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LOCALLY TRIVIAL C^r GROUPOIDS AND THEIR REPRESENTATIONS

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We develop a theory of representations of a locally trivial C^r groupoid, Z, on a C^r fiber bundle, E (with fiber Y, Lie group G, and base space $M =$ the set of units of Z).

A covariant functor, A, is defined, sending E into a locally trivial C^r groupoid $A(E) =$ the groupoid of admissible maps between fibers of E, with a natural C^r structure. A C^r bundle map $h: E \to E'$ is sent into a C^r isomorphism $A(h): A(E) \to A(E')$. Properties of the functor A are studied.

A C^r representation of Z on E is defined as a C^r homomorphism $\rho: Z \to A(E)$. Let Z_{ee} be the group of elements in Z with e as the left and right unit. We obtain the important result that a C^r homomorphism $\rho_e: Z_{ee} \to A(E_e)$ has an (essentially) unique extension to a C^r representation of Z on a C^r fiber bundle E', where E' is determined by Z and ρ_e. This leads to interesting applications in differential geometry. The representations of L^k, the (locally trivial C^∞) groupoid of invertible k-jets of C^∞ maps of a C^∞ manifold, M, into itself, provide (but are not the same as) natural fiber bundles of order k in the sense of Nijenhuis.

The correspondence obtained in 4.93 and 4.94 between transitive C^r representations of Z and closed subgroups of Z_{ee} is related to the classification of geometric objects for $Z = L^k$ and to the classification of covering spaces for $Z =$ the fundamental groupoid.

In § 5 we discuss some examples-the universal covering groupoid of Z, and the fundamental groupoid.

Locally trivial C^r groupoids are as defined in [2], except that we deal only with sets in the usual sense, and our groupoids are always transitive, i.e. groupoids in the sense of Brandt. We find it useful to introduce the notion of a C^r coordinate groupoid in § 2, and show that locally trivial C^r groupoids arise from C^r coordinate groupoids like fiber bundles arise from coordinate bundles.

In § 1 we give a definition for a groupoid which is somewhat more elaborate than the usual one, but it introduces the notation used throughout the paper and it includes some essential properties of groupoids.

1. A groupoid, Z over M (in the sense of Brandt), consists of the following:

1.1. A set, Z, with a distinguished subset, M, called the set of *units* in Z.

1.2. A map $l \times r: Z \to M \times M$, mapping Z onto $M \times M$. We introduce the following notation: $Z_{qp} = (l \times r)^{-1}(q, p)$ and $Z_B = (l \times r)^{-1}(B)$ for $B \subseteqq M \times M$. The subscripts on $\Phi_{qp} \in Z$ mean that $l \times r(\Phi_{qp}) = (q, p) = $ (*left unit* of Φ_{qp}, *right unit* of Φ_{qp}). We require that $l \times r(q) = (q, q)$ for $q \in M$, and will sometimes write 1_{qq} for q.

1.3. A law of composition, $(\cdot): D \to Z$, where $D = (l \times r \times l \times r)^{-1}(M \times \Delta \times M) \subseteqq Z \times Z$, and Δ is the diagonal in $M \times M$, satisfying the following conditions:

1.31. $\Psi_{rq} \cdot \Phi_{qp} \in Z_{rp}$.

1.32. (associative law) $\Lambda_{sr} \cdot (\Psi_{rq} \circ \Phi_{qp}) = (\Lambda_{sr} \circ \Psi_{rq}) \circ \Phi_{qp}$.

1.33. (units and inverse) For $\Phi_{qp} \in Z$, $\Phi_{qp} = \Phi_{qp} \circ 1_{pp} = 1_{qq} \circ \Phi_{qp}$, and there exists an element $(\Phi_{qp})^{-1} \in Z_{pq}$ such that $\Phi_{qp} \circ (\Phi_{qp})^{-1} = 1_{qq}$ and $(\Phi_{qp})^{-1} \circ \Phi_{qp} = 1_{pp}$.

1.4. DEFINITION. A function $\rho: Z \to Z'$ (where Z' is also a groupoid) is called a *homomorphism* if and only if

1.41. $\rho(M) \subseteqq M' = $ units of Z'. We will write $\bar{\rho}$ for the restriction $\rho|_M: M \to M'$.

