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INEQUALITIES FOR FUNCTIONS REGULAR AND BOUNDED IN A CIRCLE

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This paper is concerned with functions $w = f(z)$ regular and satisfying the inequality $|f(z)| < 1$ in $|z| < 1$. This class of functions will be denoted E.

We determine conditions on z_1, z_2, z_3 and w_1, w_2, w_3 for

$$w_k = f(z_k)\,(k = 1, 2, 3)$$

to be possible with an $f(z)$ of E. In particular to map the vertices of the equilateral triangle $z_k = re^{2k\pi i/3}$ into the vertices of another taken in the opposite direction $w_k = \rho e^{-2k\pi i/3}$ we must have $\rho \leqq r^2$. The extremal function associated with this problem is $w = z^2$. It seems convenient to discuss the fixed point if any of the mapping of $|z| < 1$ into $|w| < 1$. We include a simple proof of the theorem of Denjoy and Wolff that if no such fixed point exists then there is a unique distinguished fixed point on $|z| = 1$. We give several results restricting the position of the interior or distinguished boundary fixed point in terms of the location of a zero of $f(z)$ or the value $f(0)$.

The theorem of Pick asserts that if $f(z)$ is in E then $D(f(z_1), f(z_2)) \leqq D(z_1, z_2)$ where the nonEuclidean distance

$$D(z_1, z_2) = \frac{1}{2}\log\frac{1 + d(z_1, z_2)}{1 - d(z_1, z_2)} \text{ with } d(z_1, z_2) = \left|\frac{z_1 - z_2}{1 - \bar{z}_2 z_1}\right| .$$

Equality holds if and only if f sets up a Möbius transformation. It follows from Pick's theorem that there can be at most one fixed point of $w = f(z)$ in $|z| < 1$ unless $f(z) \equiv z$. It is usually sufficient when f has an interior fixed point at $z = \alpha (\neq 0)$ to suppose $0 < \alpha < 1$.

Our first four theorems give information about the relative positions of zeros of f, an interior fixed point, and the value $f(0)$. We exclude the case where $f(z) \equiv z$.

THEOREM 1. *Let $f \in E$ and $f(0) \neq 0$. Then f has no zeros in $|z| < |f(0)|$; and has a zero on $|z| = |f(0)|$ if and only if f determines a Möbius transformation.*

Proof. The image of $|z| \leqq |f(0)|$, which we denote by C under the transformation $w = (z + f(0))/(1 + \overline{f(0)}z)$ is a circular disc C' having nonEuclidean center $f(0)$ with boundary passing through the origin. The function $w = f(z)$ takes the closed disc C inside C' in the case f

is not a Möbius transformation so that $f(z) \neq 0$ for $z \in C$. If f is linear, nonEuclidean distances are preserved and $f(0)$ is on the boundary of C'.

THEOREM 2. *Let $f \in E$ and let $z = \alpha$ be a fixed point of f with $0 < \alpha < 1$. Then f has no zeros inside $|z - \alpha/(1 + \alpha^2)| = \alpha/(1 + \alpha^2)$ and has a zero on the boundary if and only if f determines a Möbius transformation.*

Proof. The conclusion follows directly from Pick's theorem since the circle described is the nonEuclidean circle with nonEuclidean center $z = \alpha$.

If $f(0)$ is known in addition to the existence of an interior fixed point $\alpha(\neq 0)$, then these two results can be combined to give a larger region which is zero-free, namely the union of the two closed discs. The boundary zero of f occurs at $z = \overline{f(0)}$ when f is a Möbius transformation.

THEOREM 3. *If $f \in E$ and $f(0) \neq 0$, then there can be no fixed point interior to the circle C_1: $|z| = (1 - \sqrt{1 - |f(0)|^2})/|f(0)|$; and a fixed point on the boundary at $z_0 = e^{i\arg f(0)}(1 - \sqrt{1 - |f(0)|^2})/|f(0)|$ only if f determines a Möbius transformation.*

Proof. The nonEuclidean midpoint of the segment from 0 to $f(0)$ is z_0 (See Figure 1). A displacement of all points inside C_1 by $w = f(z)$ insures there can be no fixed point interior to C_1. The boundary case is clear.

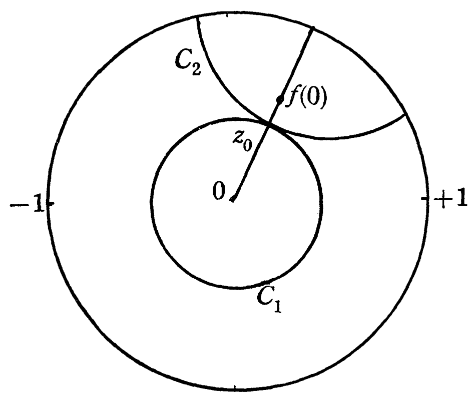

Figure 1.

