result gives a more easily applied criterion for $M_1 \# M_2$ to be irreducible.

THEOREM 2.5. *Let $K(\psi_1) = K$ and assume that $(m_K(N_1), m_K(N_2)) = 1$. Then $M_1 \# M_2$ is irreducible.*

Proof. Since $K(\psi_1) = K$, conditions (b) and (c) of Thereom 2.3 are satisfied so we need only prove that $m_K(N_1 \# N_2) = m_K(N_1)m_K(N_2)$. It follows from Theorem 2.2 that $m_K(N_1)m_K(N_2) \geqq m_K(N_1 \# N_2)$. Since $(m_K(N_1), m_K(N_2)) = 1$ we need only show that both $m_K(N_1)$ and $m_K(N_2)$ divide $m_K(N_1 \# N_2)$. By Theorem 1.4 (c) we may assume that K has characteristic zero. Let F be a maximal subfield of the division algebra component of $A_1 \otimes_K A_2$. Then $N_1^* \# N_2^*$ is realizable in F and $(F : K(\psi_1, \psi_2)) = m_K(N_1 \# N_2)$ [3, Th. 68.6]. In view of Theorem 1.4 (e) it is sufficient to prove that N_1^* and N_2^* are realizable in F. Let B_i be the simple component of $A_i \otimes_K F$ corresponding to N_i^*. Since $N_1^* \# N_2^*$ is realizable in F, $B_1 \otimes_F B_2 \cong (F)_r$. Therefore B_1 and B_2 are inverse isomorphic elements of the Brauer group of F and hence their division algebra components have the same index. If $B_i = (D_i)_{n(i)}$ we have $m(D_1) = m(D_2)$. From Theorem 1.4 (c) we have $m_F(N_1^*) = m_F(N_2^*)$. Since $m_F(N_1^*)$ divides $m_K(N_1)$ and $(m_K(N_1), m_K(N_2)) = 1$, $m_F(N_1^*) = m_F(N_2^*) = 1$.

COROLLARY 2.6. *Let the orders of G_1 and G_2 be relatively prime. If $K(\psi_1) = K$, then $M_1 \# M_2$ is irreducible.*

Proof. By Theorem 1.4 (d) $m_K(N_i)$ divides $(N_i : E)$, $i = 1, 2$. For K of characteristic zero, $(N_i : E)$ divides the order of G_i [3, Th. 33.7]. For K of characteristic p we have $m_K(N_i) = 1$, $i = 1, 2$. In both cases Corollary 2.6 is immediate from the preceding theorem.

Given an irreducible $K(G)$-module M, it is natural to ask when M is isomorphic to $M_1 \# M_2$ for irreducible $K(G_i)$-modules M_i, $i = 1, 2$. If such M_i exist, $i = 1, 2$, we say that M is *factorizable*. If M is a $K(G)$-module, we denote by M_{G_i} the left $K(G_i)$-module obtained by restriction of the set of operators on M from $K(G)$ to $K(G_i)$, $i = 1, 2$.

THEOREM 2.7. *Let M be an irreducible $K(G)$-module. Then $M_{G_i} \cong e_i M_i$ for irreducible $K(G_i)$-modules M_i, $i = 1, 2$. M is factorizable if and only if $M_1 \# M_2$ is irreducible, in which case $M \cong M_1 \# M_2$.*

Proof. $M_{G_i} \cong e_i(M_i \oplus M_i^{(g)} \oplus \cdots \oplus M_i^{(h)})$, where the $M_i^{(g)}$ are conjugates by elements of G of the irreducible $K(G_i)$-module M_i, $i = 1, 2$ [3, Th. 49.2]. Since G_1 and G_2 commute, all conjugates of M_i are equivalent so $M_{G_i} \cong e_i M_i$, $i = 1, 2$. Let