1.42. The following diagram commutes

$$\begin{array}{ccc} Z & \xrightarrow{\rho} & Z' \\ \big\downarrow{\scriptstyle l \times r} & & \big\downarrow{\scriptstyle l' \times r'} \\ M \times M & \xrightarrow[\bar{\rho} \times \bar{\rho}]{} & M' \times M' \end{array}$$

1.43. From 1.42, $\rho \times \rho(D) \subseteqq D'$. We require that

$$\begin{array}{ccc} D & \xrightarrow{\rho \times \rho} & D' \\ \big\downarrow{\scriptstyle (\circ)} & & \big\downarrow{\scriptstyle (\circ)} \\ Z & \xrightarrow[\rho]{} & Z' \end{array}$$

commute.

We establish the following notation.

1.45. Unless otherwise indicated, the r in C^r may be any integer $\geqq 1$, or ∞, or ω (for the analytic case), but is fixed throughout the discussion.

A C^r coordinate bundle, E consists of a Lie group G (the group of E) acting effectively on a C^r manifold Y (the fiber of E), so that the evaluation map $G \times Y \to Y$ is a C^r map, and a projection $\pi: E \to M$ sending the total space of E (also called E) onto the C^r manifold M (the base space of E), and an indexed set of coordinate functions $\phi_j: V_j \times Y \to E_{V_j} = \pi^{-1}(V_j)$ as in [9, pp. 7 and 8], with the transition functions $g_{ij}: V_i \cap V_j \to G$ required to be C^r maps. Then a canonical C^r manifold structure is defined for the total space E, using the C^r structures of M and Y and the coordinate functions, so that π and the coordinate functions are C^r maps. C^r fiber bundles are then defined in terms of C^r coordinate bundles, as in [9, Ch. I]. We use E and E' to denote C^r fiber bundles with group G, fiber Y, base space M, and total spaces E and E' respectively.

For a function $h: E \to E'$ we assume $h(E_p) = E'_p$, that $h_p = h \mid E_p$ is bijective, and that h_p and h_p^{-1} are both C^r maps. C^r admissible maps and C^r bundle maps are as defined in [9], replacing the topological requirements by the appropriate C^r requirements. We do not always require that h be admissible, or even a C^r map, in this paper.

Given a C^r fiber bundle E, we form the set of maps

1.5 $$P(E) = \{\Phi_{q0}: Y \to E_q \mid q \in M \text{ and } \Phi_{q0} \text{ is an admissible map}\}\ ,$$

and will regard $P(E)$ as the associated principal bundle of E.

Consider the set of maps

1.6 $$A(E) = \{\Phi_{qp}: E_p \to E_q \mid (q, p) \in M \times M \text{ and } \Phi_{qp} \text{ is an admissible map}\}\ .$$

Then $A(E)$ has a natural groupoid structure over M, with $l \times r(\Phi_{qp}) = (q, p)$ and the composition defined as the composition of maps.

2. The study of locally trivial C^r groupoids is simplified by consideration of C^r coordinate groupoids. A C^r coordinate groupoid (Z, Σ_e) over M, consists of the following:

2.1. A groupoid, Z (over M), and a C^r manifold structure for M.

2.2. A distinguished point $e \in M$ and a Lie group structure for the group Z_{ee}.

2.3. A set of functions $\Sigma_e = \{\alpha: U_\alpha \to Z_{U_\alpha \times e}\}$ U_α is open in M and $l \circ \alpha =$ identity map, satisfying

2.31. $\bigcup_{\alpha \in \Sigma} U_\alpha = M$.

2.32. For α and $\beta \in \Sigma_e$, the map

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \to Z_{ee}\ ; \qquad g_{\alpha\beta}(q) = (\alpha(q))^{-1} \cdot \beta(q)\ ,$$

is a C^r map.

Given a C^r coordinate groupoid (Z, Σ_e) we obtain a canonical C^r fiber bundle structure for Z.

For α and $\beta \in \Sigma_e$, the map

2.4 $$C_{\alpha\beta}: U_\alpha \times Z_{ee} \times U_\beta \to Z_{U_\alpha \times U_\beta}$$

is defined by $C_{\alpha\beta}(q, \Phi, p) = \alpha(q) \circ \Phi \circ (\beta(p))^{-1}$.