If f is known to have an interior fixed point, an improvement over Theorem 3 can be made as to its location based on a knowledge of $f(0)$. This is indicated in:

THEOREM 4. *Let $f \in E$. If f has a fixed point $z = \alpha (\neq 0)$, in $|z| < 1$, then α lies inside the circle C_2 passing through z_0* (Figure 1) *with center at the geometric inverse of $f(0)$, relative to the unit circle and is on the boundary if and only if f sets up an elliptic Möbius transformation.*

Proof. This is a direct consequence of the inequality $D(\alpha, 0) \geqq D(\alpha, f(0))$ where the point $z = \alpha$ is considered variable and $f(0)$ is fixed. The assumed interior fixed point is nearer $f(0)$ than the origin in the nonEuclidean sense, except when f is linear. This requires an investigation of the nonEuclidean perpendicular bisector of the radial segment from 0 to $f(0)$. Straight lines of the Poincaré model are Euclidean circles orthogonal to the unit circle. The Euclidean circle C_2 passing through the point z_0 and orthogonal to $|z| = 1$ is the one described in the statement of the theorem.

Theorem 4 provides a simple proof of the Theorem of Denjoy on the fixed points of analytic transformations of the unit circle into itself [3]. It is convenient to develop the argument by formulating several variants of Theorem 4. $f(z)$ is supposed to belong to E.

THEOREM 4 A. *If $f(0) \neq 0$ and $\arg f(0) = \theta$, then any interior fixed point must lie in the half plane $R(e^{i\theta}z) > 0$.*

The half plane evidently contains the circle C_2 of Theorem 4.

THEOREM 4 B. *The nonEuclidean bisector of the nonEuclidean segment joining x and $f(x)$ divides the unit circle into two parts. Any interior fixed point must lie in the part containing $f(x)$ unless the function sets up an elliptic linear transformation when the fixed point must lie on the bisector.*

This statement is equivalent to that of Theorem 4. We have only to apply Theorem 4 to $w = Tf(T^{-1}z)$ where T is a linear transformation of $|z| < 1$ into itself which carries x to the origin.

THEOREM 4 C. *If x and $f(x)$ have the same argument, then any interior fixed point must lie on the same side as $f(x)$ of the circle through x and orthogonal to the radius Ox and to $|z| = 1$.*

This follows from Theorem 4 A. We consider $w = Tf(T^{-1}z)$ where T carries x to the origin and the diameter through x into itself.

Now consider $z = g(w)$ the solution of $wf(z) = z$. From Rouché's theorem $g(w)$ is regular and one valued for w in $|w| < 1$. Let $0 < w < w' < 1$. Apply Theorem 4 C to $F(z) = w'f(z)$. Let $\alpha = g(w)$. We know that $F(\alpha) = w'\alpha/w$. Any fixed point of $F(z)$ and that is to say $g(w')$ must lie in the smaller part of the unit circle partitioned as in Theorem 4 C. If $g(w)$ does not tend to a fixed point of $|w| < 1$ as $w \to 1$ by positive values, it must converge to a point of $|z| = 1$. This point on $|z| = 1$ is the Denjoy distinguished fixed point. Calling such points D fixed points it is clear that Theorem 4 applies to these as well as to interior fixed points.

We shall next be concerned with special cases of three point interpolation by $f \in E$. The problem first considered is that in which we require the vertices of an isosceles triangle to be mapped by f into vertices of another isosceles triangle.

THEOREM 5. *A necessary and sufficient condition for the existence of a function $f \in E$ taking points z_0, 0, $\bar{z}_0$ into w, 0, $\bar{w}$, respectively, is that $w = f(z_0)$ lies in lens $B = \{t \mid t = z_0\zeta, \zeta \in A\}$, where A is the lens formed by the two circular arcs passing through -1, z_0, $+1$ and -1, $\bar{z}_0$, $+1$.*

Proof. This follows from an inequality of G. Julia [4, 74-78] which for our problem is expressed by $D(w/z_0, \bar{w}/\bar{z}_0) \leqq D(z_0, \bar{z}_0)$. Since D is a monotone increasing function of d, it is sufficient for our purpose to use d and we shall refer to this as the nonEuclidean distance.

Let $\delta = |(z_0 - \bar{z}_0)/(1 - z_0^2)|$ and $\zeta = w/z_0 = x + iy$. Then the basic inequality becomes $|(\zeta - \bar{\zeta})/(1 - \zeta^2)| \leqq \delta$ or $2|y|/|1 - (x + iy)^2| \leqq \delta$. On squaring and simplifying we have $4y^2(1 - \delta^2) \leqq \delta^2(1 - \{x^2 + y^2\})^2$. After taking square roots and rearranging we obtain

$$x^2 + \left(|y| + \frac{\sqrt{1 - \delta^2}}{\delta}\right) \leqq \left(\frac{1}{\delta}\right)^2 .$$

If $y \geqq 0$, ζ lies on or below one circular arc; for $y < 0$, ζ lies on or above the other arc, the reflection of the first in the real axis. These arcs form the boundary of a lens. To see that the boundary curves pass through z_0 and $\bar{z}_0$, consider the case of equality $|\zeta - \bar{\zeta}|/|1 - \zeta^2| = |z_0 - \bar{z}_0|/|1 - z_0^2|$. This equation describes the locus of a point which is a fixed nonEuclidean distance from its conjugate, in this case the nonEuclidean distance being $d(z_0, \bar{z}_0)$. The lens just described is labeled A in Figure 2. To complete the proof one notes that $w = z_0\zeta$, for $\zeta \in A$, is the set of points of lens B.

A slightly more general result than Theorem 5 can be obtained. We require f to be real at a real point h as well as to take conjugate values at the conjugate pair $z_0, \bar{z}_0$.

THEOREM 6. *A necessary and sufficient condition for the existence of a function $f \in E$ taking z_0, h, $\bar{z}_0$ into w, h', $\bar{w}$, respectively, where h and h' are real numbers, is that $w = f(z_0)$ lies in a lens*

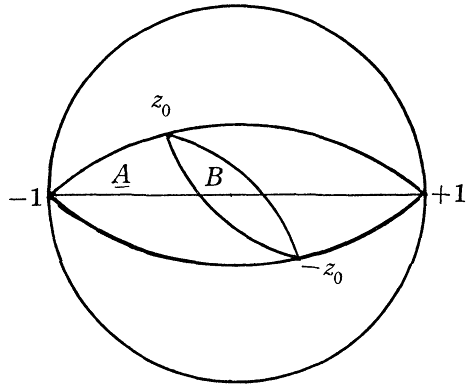

Figure 2.

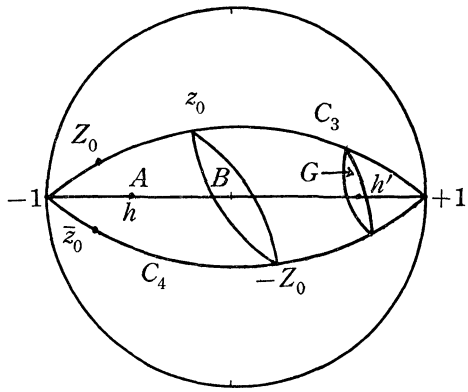

Figure 3.

$G = \{t \mid t = (W + h')/(1 + h'W),\ W \in B\}$, *where*

$$B = \left\{W \mid W = Z_0\zeta,\ \zeta \in A,\ Z_0 = \frac{z_0 - h}{1 - hz_0}\right\}$$

and A *is the lens described in Theorem* 5.

Proof. The proof depends on the fact that the composition of functions in E is again in E. The transformation $Z = (z - h)/(1 - hz)$ takes h to zero with z_0 and $\bar{z}_0$ going to conjugate points $Z_0 = (z_0 - h)/(1 - hz_0)$ and $\bar{Z}_0$. Since this transformation preserves nonEuclidean distances, z_0 is moved to Z_0 on circular arc C_3 which passes through -1, z_0, $+1$. By Theorem 5, a necessary and sufficient condition for the existence of a function of the class E taking 0 to 0 and Z_0, $\bar{Z}_0$ into conjugate points, say W and $\bar{W}$, is that W lies in lens B described in the statement of the theorem. Denote by G the image of B under the transformation $w = (W + h')/(1 + h'W)$. We conclude that $w = f(z_0)$ must lie in G, the lens enclosing h' with end points $(Z_0 + h')/(1 + h'Z_0)$ and $(h' - Z_0)/(1 - h'Z_0)$ on C_3 and C_4, respectively.

If $f(z)$ is real for all h, $-1 < h < +1$, we have the Carathéodory theorem [1, 53] which asserts that if $f \in E$ and if, furthermore, f is real for z real, then a point z inside lens A has its image $f(z)$ also in this lens.

Finally, we investigate the Julia inequality in the case of a reversed equilateral triangle.

THEOREM 7. *A necessary and sufficient condition for the existence of a function $f \in E$ which maps the vertices of the equilateral triangle, r, $r\omega$, $r\omega^2$ into the vertices of the reversed equilateral triangle ρ, ρw^2, $\rho\omega$, respectively, is that $\rho \leqq r^2$.*

Proof. The result is obtained by investigating the Julia condition: $D(A_2'/a_2', A_3'/a_3') \leqq D(a_2', a_3')$, where

$$a_2' = \frac{r(\omega - 1)}{1 - r^2\omega}, \quad a_3' = \frac{r(\omega^2 - 1)}{1 - r^2\omega^2}, \quad A_2' = \frac{\rho(\omega^2 - 1)}{1 - \rho^2\omega^2}, \quad A_3' = \frac{\rho(\omega - 1)}{1 - \rho^2\omega}$$

and simplifying the somewhat involved expression. The computation is omitted.

In the extreme case $\rho = r^2$, the function $w = z^2$ performs the required interpolation.

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