$$M^E \cong m_K(N)(\sum_j \sigma_j N),\ \sigma_j \in \mathscr{G}(E \mid K) ,$$

where N is an irreducible $E(G)$-module. Since N is factorizable, we have $N \cong N_1 \# N_2$ where the N_i are irreducible $E(G_i)$-modules, $i = 1, 2$. Then $M^E \cong m_K(N)(\sum_j \sigma_j(N_1 \# N_2)$. Since

$$N_{G_i} \cong f_i N_i,\ (M^E)_{G_i} \cong f_i m_K(N)(\sum_j \sigma_j N_i) ;$$

so $(M_{G_i})^E \cong (M^E)_{G_i} \cong f_i m_K(N)(\sum_j \sigma_j N_i)$. Therefore

$$(M_i)^E \cong \frac{f_i m_K(N)}{e_i} (\sum_j \sigma_j N_i) ,$$

so $N_1 \# N_2$ is a component of both M^E and $(M_1 \# M_2)^E$. If $M \cong M'_1 \# M'_2$, $\{M'_i\}$ irreducible $K(G_i)$-modules, then $M_{G_i} \cong k_i M'_i$; and so $(M'_i)^E$ and $(M_i)^E$ both have N_i as a component, $i = 1, 2$. Therefore $M_i \cong M'_i$, $i = 1, 2$; and similarly, if $M_1 \# M_2$ is irreducible, we have $M \cong M_1 \# M_2$.

The well known theory of central simple algebras over algebraic number fields has an interesting application to outer tensor products. Let K be an algebraic number field, G an arbitrary finite group, and E a finite normal extension of K which is a splitting field for G. We denote by $G^{(r)}$ the direct product of G with itself r times. Let N be an irreducible $E(G)$-module. $N^{(r)}$ will denote the $E(G^{(r)})$-module $N \# N \# \cdots \# N$, the outer tensor product of N with itself r times. Let ψ be the character of N.

THEOREM 2.8. *$m_K(N)$ is the smallest integer r such that $N^{(r)}$ is realizable in $K(\psi)$.*

Proof. Since $m_K(N) = m_{K(\psi)}(N)$ we may assume that $K(\psi) = K$. Let A be the simple component of $K(G)$ corresponding to N. Then $A^{(r)} = A \otimes_K A \otimes_K \cdots \otimes_K \cdots \otimes_K A$, r times, is the simple component of $K(G^{(r)})$ corresponding to $N^{(r)}$. $N^{(r)}$ is realizable in K if and only if $A^{(r)} \cong (K)_s$. Let t be the exponent of A. Then $A^{(r)} \cong (K)_s$ if and only if t divides r. Let D be the division algebra component of A. Since A is central simple over an algebraic number field, $t = m(D)$ [4, Satz 7, p. 119]. The desired conclusion now follows from Theorem 1.4 (b).

3. Derivable division algebras. Let D be a division algebra and K a subfield of the center of D.

DEFINITION 3.1. *D is K-derivable* if $D \cong \mathrm{Hom}_{K(G)}(M, M)$ for some finite group G and irreducible $K(G)$-module M.

Theorems 2.3 and 2.5 have an immediate application to the theory of derivable division algebras.

THEOREM 3.2. *Let D_1 and D_2 be K-derivable division algebras. Let L_1, L_2 be the centers of D_1 and D_2 respectively. If D_1 is central over K, i.e. $L_1 = K$, and if $(m(D_1), m(D_2)) = 1$, then $D_1 \otimes_K D_2$ is a division algebra. In general, the following conditions are necessary and sufficient for $D_1 \otimes_K D_2$ to be a division algebra:*

(a) $m(D_1)m(D_2) = m(D_3)$, where D_3 is the division algebra component of a simple component of $(D_1 \otimes_K D_2) \otimes_K L_1 \cdot L_2$, $L_1 \cdot L_2$ being a composite of L_1 and L_2 over K.

(b) Let E be a finite normal extension of K which is a splitting field for D_1 and D_2. Then $\mathscr{G}(E \mid K) = \mathscr{H}_1\mathscr{H}_2$ where $\mathscr{H}_i = \mathscr{G}(E \mid L_i)$, $i = 1, 2$.

