PACIFIC JOURNAL OF MATHEMATICS
Vol. 20, No. 3, 1967

COMPLETE DISTRIBUTIVITY IN LATTICE-ORDERED GROUPS

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Throughout this note let G be a lattice-ordered group ("1-group"). G is said to be *representable* if there exists an 1-isomorphism of G into a cardinal sum of totally ordered groups ("o-groups"). The main result of § 3 establishes five conditions in terms of certain convex 1-subgroups each of which is equivalent to representability. In § 4 it is shown that there is an 1-isomorphism of G onto a subdirect product of 1-groups where each 1-group is a transitive 1-subgroup of all o-permutations of a totally ordered set and that this 1-isomorphism preserves all joins and meets if and only if G possesses a collection of closed prime subgroups whose intersection contains no nonzero 1-ideal. Both theorems lead to results concerning complete distributivity.

G is completely distributive if

$$\bigwedge_{i \in I} \bigvee_{j \in J} g_{ij} = \bigvee_{f \in J^I} \bigwedge_{i \in I} g_{if(i)}$$

where $g_{ij} \in G$ and provided the indicated joins and meets exist. Weinberg [12] has given an equivalent condition to complete distributivity involving arbitrary joins of elements of G (see Proposition 3.5). In [4] Conrad shows that a representable 1-group G is completely distributive if and only if the ideal radical $L(G)$ is zero (in this paper it was denoted by $R(G)$). Using this result we are able to show (Proposition 3.8) that for representable 1-groups the Weinberg condition may be reduced to a condition involving only the joins of pairs of elements. This has been shown by Bernau ([1], Theorem 8) for Archimedean 1-groups. Holland [7] has shown that each 1-group is 1-isomorphic to a subdirect product of 1-groups $\{A_\lambda \mid \lambda \in \Lambda\}$ where each A_λ is a transitive 1-subgroup of the 1-group of all o-permutations of a totally ordered set. Theorem 4.6 generalizes the known result for representable 1-groups (see [12] or [4]).

2. Notation and terminology.

For the standard definitions and results concerning 1-groups the reader is refered to [2] and [5]. A convex 1-subgroup M of G is called *prime* if whenever a and b belong to G^+ and not M, then $a \wedge b > 0$. A convex 1-subgroup (1-ideal) that is maximal with respect to not containing some g in G is called a *regular subgroup* (*regular* 1-*ideal*). Let $\Gamma(\Gamma_1)$ be an index set for the collection $G_\gamma(I_\lambda)$ of regular subgroups (regular 1-ideals) of G. We shall frequently identify these subgroups with their indices. For each $\gamma \in \Gamma(\lambda \in \Gamma_1)$ there exists a unique convex 1-subgroup G^γ (1-ideal I^λ) of G that covers $G_\gamma(I_\lambda)$. If g belongs to G^γ but not G_γ (I^λ but not I_λ), then $\gamma(\lambda)$ is said to be a *value* (*ideal value*) of g. Each regular subgroup is prime. For completeness we state the following theorem, a proof of which may be found in [3].

THEOREM 2.1. *For a convex* 1*-subgroup* M *of* G *the following are equivalent.*

(1) M *is prime.*

(2) *If* a *and* b *belong to* G^+ *but not* M, *then* $a \wedge b$ *belongs to* G^+ *but not* M.

(3) *If* $M \supseteqq A \cap B$, *where* A *and* B *are convex* 1*-subgroups of* G, *then* $M \supseteqq A$ *or* $M \supseteqq B$.

(4) *If* $A \supset M$ *and* $B \supset M$, *where* A *and* B *are convex* 1*-subgroups of* G, *then* $A \cap B \supset M$.

(5) *The lattice* $r(M)$ *of right cosets of* M *is totally ordered.*

(6) *The convex* 1*-subgroups of* G *that contain* M *form a chain.*

(7) M *is the intersection of a chain of regular subgroups.*

If M *is normal, then each of the above is equivalent to*

(8) G/M *is an* o*-group.*

It follows from (6) that the intersection of a chain of prime subgroups of G is prime and hence each prime subgroup exceeds a minimal prime subgroup. If S is a subset of G, then $[S]$ will denote the subgroup of G generated by S. Again using (6) we state a trivial observation.