2.5. THEOREM. *A C^r coordinate groupoid, (Z, Σ_e) may be regarded as a C^r coordinate bundle over $M \times M$, with projection $l \times r: Z \to M \times M$, fiber Z_{ee}, and group $Z_{ee} \times Z_{ee}$ acting on Z_{ee} by left and right composition, i.e. $(\Phi, \Psi)\Gamma = \Phi \cdot \Gamma \cdot \Psi^{-1}$, and coordinate functions $C_{\alpha\beta}$ as given in 2.4.*

Proof. For γ and $\delta \in \Sigma_e$, we obtain

$$C_{\gamma\delta}(q, \Phi, p) = C_{\alpha\beta}(q, g_{\alpha\gamma}(q) \cdot \Phi \cdot (g_{\beta\delta}(p))^{-1}, p)$$

for $q \in U_\alpha \cap U_\gamma$ and $p \in U_\beta \cap U_\delta$. By 2.32, the transition functions are C^r maps.

A canonical C^r structure for Z is obtained via the $C_{\alpha\beta}$ coordinate functions from the C^r structures of Z_{ee} and M. With this C^r structure Z becomes a (LT) locally trivial C^r groupoid in the sense of Ehresman [1 and 2], as we show in the next theorem.

2.6. THEOREM. *Using the canonical C^r structure for Z (as determined from Σ_e) the composition $(\cdot): D \to Z$ and inverse $(-1): Z \to Z$ are C^r maps. The function $(q, p) \to \alpha(q) \cdot (\beta(p))^{-1}$, for $\alpha, \beta \in \Sigma_e$ is a C^r local section in $l \times r: Z \to M \times M$.*

Proof. The result that $(\cdot)$ and (-1) are C^r maps follows from the corresponding properties of the Lie group Z_{ee}, using the $C_{\alpha\beta}$ maps.

Conversely, given a LTC^r groupoid Z over M and a point $e \in M$, we obtain a C^r coordinate groupoid (Z, Σ_e). We define

2.7. $\Sigma_e =$ the set of local C^r sections in $l: Z_{M \times e} \to M$, regarding $Z_{M \times e}$ as a closed submanifold in Z. Z_{ee} has a natural Lie group structure (see [5]). Then Σ_e satisfies 2.31 by the locally trivial condition on Z and satisfies 2.32 by the C^r condition on composition and inverse. Accordingly (Z, Σ_e) forms a C^r coordinate groupoid, and the canonical C^r structure for Z obtained via the $C_{\alpha\beta}$ maps is the same as the original C^r structure for Z.

If Z' is a subgroupoid of Z with the same set of units, and Z' is also a C^r submanifold of Z, then Z' is a locally trivial C^r groupoid.

We establish the following notation.

2.71. $\rho: Z \to Z'$ will be called a C^r *homomorphism*, if and only if Z and Z' are locally trivial C^r groupoids and ρ is a homomorphism as in (1.4), and ρ is also a C^r map. ρ will be called a C^r *isomorphism* if and only if ρ is a bijective C^r homomorphism, and $\bar{\rho}: M \to M'$ is the identity map. As a consequence of (4.1), ρ is a C^r isomorphism implies ρ^{-1} is also a C^r isomorphism.

3. We obtain a natural C^r coordinate groupoid structure for $A(E)$ (defined in 1.6), making $A(E)$ a LTC^r groupoid.

3.1. CONSTRUCTION. The Lie group structure for $A_{ee}(E)$ is obtained via the isomorphism $I: G \to A_{ee}(E)$ induced by an admissible map $i: Y \to E_e$. Σ_e is defined as the set of maps $\{\phi \times i\}$ where $\phi: V \times Y \to E_V$ is a coordinate function for E, $i: Y \to E_e$ is an admissible map, and $\phi \times i: U \to A(E)_{U \times e}$ is given by $(\phi \times i)(q) = \phi(q, i^{-1}(\ ))$.

3.2. REMARKS. A may be regarded as a convariant functor sending^r a C' fiber bundle E into LTC^r groupoid $A(E)$, and sending a C^r bundle map $h: E \to E'$ into a C^r isomorphism $A(h): A(E) \to A(E')$; $A(h)(\Phi_{qp}) = h_q \circ \Phi_{qp} \circ h_p^{-1}$. We summarize some properties of the functor A below.

3.21. If $E = E''$, except that the group G'' of E'' is a Lie subgroup of G, then the identity map $h: E'' \to E$ defines the inclusion $A(h): A(E'') \subseteqq A(E)$ of $A(E'')$ as a LTC^r subgroupoid of $A(E)$.

3.22. Suppose $f: M' \to M$ is a C^r map, and $f^*(E)$ is the pullback of E over M'. $f \times f: M' \times M' \to M \times M$ defines a pullback $(f \times f)^* A(E)$ which is then a LTC^r groupoid over M'. There is a natural C^r isomorphism $A(f^*(E)) \cong (f \times f)^*(A(E))$. In particular, if N is a submanifold of M, the inclusion $h: E_N \subseteqq E$ defines the inclusion $A(h): A(E_N) \subseteqq A(E)$.

Given a locally trivial C^r groupoid, Z, and a Lie group isomorphism $I: G \to Z_{ee}$, we obtain a C^r principal G bundle, Z_I by the following.

3.4. CONSTRUCTION. For $\alpha \in \Sigma_e$ (as in 2.7), we define $\alpha \times I: U_\alpha \times G \to Z_{U_\alpha \times e}$ by $\alpha \times I(q, \Phi) = \alpha(q) \cdot I(\Phi)$. Let Z_I denote the C^r fiber bundle with projection l, total space $Z_I = Z_{M \times e}$, group G acting on G by left composition, and coordinate functions $\alpha \times I$.

The following theorem is easily verified.

3.5. THEOREM. *Left composition by $\Phi_{qp} \in Z$ is an admissible map $L(\Phi_{qp}): (Z_I)_p \to (Z_I)_q$, and $L: Z \to A(Z_I)$ is a C^r isomorphism.*

It is interesting to note that the principal bundles obtained for different choices of the isomorphism, I, may not be fiber bundle equivalent (see 4.41).

3.51. THEOREM. *Suppose the admissible map $i: Y \to E_e$ induces the isomorphism $I: G \to A_{ee}(E)$. Then there is a natural C^r bundle map (right composition by i) $R_i: A_I(E) \to P(E)$.*

4. A homomorphism $\rho: Z \to Z'$ induces a locally trivial C^r structure in Z' under certain conditions. This idea is made precise in the following theorem.

4.1. THEOREM. *Let Z and Z' be two groupoids with the same set of units, M. Suppose Z is an LTC^r groupoid and that Z'_{ee} is a Lie group. If $\rho: Z \to Z'$ is a homomorphism satisfying*

4.11. *$\bar{\rho}: M \to M$ is the identity map*

4.12. *$\rho|_{Z_{ee}}: Z_{ee} \to Z'_{ee}$ is a Lie group homomorphism then there is a unique LTC^r groupoid structure for Z' such that ρ is a C^r homomorphism.*

Proof. We define $\Sigma'_e = \{\alpha' = \rho \circ \alpha \mid \alpha \in \Sigma_e$ (as in 2.7)$\}$. Since $g_{\alpha'\beta'}(q) = \rho \circ g_{\alpha\beta}(q)$, we see that Σ'_e satisfies 3.42. Conversely, if Z' has an LTC^r structure and h is a C^r map, then Σ'_e is a set of local C^r sections in $l: Z'_{M\times e} \to M$.

4.13. COROLLARY. *Let Z and Z' be LTC^r groupoids over M, and $\rho: Z \to Z'$ a C^r homomorphism such that $\bar{\rho} =$ identity map.* $\ker h = \{\Phi \in Z: \rho(\Phi) = 1'_{ee} \in Z'\}$ *is a closed normal subgroup in Z_{ee}. There are unique LTC^r groupoid structures on $Z/\ker\rho$ and $\rho(Z)$, such that the canonical maps π, ρ' and i are C^r homomorphisms;*

$$Z \xrightarrow{\pi} Z/\ker h \xrightarrow{\rho'} h(Z) \xrightarrow{i} Z' .$$

4.14. COROLLARY. *Let $\rho: Z \to Z'$ and $\rho': Z \to Z''$ be C^r homomorphisms such that $\bar{\rho}$ and $\bar{\rho}'$ are the identity maps. Suppose the restrictions of ρ and ρ' to Z_{ee} (denoted by ρ_e and ρ'_e respectively) commute with a Lie group homomorphism $\rho''_e: Z'_{ee} \to Z''_{ee}$, i.e. $\rho''_e \circ \rho_e = \rho'_e$. Then ρ''_e has a unique extension to a C^r homomorphism $\rho'': Z' \to Z''$ such that $\rho'' \circ \rho = \rho'$.*

Proof. $\Psi'_{qp} \in Z'$ may be written as the composition $\rho(\Phi_{qe}) \circ \Lambda_{ee} \circ \rho(\Gamma_{ep})$. We define $\rho''(\Psi'_{qp}) = \rho'(\Phi_{qe}) \circ \rho''_e(\Lambda_{ee}) \circ \rho'(\Gamma_{ep})$ and note that the result is independent of the particular choice of Φ_{qe} and Γ_{ep}. Then ρ'' has the desired properties.

We next consider representations of LTC^r groupoids, as defined below.

4.2. DEFINITION. A C^r representation of Z on E is defined to be a C^r homomorphism $\rho: Z \to A(E)$, such that $\bar{\rho}: M \to M$ is the identity map.

There are two notions of equivalence which are of interest in connection with groupoid representations. One of these, called weak equivalence involves maps between fiber bundles which may not be admissible.

4.3. DEFINITION. Given $h: E \rightarrow E'$ (see 1.45) we define $A(h)(\Phi_{qp}) = h_q \circ \Phi_{qp} \circ h_p^{-1}$, for $\Phi_{qp} \in A(E)$. h is called *weakly admissible* if and only if $A(h)$ sends $A(E)$ into $A(E')$.

We remark that if h is weakly admissible then

4.31. $A(h): A(E) \rightarrow A(E')$ is a groupoid isomorphism.

4.32. $h_e: E_e \rightarrow E'_e$ is admissible implies that h_p is admissible for all $p \in M$.

Bundle maps are related to groupoid isomorphism in the following theorem.

4.4. THEOREM. *If $h_e: E_e \rightarrow E'_e$ is admissible (resp. weakly admissible) and $A(h_e)$ extends to a C^r isomorphism $H: A(E) \rightarrow A(E')$, then h_e has a unique extension to a C^r bundle map (resp. C^r weakly admissible map) $h: E \rightarrow E'$ such that $A(h) = H$.*

Proof. We define $h_q = H(\Phi_{qe}) \circ h_e \circ \Phi_{qe}^{-1}$ for some $\Phi_{qe} \in (A(E))_{qe}$ and note that h_q is independent of the choice of Φ_{qe}.

As an application of 4.4 we obtain the following corollary, relating to the construction in 3.4. (Assume G is isomorphic to Z_{ee}.)

4.41. COROLLARY. *The set of C^r principal G bundles Z_I obtained via* 3.4 $\bmod C^r$ *bundle equivalence is in one to one correspondence with the cosets in* $\operatorname{Aut} Z_{ee}/\operatorname{Ext}(Z, Z_{ee}) \circ \operatorname{Int} Z_{ee}$, *where* $\operatorname{Aut} Z_{ee}$ *is the set of Lie group automorphisms of* Z_{ee}, $\operatorname{Int} Z_{ee}$ *is the group of inner automorphisms in* $\operatorname{Aut} Z_{ee}$, *and* $\operatorname{Ext}(Z, Z_{ee})$ *is the group of automorphisms in* $\operatorname{Aut} Z_{ee}$ *that extend to C^r automorphisms of Z, leaving M pointwise fixed.*

Proof. For $J \in \operatorname{Aut} Z_{ee}$, there is a C^r bundle equivalence $h: Z_I \rightarrow Z_{J_0 I}$ if and only if there exists $\Phi \in Z_{ee}$ such that $A(J \circ L_\Phi) \in \operatorname{Ext}(Z, Z_{ee})$, where L_Φ is left composition by Φ. $A(J \circ L_\Phi) = S \circ J$, where S sends $\Gamma \in Z_{ee}$ into $J(\Phi)\Gamma(J(\Phi))^{-1}$, so $S \in \operatorname{Int} Z_{ee}$.

Let ρ and ρ' be C^r representations of Z on E and E' respectively. $h: E \rightarrow E'$ is an *equivalence* (resp. *weak equivalence*) for ρ and ρ' if and only if h is admissible (resp. weakly admissible) and $A(h) \circ \rho = \rho'$.

4.5. THEOREM. *An equivalence, h, for ρ and ρ' as above is a C^r bundle map.*

Proof. We apply Theorem 4.1 to show that $A(h)$ is a C^r isomorphism, then theorem 4.4 shows that h is a C^r bundle map.

The representation ρ is said to be *equivalent* to ρ', written $\rho \cong \rho'$, if and only if there is an equivalence $h: E \rightarrow E'$ for ρ and ρ'. ρ is said to be *weakly equivalent* to ρ', written $\rho \sim \rho'$, if and only if there is a weak equivalence for ρ and ρ'. The relations $\cong$ and $\sim$ are equivalence relations in the usual sense.

In the special case $M = \{e\}$, a groupoid Z over M is just a group. The definitions for groupoid representations and equivalences then yield a theory of group representations, as illustrated below.

Let $Y = C^n$ and $G = U(n) =$ the $n \times n$ unitary matrices. Then the fiber bundle, E over $\{e\}$, may be regarded as an n-dimensional Hilbert space, and a C^r representation of Z on E as a unitary representation. If E' is another C^r fiber bundle over $\{e\}$, with group $U(n)$ and fiber C^n, then $h: E \rightarrow E'$ is admissible if and only if h is a unitary map of E onto E' (regarding E and E' as Hilbert spaces). Hence a representation equivalence is a unitary equivalence in the usual sense. We note that an antilinear map is weakly admissible, so a representation of Z on E is weakly equivalent to its dual.

Accordingly, the group G of the bundle plays an important role in the representation theory, in determining the nature of the representations allowed, and in the equivalence of the representations.

Returning to the general case, let ρ and ρ' be C^r representations of Z on E and E' respectively. The restrictions of ρ and ρ' to Z_{ee} (denoted by ρ_e and ρ'_e respectively) define representations on E_e and E'_e respectively. The following result shows that ρ is essentially determined by ρ_e.

4.6. THEOREM. *$\rho \sim \rho'$ if and only if $\rho_e \sim \rho'_e$. $\rho \cong \rho'$ if and only if $\rho_e \cong \rho'_e$.*

Proof. A representation equivalence $h_e: E_e \rightarrow E'_e$ for ρ_e and ρ'_e extends to a representation equivalence for ρ and ρ' as follows. $A(h_e)$ extends to the C^r isomorphism $H: A(E) \rightarrow A(E')$ by 4.14, and H defines an extension h of h_e by 4.4, having the desired properties. The proof for weak equivalence is similar.

Next, we see that any C^r representation $\rho_e: Z_{ee} \rightarrow A(E_e)$ extends in a natural way to a C^r representation $\rho: Z \rightarrow A(E')$, where E' is constructed from Z and ρ_e.

Suppose E_e is a fiber bundle over $\{e\}$, with group G and fiber Y, and that ρ_e is a C^r representation of Z_{ee} on E_e. We define

$$E' = \{Z_{M \times e} \times E_e\}/\sim \ ,$$

where $(\Phi_{qe}, v) \sim (\Psi_{qe}, w)$ if and only if $\rho_e(\Psi_{qe}^{-1} \circ \Phi_{qe})v = w$. Regarding $Z_{M \times e}$ as a principal Z_{ee} bundle, E' is simply the associated bundle. The projection π is given by $\pi(\Phi_{qe}, v) = q$. The map $E_e \rightarrow E'_e$; $v \rightarrow 1_{ee}, v$, identifies E_e with E'_e. The following result is easily verified.

4.7. THEOREM. *The map* ρ', $\rho'(\Psi_{pq})(\Phi_{qe}, v) = (\Psi_{pq}\cdot\Phi_{qe}, v)$, *is a* C^r *representation of* Z *on* E', *and the restriction* $\rho'_e = \rho'|_{Z_{ee}}$ *is equivalent to* ρ_e.

4.71. REMARK. If ρ'' is another extension of ρ_e to a C^r representation of Z on E'', then by 4.6 $\rho'' \cong \rho'$, and by 4.5 E' and E'' are equivalent as C^r fiber bundles.

We next present an example relating groupoid representations to the notion of "natural fiber bundles of order k" introduced by Nijenhuis.

4.72. EXAMPLE. Let $P^k = P^k(M)$ be the bundle of k-frames on a C^∞ manifold, M, as defined in [4]. Then there is a natural map (left composition) $L: L^k \to A(P^k)$, where $L^k =$ the groupoid of invertible k-jets of C^∞ maps of M into itself. We give L the unique LTC^r structure such that L^{-1} is a C^r isomorphism. Then L is a C^r representation of L^k on P^k. Let ρ be a C^r representation of L^k on E. The subgroupoid $\rho(L^k) \subseteqq A(E)$ defines a reduction of the group of E. If we assume the group of E has been reduced so that $\rho(L^k) = A(E)$ and ignore the LTC^r structure on L^k, then ρ may be regarded as a natural fiber bundle of order k as defined by Nijenhuis in [7]. The author feels that in view of the results obtained in this paper and the motivation given in [7] for defining "natural bundles", that a natural fiber bundle of order k should be defined as a C^r representation of L^k.

Next, we examine equivalences and weak equivalences of representations from another point of view.

4.8. DEFINITION. We define $G^\wedge = \{\Phi: Y \to Y \mid \Phi$ is bijective, and Φ and Φ^{-1} are C^r maps, and $\Phi G \Phi^{-1} = G\}$.

The following conclusions are easily verified.

4.81. $G \subseteqq G^\wedge$, $\Phi \in G^\wedge$ defines an automorphism of G by $\Gamma \in G \to \Phi \circ \Gamma \circ \Phi^{-1}$.

4.82. $h: E \to E'$ is weakly admissible if and only if $\Psi_p^{-1} \circ h_p \circ \phi_p \in G^\wedge$ for admissible maps Ψ_p and ϕ_p, for all p in M.

Let $\rho_e: Z_{ee} \to A(E_e)$ be a C^r homomorphism, and $i: Y \to E_e$ be a weakly admissible map (i.e. i induces a C^r isomorphism $I: G \to A(E_e)$, $I(\Phi) = i^{-1} \circ \Phi \circ i$, and i and i^{-1} are C^r maps). We define

4.81. $\{\rho_e\} = \{I^{-1} \circ \rho_e \mid i$ admissible$\}$

4.82. $[\rho_e] = \{I^{-1} \circ \rho_e \mid i$ is weakly admissible$\}$.

The following is easily verified.

4.83. THEOREM. $\rho_e \cong \rho'_e$ *if and only if* $\{\rho_e\} = \{\rho'_e\}$ *and* $\rho_e \sim \rho'_e$ *if and only if* $[\rho_e] = [\rho'_e]$.

4.9. DEFINITION. A C^r representation ρ of Z on E is called *transitive* if and only if

4.91. $\rho(Z_{ee}) = A_{ee}(E)$

4.92. G acts transitively on Y.

The classification of transitive C^r representations of Z given by 4.93 and 4.94 applies to the theory of geometric objects for $Z = L^k$, and is well known in that application (cf. [6]).

4.93. THEOREM. *Suppose ρ and ρ' are transitive C^r representations of Z on E and E' respectively. Let G_y be the isotropy group at $y \in Y$. Then $\{(I^{-1} \circ \rho_e)^{-1}(G_y)\} = \{H_i\}$ is a conjugacy class of closed subgroups of Z_{ee}, independent of y. If $\{H'_j\}$ is the corresponding class of subgroups for ρ', then $\rho \sim \rho'$ if and only if $\{H_i\} = \{H'_j\}$.*

Proof. $\{\rho_e\}$ is just the conjugacy class of $I^{-1} \circ \rho_e$ under inner automorphism of G. $\Phi \circ G_y \circ \Phi^{-1} = G_{\Phi y}$ for all $\Phi \in G^\wedge$, so $\rho \sim \rho'$ implies that $\{H_i\} = \{H'_j\}$ by 4.6 and 4.83. If $\{H_i\} = \{H'_j\}$, choose admissible maps i and j so that $H_i = H'_j$. Then $F = J^{-1} \circ \rho'_e \circ \rho_e^{-1} \circ I: G \to G$ is a C^r isomorphism (assuming $\ker \rho_e = I_{ee}$, otherwise replace Z by $Z/\ker \rho_e$), and $F(G_y) = G_y$. Then F induces a map $F': Y \to Y$ (via the identification $G/G_y \cong Y$), and $j \circ F' \circ i^{-1}: E_e \to E'_e$ is the desired weak equivalence for ρ_e and ρ'_e. Hence $\rho \sim \rho'$ by 4.6.

A subgroup H_i obtained as in 4.93 contains no nontrivial normal subgroup in Z_{ee}, since G acts effectively on Y. Suppose H is a closed subgroup of Z_{ee} containing no nontrivial normal subgroups of Z_{ee}. Then define $E = Z_J/H$ (see 3.4) where J is a Lie group isomorphism $G \to Z_{ee}$. We then obtain the result.

4.94. THEOREM. *The left composition $L: Z \to A(Z_J/H)$ is a transitive C^r representation. The conjugacy class $\{H_i\}$ obtained for $L = \rho$ in 4.93 is the conjugacy class of H.*

4.95. REMARKS. Taking $H = 1_{ee}$, we see that the "left regular" transitive C^r representations $L: Z \to A(Z_I)$, obtained in 3.4 and 3.5, for various choices of I, are all weakly equivalent.

5. For any C^r manifold X, we define $\Pi(X) = \{\sigma: [0, 1] \to X \mid \sigma$ is continuous$\}/\sim$, where $\sigma \sim \tau$ if and only if σ is homotopic to τ with end points held fixed.

Assuming M is connected, $\Pi(M) = \Pi$ is the *fundamental groupoid* for M, with the natural composition of curves and LTC^r structure. Π_{ee} is the (discrete) fundamental group at e, usually written $\pi_1(M, e)$, and $l: \Pi_{M \times e} \to M$ is a universal covering of M.

We obtain a universal covering groupoid, $Z^\sim$, for a connected LTC^r groupoid, Z, over a connected C^r manifold, M. Let $Z^\sim = \Pi_{Z\times 1_{ee}}(Z)$ and $M^\sim = \Pi_{M\times e}(M)$. Then $M^\sim$ may be regarded as a subset of $Z^\sim$, and we obtain the following result.

5.1. THEOREM. *There is a natural LTC^r groupoid structure for $Z^\sim$, with $l\times r\colon Z^\sim \to M^\sim\times M^\sim$ defined by $l\times r(\sigma) = (l\circ\sigma/\sim, r\circ\sigma/\sim)$, such that $\pi\colon Z^\sim \to Z$; $\pi(\sigma) = \sigma(1)$, is a C^r homomorphism and $\pi/Z^\sim_{ee}\colon Z^\sim_{ee} \to Z_{ee}$ makes $Z^\sim_{ee}$ the universal covering group of Z^0_{ee} (where $e\in M^\sim$ is the identity curve at e, and Z^0_{ee} is the connected component of 1_{ee} in Z_{ee}).*

Proof. Let σ and τ be representative curves for $\sigma/\sim$ and $\tau/\sim$ in $Z^\sim$. If $r\circ\sigma \sim l\circ\tau$, then we can find new representatives, $\bar\sigma$ and $\bar\tau$, such that $r\circ\bar\sigma = l\circ\bar\tau$, $\bar\sigma\sim\sigma$ and $\bar\tau\sim\tau$ (see [3, Ch. 3]. Then the composition, $\sigma\circ\tau$, is defined by $\sigma\circ\tau(t) = \bar\sigma(t)\circ\bar\tau(t)$.

5.2. THEOREM. *Let Z be a LTC^r groupoid over a connected C^r manifold M, and suppose that Z_{ee} is discrete. Then there is a unique C^r homomorphism*

$$\delta\colon \Pi = \Pi(M) \to Z\ ,$$

mapping Π onto the connected component of 1_{ee} in Z.

Proof. Let $\sigma^\sim_{qe}$ be the unique lift of $\sigma_{qe}\in\Pi_{qe}$ to $Z_{M\times e}$, starting from 1_{ee}. Then define $\delta(\sigma_{qe}) = \sigma^\sim_{qe}(1)$, and extend δ as a homomorphism to all of Π.

REFERENCES

1. C. Ehresmann, *Especes de structures locales* (a translation of "Gattungen von lokalen Strukturen"), Seminaire de Topologie et Geometrie Differentielle, Vol. III.
2. ———, *Categories topologiques et categories differentiables*, Colloque de Geometrie Differentielle Globale, Bruxelles, 1958.
3. S. Hu, *Homotopy Theory*, Academic Press, 1959.
4. Kobayashi, S., *Canonical forms on frame bundles of higher order contact*, Differential Geometry, Amer. Math. Soc. 1961.
5. D. Montgomery and L. Zippen, *Topological Transformation Groups*, Interscience, New York, 1955.
6. A. Nijenhuis, *Theory of the Geometric Object*, Thesis, Amsterdam, 1952.
7. ———, *Geometric aspects of formal differential operations on tensor fields*, Proc. Int. Cong. Mathematicians, 1958.
8. K. Nomizu, *Lie Groups and Differential Geometry*, The Mathematical Society of Japan, 1956.
9. N. Steenrod, *The Topology of Fiber Bundles*, Princeton University Press, 1951.

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