(c) $(L_1 : K)(L_2 : K) = (L_1 \cdot L_2 : K)$.

Proof. Let $D_i \cong \mathrm{Hom}_{K(G_i)}(M_i, M_i)$, $i = 1, 2$. Set $G = G_1 \times G_2$. Then

$$\mathop{\mathrm{Hom}}_{K(G)}(M_1 \# M_2, M_1 \# M_2) \cong \mathop{\mathrm{Hom}}_{K(G_1)}(M_1, M_1) \mathop{\otimes}_K \mathop{\mathrm{Hom}}_{K(G_2)}(M_2, M_2) \cong D_1 \mathop{\otimes}_K D_2 .$$

If $M_1 \# M_2$ is an irreducible $K(G)$-module, then $D_1 \otimes_K D_2$ is a skewfield [3, Th. 26.8]. Conversely, if $D_1 \otimes_K D_2$ is a skewfield, then $M_1 \# M_2$ is indecomposable. By Theorem 2.2 $M_1 \# M_2$ is irreducible. Theorems 2.3 and 2.5 now yield the desired result.

If K is an infinite field of prime characteristic, there will, in general, exist division algebras central over K which are not fields. Theorem 1.4 (c) proves that such division algebras are not derivable. We now consider fields of characteristic zero. If D is a K-derivable division algebra and L is the center of D, then Theorem 1.1 shows that D is L-derivable. For this reason we shall consider only central division algebras. Our final result shows the existence of infinitely many division algebras D which are not K-derivable for any subfield K of D.

Let $B(K)$ denote the Brauer group of K. Let $B_0(K)$ be the subset of $B(K)$ consisting of those classes of central simple algebras which have K-derivable division algebra components.

THEOREM 3.3. *$B_0(K)$ is a subgroup of $B(K)$. If K is an algebraic number field which is not an abelian extension of the rationals, then $B_0(K)$ has infinite index in $B(K)$.*

Proof. K is K-derivable since $K \cong \mathrm{Hom}_{K(G)}(N, N)$ with G the identity group and N the trivial $K(G)$-module. Therefore $B_0(K)$ is nonempty. Since every element of $B(K)$ has finite order, to show that $B_0(K)$ is a subgroup of $B(K)$ it is sufficient to prove that $B_0(K)$ is closed under $\otimes_K$. Let $\{A_1\}, \{A_2\} \in B_0(K)$ with $D_i \in \{A_i\}$, D_i a division algebra central over K and $D_i \cong \mathrm{Hom}_{K(G_i)}(M_i, M_i)$, $i = 1, 2$. Let $G = G_1 \times G_2$. Then $D_1 \otimes_K D_2 \cong \mathrm{Hom}_{K(G)}(M_1 \# M_2, M_1 \# M_2)$. From the proof of Theorem 2.2 we see that $M_1 \# M_2 \cong kN$, N an irreducible $K(G)$-module. Let $\mathrm{Hom}_{K(G)}(N, N) = D_3$, D_3 a division algebra central over K. Then $\mathrm{Hom}_{K(G)}(M_1 \# M_2, M_1 \# M_2) \cong (D_3)_k$ so $D_1 \otimes_K D_2 = (D_3)_r$. Therefore $A_1 \otimes_K A_2 \cong (D_3)_s$, so $B_0(K)$ is a subgroup of $B(K)$.