COROLLARY 2.2. *Let* $M_1, \cdots, M_n$ *be convex* 1*-subgroups of* G *such that* M_1 *is prime. Then* $[\bigcup_{i=1}^n M_i] = [M_1 \cup M_k]$ *for some* k, $1 \leqq k \leqq n$.

For $0 \neq g$ in G let $R_g(L_g)$ be the subgroup of G that is generated by the set of all convex 1-subgroups (1-ideals) not containing g. Then $R_g(L_g)$ is a convex 1-subgroup (1-ideal) of G and we define the *radical* and the *ideal radical* of G respectively to be

$$R(G) = \cap R_g \qquad (0 \neq g \in G)$$
$$L(G) = \cap L_g \qquad (0 \neq g \in G) .$$

Clearly $L_g \subseteqq R_g$ for all g in G so $L(G) \subseteqq R(G)$. A regular subgroup G_γ (regular 1-ideal I_λ) is called an *essential subgroup* (*essential* 1*-ideal*) if there exists $0 \neq h$ in G such that $R_h \subseteqq G_\gamma(L_h \subseteqq I_\lambda)$. In [4] it was shown that $L(G)$ is the intersection of all essential 1-ideals of G and a similar proof shows that $R(G)$ is the intersection of all essential subgroups of G. *In particular* $R(G)$ *is an* 1-*ideal of* G.

A convex 1-subgroup C of G is said to be *closed* if whenever $\{g_\alpha \mid \alpha \in A\} \subseteqq C$ such that $\bigvee g_\alpha$ exists, then $\bigvee g_\alpha \in C$. If $a \in G$, then the *polar of* a is defined to be $p(a) = \{x \in G \mid |x| \wedge |a| = 0\}$. $p(a)$ is a closed subgroup of G. If $S \subseteqq G$, then we define the polar of S to be $p(S) = \bigcap p(a)(a \in S)$. If C is a convex 1-subgroup of G, then $r(C)$ will denote the set of right cosets of C and this set is partially ordered by $C + x \leqq C + y$ if $c + x \leqq y$ for some c in C. Then $r(C)$ is a distributive lattice and $C + x \vee C + y = C + x \vee y$ and dually. The empty set will be denoted by $\square$, $A \backslash B$ denotes the set of elements in A but not in B, and $A \subset B$ denotes that A is a proper subset of B.

3. Representable 1-groups.

Sik [11] proved that an 1-group is representable if and only if all polars are normal. Also in [10] Sik has announced the equivalence of (1) and (4) of Theorem 3.1. The author wishes to thank A. H. Clifford who read a rough draft of this paper and made several valuable suggestions. In particular Clifford noted that in the proof of (1) implies (3), (2) had been proven.

THEOREM 3.1. *For an* 1-*group* G *the following are equivalent.*

(1) *G is representable.*

(2) *If M is a prime subgroup of G, then the maximal* 1-*ideal of G contained in M is prime.*

(3) *M and $g + M - g$ are comparable for all prime subgroups M of G and for all g in G.*

(4) *Each minimal prime subgroup is normal.*

(5) *M and $g + M - g$ are comparable for all regular subgroups M of G and for all g in G.*

(6) *Each regular subgroup M of G contains a prime subgroup N such that N is normal in G.*

Proof. (1) implies (2). Let M be a prime subgroup and let J be the subgroup generated by the collection of all 1-ideals of G that are contained in M. Then J is an 1-ideal of G. Since M is prime, $p(a) \subseteqq M$ for each $a \in G^+ \backslash M$. Suppose (by way of contradiction) that J is not prime. Then there exists b, c in $G^+ \backslash J$ such that $b \wedge c = 0$. Therefore $b, c \in M$. Choose $0 < a \in G \backslash M$. $b \notin J$ implies $a \wedge b > 0$ and $c \in p(a \wedge b)$ implies $p(a \wedge b) \backslash J \neq \square$. Since J is maximal in M, there exists $0 < z \in p(a \wedge b) \backslash M$ and since M is prime, $a \wedge z \in G^+ \backslash M$. But then $b \in p(a \wedge z) \subseteqq J$, a contradiction. Therefore J is a prime subgroup of G.

(2) implies (3). Let M be a prime subgroup of G and let $g \in G$. By (2) the maximal 1-ideal J of G contained in M is prime. Therefore $J = g + J - g \subseteqq g + M - g$. By (6) of Theorem 2.1 it follows that

M and $g + M - g$ are comparable.

(3) implies (4) since inner automorphisms of G preserve minimal primes.

(4) implies (5). Let M be a regular subgroup and let N be a minimal prime subgroup such that $N \subseteqq M$. Then $N = g + N - g \subseteqq g + M - g$. Thus $g + M - g$ and M are comparable by (6) of Theorem 2.1.

(5) implies (6). Let M be a regular subgroup of G. By (5) $N = \bigcap \{g + M - g \mid g \in G\}$ is the intersection of a chain of regular subgroups, hence by Theorem 2.1 is prime. Clearly N is normal in G and is contained in M.

(6) implies (1). For each $0 \neq a \in G$, let M_a be a value of a. By (6) there exists a prime 1-ideal N_a such that $N_a \subseteqq M_a$. By (8) of Theorem 2.1, G/N_a is an o-group. The mapping $x \to (\cdots, N_a + x, \cdots)$ is an 1-isomorphism of G into the cardinal sum of the o-groups $G/N_a (0 \neq a \in G)$. Thus G is representable.

COROLLARY 3.2. *If G is a representable* 1-*group, then $N(G_\gamma) = N(G^\gamma)$ for each $\gamma \in \Gamma$, where $N(X)$ denotes the normalizer of X in G. Hence G_γ is normal in G^γ for each $\gamma \in \Gamma$.*

Proof. For γ in Γ and x in G, $x + G^\gamma - x$ covers $x + G_\gamma - x$. Thus if $x \in N(G_\gamma)$ it follows that $x + G^\gamma - x = G^\gamma$. Conversely if $x \in N(G^\gamma)$, $G^\gamma = x + G^\gamma - x$ covers $x + G_\gamma - x$. By the theorem G_γ and $x + G_\gamma - x$ are comparable. Thus $x + G_\gamma - x = G_\gamma$.

COROLLARY 3.3. *Let G be a representable* 1-*group and let $0 \neq g \in G$. Then the mapping $G_\gamma \to I_\gamma = \bigcap \{x + G_\gamma - x \mid x \in G\}$ is a one to one mapping of the set of all values of g onto the set of all ideal values of g. Moreover, I_γ is prime and is the largest* 1-*ideal of G contained in G_γ. $I^\gamma = \bigcup \{x + G^\gamma - x \mid x \in G\}$, hence $I_\gamma \subseteqq G_\gamma \subset G^\gamma \subseteqq I^\gamma$. Finally G_γ is an essential subgroup if and only if I_γ is an essential* 1-*ideal, and if I_γ is essential, then G_λ is essential for all $\lambda \in \Gamma$ such that $G_\lambda \supseteqq I_\gamma$.*

Proof. The first part of this corollary follows trivially from Theorem 2.1 and 3.1. We prove only the last sentence. Suppose G_γ is an essential subgroup. Then $G_\gamma \supseteqq R_h \supseteqq L_h$ for some $0 < h$ in G. Since L_h is an 1-ideal of G, $L_h \subseteqq I_\gamma$. Hence I_γ is an essential 1-ideal. Conversely suppose I_γ is an essential 1-ideal. Then $L_h \subseteqq I_\gamma$ for some $0 < h$ in G. Let I_β be an ideal value of h.

Case 1. $I_\beta = I_\gamma$. Then h has only one value, namely G_β, and G_λ is essential for all $\lambda \in \Gamma$ such that $G_\lambda \supseteqq G_\beta$. If $I_\beta \subseteqq G_\lambda \subset G_\beta$, then pick $0 < x \in G^\lambda \backslash G_\lambda$. Since G_λ is prime, $x \wedge h \in G^\lambda \backslash G_\lambda$ and $x \wedge h$ has G_λ for its only value. Thus $R_{x \wedge h} \subseteqq G_\lambda$ and so G_λ is essential.

Case 2. $I_\beta \subset I_\gamma$. Then $I_\beta \subseteqq G_\beta \subset I^\beta \subseteqq I_\gamma$. Therefore $R_h \subseteqq I_\gamma$ and G_λ is essential for all $\lambda \in \Gamma$ such that $G_\lambda \supseteqq I_\gamma$.

COROLLARY 3.4. *For a representable* 1-*group* G, $R(G) = L(G)$.

Proof. As observed in § 2, $L(G) \subseteqq R(G)$. Let $0 \neq g \in G$. If g has at least two values, then by the preceding corollary we have $I_\alpha \subseteqq G_\alpha \subset L_g$ where α is a value of g. Thus $R(G) \subseteqq R_g \subseteqq L_g$. Suppose that g has only one value, say α. Then $L_g = I_\alpha$ and I_α is an essential 1-ideal. Thus G_λ is essential for all $\lambda \in \Gamma$ such that $G_\lambda \supseteqq I_\alpha$ by Corollary 3.3. Moreover $L_g = \bigcap \{G_\lambda \mid G_\lambda \supseteqq I_\alpha\} \supseteqq R(G)$, as $R(G)$ is the intersection of all essential subgroups of G. Thus $R(G) \subseteqq L_g$ for all $0 \neq g$ in G and it follows that $R(G) \subseteqq L(G)$.

It was pointed out in [4] that in general these radicals are not the same. Also in [4] Conrad showed that an 1-group is representable if and only if each regular 1-ideal is prime. It is easy to construct examples to show that the converse to Corollary 3.2 and the converse to Corollary 3.4 are not true.

PROPOSITION 3.5. (*Weinberg* [12]). An 1-group G is completely distributive if and only if for each $0 < g$ in G there exists $0 < g^*$ in G such that $g = \bigvee g_\alpha (\alpha \in A)$, $g_\alpha \in G^+$ implies $g^* \leqq g_\alpha$ for some $\alpha \in A$.

For g in G let $L(g)$ denote the 1-ideal of G generated by g. We shall call h in G t-subordinate to g if whenever $|g| = g_1 \vee g_2$, $0 \leqq g_i \in G$, then $h \in L(g_i)$ for $i = 1$ or $i = 2$. We shall use the notation $h \prec g$ to signify that h is t-subordinate to g. Let $T(G) = \{g \in G \mid h \prec g$ implies $h = 0\}$. In [6] Fuchs defines h to be subordinate to g if whenever $|g| = g_1 \vee \cdots \vee g_n$, $0 \leqq g_i \in G$ implies $h \in L(g_i)$ for some i. There he shows $\{g \in G \mid h$ is subordinate to g implies $h = 0\} = L(G)$. A proof of this given by a trivial modification of the proof of the next lemma. The hypothesis of representability enables us to cut n down to 2.

LEMMA 3.6. *Let G be a representable* 1-*group and let* $0 \neq h \in G$. *Then h is not t-subordinate to g in G if and only if* $g \in L_h$.

Proof. Suppose h is not t-subordinate to g. Then there exists $0 \leqq g_1, g_2$ in G such that $|g| = g_1 \vee g_2$ and $h \notin L(g_1) \cup L(g_2)$. Therefore $g \in [L(g_1) \cup L(g_2)] \subseteqq L_h$. Conversely suppose $g \in L_h$. Let Δ be the set of ideal values of h. Then $L_h = [\bigcup \{I_\delta \mid \delta \in \Delta\}]$ and so $g \in L_h$ implies $g \in [\bigcup_{i=1}^n \{I_{\delta_i} \mid \delta_i \in \Delta\}]$. By Corollary 3.3 each I_{δ_i} is prime. Thus by Corollary 2.2, $g \in [I_{\delta_1} \cup I_{\delta_k}]$ for some k, $1 \leqq k \leqq n$, say $k = 2$. Then

$|g| = g_1 \vee g_2$ ([4], Lemma 4) where $0 \leqq g_i \in I_{\delta_i}$. Since the I_δ's are ideal values of h we have $h \notin I_{\delta_1} \cup I_{\delta_2} \supseteqq L(g_1) \cup L(g_2)$. Therefore h is not t-subordinate to g.

PROPOSITION 3.7. If G is a representable 1-group, then $T(G) = L(G) = R(G)$.

Proof. Let $g \in T(G)$. Then for each $0 \neq h$ in G, h is not t-subordinate to g. Thus $g \in L_h$ for all $0 \neq h \in G$ and so $g \in L(G)$. Conversely if $g \in L(G)$, then $g \in L_h$ for all $0 \neq h$ in G. Therefore h is not t-subordinate to g for all $0 \neq h \in G$ and so $g \in T(G)$.

PROPOSITION 3.8. Let G be a representable 1-group. Then G is completely distributive if and only if for each $0 < g$ in G there exists $0 < g^*$ in G such that whenever $g = g_1 \vee g_2$, $g_i \in G^+$, then $g^* \leqq g_1$ or $g^* \leqq g_2$.

Proof. Suppose the condition is satisfied. Then for each $0 < g$ in G, g^* is t-subordinate to g. Therefore $0 = T(G) = L(G)$. By the theorem in [4], G is completely distributive. The converse follows trivially from the Weinberg condition stated in Proposition 3.5.

4. The Holland representation. For each $\lambda \in \Lambda$ let T_λ be a totally ordered set and let $P(T_\lambda)$ be the o-permutation group on T_λ. Let $H = \prod P(T_\lambda)(\lambda \in \Lambda)$ be the large cardinal product of the $P(T_\lambda)$ and let σ_λ denote the projection map of H onto the 1-group $P(T_\lambda)$. The pair (σ, H) is an *H-representation* of an 1-group G if σ is an 1-isomorphism of G into H such that $G\sigma\sigma_\lambda$ acts transitively on T_λ for all λ in Λ. The main result of [7] is that each 1-group has an H-representation. A set $\{C_\lambda \mid \lambda \in \Lambda\}$ of prime subgroups of G is an *H-kernel* if $\bigcap \{C_\lambda \mid \lambda \in \Lambda\}$ contains no nonzero 1-ideal of G. The H-representation (σ, H) is called *complete* if σ preserves all joins and meets that exist in G. In the 1-groups $P(T_\lambda)$ it is convenient to use multiplicative notation for the group operation since composition of function is the group operation and $f \in P(T_\lambda)$ is defined to be positive if $tf \geqq t$ for all t in T_λ.

To prove a convex 1-subgroup is closed it is not difficult to show it suffices to consider only positive elements. Clearly the intersection of closed subgroups is closed.

PROPOSITION 4.1. If G_γ is an essential subgroup of an 1-group G, then G_γ is closed.

Proof. Suppose (by way of contradiction) that there exists

$\{g_\alpha \in G_\gamma^+ \mid \alpha \in A\}$ such that $g = \bigvee g_\alpha \notin G_\gamma$. Let λ be a value of g such that $G_\gamma \subseteqq G_\lambda$.

Case 1. There exists $0 < h$ in G such that $R_h \subseteqq G_\gamma$ and $h \in G_\gamma$. Then $G_\lambda - h + g - g_\alpha = G_\lambda + g - g_\alpha > G_\lambda$ for all α and λ is a value of $-h + g - g_\alpha$. Let δ be any other value of $-h + g - g_\alpha$. Then $h \in G_\delta$, for otherwise $G_\delta \subseteqq R_h \subseteqq G_\gamma \subseteqq G_\lambda$ and so $\delta = \lambda$. Thus

$$G_\delta - h + g - g_\alpha = G_\delta + g - g_\alpha > G_\delta .$$

Therefore $-h + g - g_\alpha > 0$ ([3], p. 114) for all α. This implies $-h + g \geqq \bigvee g_\alpha = g$, a contradition.

Case 2. For all $h > 0$ such that $R_h \subseteqq G_\gamma$, $h \notin G_\gamma$. Thus γ is the only value of h. Now $0 \leqq h \wedge g_\alpha \in G_\gamma$ for all α in A. Suppose $0 < h \wedge g_\alpha$ for some α and let G_β be a value of $h \wedge g_\alpha$. Then $h \notin G_\beta$ so $G_\beta \subseteqq G_\gamma$. Since $h \wedge g_\alpha \in G_\gamma$ we have $G_\beta \subset G_\gamma$. Thus $R_{h \wedge g_\alpha} \subseteqq G_\gamma$. But this is impossible by our assumption. Thus $0 = h \wedge g_\alpha$ for all α in A. $0 < g, h \notin G_\gamma$ implies $g \wedge h > 0$ as G_γ is prime. But then $0 = \bigvee (h \wedge g_\alpha) = h \wedge (\bigvee g_\alpha) = h \wedge g > 0$, a contradiction. This completes the proof of the proposition.

COROLLARY 4.2. *For an* 1-*group* G, $R(G)$ *is closed.*

Proof. $R(G)$ is the intersection of all essential subgroups of G.

COROLLARY 4.3. *If* G *is a representable* 1-*group and if* I_γ *is an essential* 1-*ideal, then* I_γ *is closed.*

Proof. By Corollary 3.3, G_λ is an essential subgroup of G for all $\lambda \in \Gamma$ such that $G_\lambda \supseteqq I_\gamma$ and $I_\gamma = \bigcap \{G_\lambda \mid G_\lambda \supseteqq I_\gamma\}$.

If L and L' are lattices and π is a mapping of L into L' such that $(a \vee b)\pi = a\pi \vee b\pi$ and $(a \wedge b)\pi = a\pi \wedge b\pi$ for all $a, b \in L$, then π is called a *lattice homomorphism.* If, in addition, π preserves all joins and meets that exist in L, then π is said to be *complete.* If π is the natural mapping of G onto the lattice $r(C)$ of right cosets of C, where C is a convex 1-subgroup of G, then π is a lattice homomorphism. The following lemma was proven in [12] for 1-ideals.

LEMMA 4.4. *Let* C *be a convex* 1-*subgroup of* G *and let* π *be the natural mapping of* G *onto* $r(C)$. *Then* π *is complete if and only if* C *is closed.*

Proof. Suppose C is closed and let $\{g_\alpha \mid \alpha \in A\} \subseteqq G$ such that $g = \bigvee g_\alpha$ exists in G. Then $C + g \geqq C + g_\alpha$ for all α. Suppose (by way of contradiction) that there exists y in G such that

$$C + g > C + y \geqq C + g_\alpha$$

for all α. Then

$$C + g = C + g \vee C + y = C + g \vee y > C + y ,$$

so $g \vee y - y \notin C$. On the other hand $C \geqq C + g_\alpha - y$ so

$$C = C + g_\alpha - y \vee C = C + (g_\alpha - y) \vee 0$$

for all α. Thus $(g_\alpha - y) \vee 0 \in C$ for all α. Therefore

$$(g \vee y) - y = (\bigvee g_\alpha) \vee y - y = (\bigvee (g_\alpha - y)) \vee 0 = \bigvee ((g_\alpha - y) \vee 0) .$$

Since C is closed, $g \vee y - y \in C$, a contradiction. The converse is trivial.

The next lemma can be proven by a direct application of Proposition 3.5 and the proof will be omitted.

LEMMA 4.5. *Let $H = \prod H_\lambda (\lambda \in \Lambda)$ be the large cardinal product of the* 1*-groups H_λ. Then H is completely distributive if and only if H_λ is completely distributive for all $\lambda \in \Lambda$.*

THEOREM 4.6. *For an* 1*-group G, the following are equivalent.*

(1) *G has a complete H-representation.*

(2) *G has an H-kernel $\{C_\lambda \mid \lambda \in \Lambda\}$ where each C_λ is closed.*

Proof. (1) implies (2). Suppose (σ, H) is a complete H-representation of G, where H, σ and σ_λ are as in the beginning of this section. For each $\lambda \in \Lambda$ pick $t_\lambda \in T_\lambda$ and let $C_\lambda = \{g \in G \mid t_\lambda g \sigma \sigma_\lambda = t_\lambda\}$. Then C_λ is a prime subgroup (see [7], Theorem 3). Suppose $0 < h \in \bigcap \{C_\lambda \mid \lambda \in \Lambda\}$. Then $h \sigma \sigma_\lambda > \theta_\lambda$ for some λ, where θ_λ denotes the identity in $P(T_\lambda)$. Then there exists s in T_λ with $s \neq t_\lambda$ and $s < sh\sigma\sigma_\lambda$. Since $G\sigma\sigma_\lambda$ acts transitively on T_λ, there exists g in G such that $t_\lambda g \sigma \sigma_\lambda = s$. Therefore $t_\lambda(g + h - g)\sigma\sigma_\lambda \neq t_\lambda$. Thus $\bigcap \{C_\lambda \mid \lambda \in \Lambda\}$ contains no nonzero 1-ideal of G and hence $\{C_\lambda \mid \lambda \in \Lambda\}$ is an H-kernel. Since polars are closed the projection map of H onto a cardinal summand is complete. Suppose $\{g_\alpha \mid \alpha \in A\} \subseteqq C_\lambda$ such that $\bigvee g_\alpha$ exists. Then

$$t_\lambda(\bigvee g_\alpha)\sigma\sigma_\lambda = t_\lambda(\bigvee (g_\alpha \sigma \sigma_\lambda)) = t_\lambda$$

by a theorem of J. T. Lloyd ([8], Theorem 1.3). Therefore $\bigvee g_\alpha \in C_\lambda$ and hence each C_λ is closed.

(2) implies (1). Let $\{C_\lambda \mid \lambda \in \Lambda\}$ be as in (2). For each $\lambda \in \Lambda$ let $P(r(C_\lambda))$ be the o-permutation group on the totally ordered set $r(C_\lambda)$ of right cosets of C_λ. For g in G and λ in Λ we define a mapping σ_λ from G into $P(r(C_\lambda))$ by $(C_\lambda + x)g\sigma_\lambda = C_\lambda + x + g$. It is easy to

verify (or see [7]) that σ_λ is an 1-homomorphism of G onto a transitive 1-subgroup of $P(r(C_\lambda))$. Let $H = \prod P(r(C_\lambda))(\lambda \in \Lambda)$ be the large cardinal product of the 1-groups $P(r(C_\lambda))$. We define a mapping σ of G into H by $g\sigma = (\cdots, g\sigma_\lambda, \cdots)$. Then σ is an 1-homomorphism of G into H and the kernel of σ,

$$K(\sigma) = \{g \in G \mid x + g - x \in C_\lambda \text{ for all } x \in G, \lambda \in \Lambda\} \subseteqq \bigcap C_\lambda(\lambda \in \Lambda) .$$

Since this intersection contains no nonzero 1-ideals, σ is an 1-isomorphism. Therefore (σ, H) is an H-representation of G. Let $\{g_\alpha \mid \alpha \in A\} \subseteqq G$ such that $\bigvee g_\alpha$ exists. Since the C_λ's are closed we have by Lemma 4.4 that for each λ in Λ,

$$\begin{aligned}(C_\lambda + x)(\bigvee g_\alpha)\sigma_\lambda &= C_\lambda + x + \bigvee g_\alpha = C_\lambda + \bigvee (x + g_\alpha) \\ &= \bigvee (C_\lambda + x + g_\alpha) = \bigvee ((C_\lambda + x)g_\alpha\sigma_\lambda) .\end{aligned}$$

For h in H the following are equivalent. $h \geqq g_\alpha\sigma$ for all α; $(h)_\lambda \geqq g_\alpha\sigma_\lambda$ for all α and all λ; $(h)_\lambda \geqq \bigvee (g_\alpha\sigma_\lambda) = (\bigvee g_\alpha)\sigma_\lambda$ for all λ; $h \geqq (\bigvee g_\alpha)\sigma$. Therefore σ is complete. This concludes the proof of the theorem.

COROLLARY 4.7. *If G satisfies* (1) *and* (2) *of the theorem, then G is completely distributive.*

Proof. J. T. Lloyd has proven ([8], Theorem 1.1) that for an ordered set T, $P(T)$ is completely distributive. Thus by Lemma 4.5, H (as above) is completely distributive. Since the H-representation is complete, joins and meets in G "agree" (i.e., under 1-isomorphism) with those in H. Thus G is completely distributive.

COROLLARY 4.8. *$R(G) = 0$ implies* (2) *of the theorem. Thus $R(G) = 0$ implies G is completely distributive.*

Proof. $R(G) = \bigcap \{G_\gamma \mid G_\gamma$ is an essential subgroup of $G\}$. By Proposition 4.1, each essential subgroup of G is closed and an essential subgroup, being regular, is prime. Thus $\{G_\gamma \mid G_\gamma$ is an essential subgroup of $G\}$ is an H-kernel as $R(G) = 0$, all of whose members are closed.

In [4] Conrad observed that $R(G) = 0$ implies G is completely distributive and gives an example to show that the converse is false. Also it is shown in [4] that for representable 1-groups the converse to Corollary 4.7 is true. The answer to this question is not known for arbitrary 1-groups. Finally, Corollary 4.8 shows that the H-representation used in [9] (Theorem 2.1) is complete, as the possession of a basis for an 1-group G implies $R(G) = 0$.

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Received May 2, 1966. This work was supported in part by National Science Foundation Grants GP1791 and 64239 and represents a portion of the author's doctoral dissertation, written under the direction of Professor Paul F. Conrad, to whom the author expresses his gratitude.

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