Assume that K is an algebraic number field which is not an Abelian extension of the rationals. Let L be the maximal abelian subfield of K. There exists a rational prime p which splits completely in K [6, Satz 114, p. 126]. (As Dr. Basil Gordon has pointed out, this result can also be proved purely algebraically.) Let L_0, K_0 denote the rings of algebraic integers of L, K, respectively. There exist prime ideals $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ of K_0 such that $\mathfrak{Y}_1 \cap L_0 = \mathfrak{Y}_2 \cap L_0$ and $\mathfrak{Y}_1 \cap Z = \mathfrak{Y}_2 \cap Z = (p)$, Z the ring of rational integers [9, Corollary, p. 287]. There exists a division algebra D central over K for which $h(D, \mathfrak{Y}_1) = 1/3$, $h(D, \mathfrak{Y}_2) = 2/3$, $h(D, \mathfrak{Y}) = 0$ for all other primes $\mathfrak{Y}$ of K, finite or infinite, where $h(D, \mathfrak{Y})$ denotes the Hasse invariant of D at $\mathfrak{Y}$ [4, Satz 9, p. 119]. We shall show that D is not K-derivable.

Suppose $D \cong \mathrm{Hom}_{K(G)}(M, M)$ for some finite group G and irreducible $K(G)$-module M. Let E be a finite normal extension of K which is a splitting field for G. Since D is central over K, $M^E = m_K(N) \cdot N$, N an irreducible $E(G)$-module. Let ψ be the character of N. Let Q be the field of rational numbers. Then $Q(\psi)$ is a subfield of K and since $Q(\psi)$ is an Abelian extension of Q, $Q(\psi)$ is a subfield of L. Let A be the simple component of $Q(\psi)(G)$ corresponding to N. Then A is a central simple $Q(\psi)$-algebra, and $A \otimes_{Q(\psi)} K \cong (D)_r$. Since $\mathfrak{Y}_1 \cap L_0 = \mathfrak{Y}_2 \cap L_0$ we have $\mathfrak{Y}_1 \cap Q(\psi)_0 = \mathfrak{Y}_2 \cap Q(\psi)_0 = \mathfrak{Y}_3$. Let $n(\mathfrak{Y}_i)$ denote the residue class degree of $\mathfrak{Y}_i$ over $\mathfrak{Y}_3$, $i = 1, 2$. Since p splits completely in K, $n(\mathfrak{Y}_1) = n(\mathfrak{Y}_2) = 1$. Let $h(A, \mathfrak{Y}_3)$ be the Hasse invariant of A at $\mathfrak{Y}_3$. Then $n(\mathfrak{Y}_i)h(A, \mathfrak{Y}_3) = h(A, \mathfrak{Y}_3)$ is the Hasse invariant of D at $\mathfrak{Y}_i$, $i = 1, 2$ [4, Satz 4, p, 113]. This contradicts the fact that $h(D, \mathfrak{Y}_1) \neq h(D, \mathfrak{Y}_2)$.

We have shown that $B_0(K)$ is a proper subgroup of $B(K)$. Suppose that $B_0(K)$ has finite index n in $B(K)$. Let $\mathfrak{Y}_4$ be a prime ideal of K_0, $\mathfrak{Y}_4$ distinct from both $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$. Let $\{A_1\}$ be the class of central simple K-algebras whose Hasse invariants are

$$h(A_1, \mathfrak{Y}_1) = 1/3n,\ h(A_1, \mathfrak{Y}_2) = 2/3n,\ h(A_1, \mathfrak{Y}_4) = (n-1)/n,\ h(A_1, \mathfrak{Y}) = 0$$

for all other primes of K, finite or infinite. Let $\{A_1\}^n$ be the nth power of $\{A_1\}$ in $B(K)$ i.e. $\{A_1\}^n = A_1 \otimes_K A_1 \otimes_K \cdots \otimes_K A_1\}$. The Hasse invariants of $\{A_1\}^n$ are precisely the Hasse invariants of $\{D\}$ [4, Satz 3, p. 112]. Therefore $\{A_1\}^n$ equals $\{D\}$ [4, Satz 8, p. 119]. But $B_0(K)$ has index n in $B(K)$ so $\{A_1\}^n \in B_0(K)$. This contradicts $\{D\} \notin B_0(K)$, so $B_0(K)$ is a subgroup of infinite index in $B(K)$.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES