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PROPERTIES OF DIFFERENTIAL FORMS IN n REAL VARIABLES

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We prove the following theorem. Let $\mathscr{L}_X$ be a homogeneous elliptic operator of the second order with constant coefficients. Let f be a Lebesgue integrable solution of

$$\mathscr{L}_X[f(X)] = 0$$

for all X in some neighborhood of the point A in the Euclidean space E_n. Let $X = (x_1, \cdots, x_n)$ and $H = (h_1, \cdots, h_n)$. Then for each $p=1, 2, \cdots$ the homogeneous polynomial $\varphi_p(H; f)$ defined by

$$\varphi_p(H; f) = \sum_{r_1+\cdots+r_n=p} \frac{h_1^{r_1} \cdots h_n^{r_n}}{r_1! \cdots r_n!} \left(\frac{\partial^p f}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}\right)_{X=A}$$

is an indefinite form, or is identically zero, and it satisfies the same differential equation $\mathscr{L}_H[\varphi_p(H; f)] = 0$ for all $H \in E_n$. Analogous differential relations are true for the solutions of homogeneous hypoelliptic equations of any order. The infinite differentiability of these solutions is called upon.

2. Forms associated with differential operators. Let E_n be the n-dimensional Euclidean vector space and let $R = (r_1, \cdots, r_n)$, a multi-index, be a point whose coordinates r_j are nonnegative integers; associated with R are the nonnegative integers $|R| = r_1 + \cdots + r_n$ and $R! = r_1! \cdots r_n!$ and the differential operator

$$(1) \qquad D_X^{|R|} = \frac{\partial^{|R|}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \quad \text{where } X = (x_1, \cdots, x_n) \in E_n \,.$$

If $H=(h_1, \cdots, h_n) \in E_n$, we define the differential operators $\mathscr{D}_p(H)$ for $p = 1, 2, \cdots$ by

$$\mathscr{D}_p(H) = \frac{1}{p!}\left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n}\right)^p = \sum_{|R|=p} \frac{1}{R!} h_1^{r_1} \cdots h_n^{r_n} D_X^{|R|} \,,$$

and we let $\mathscr{D}_0(H)$ be the identity operator. Let $\Omega \subset E_n$ be a domain and let $C^k(\Omega)$ be the class of all real-valued functions having continuous partial derivatives of order k on Ω. If $A = (a_1, \cdots, a_n) \in \Omega$ is arbitrary but fixed once and for all, and if $f \in C^k(\Omega)$ then we define $\varphi_p(H; f)$ for $p = 0, 1, \cdots, k$ to be the result of applying $\mathscr{D}_p(H)$ to f and evaluating the partial derivatives at A; thus

$$(2) \qquad \varphi_p(H; f) = \sum_{|R|=p} \frac{1}{R!} h_1^{r_1} \cdots h_n^{r_n} (D_X^{|R|} f)_{X=A} \,.$$

Hence $\varphi_p(H; f)$ is a homogeneous polynomial of degree p in

$$h_1, \cdots, h_n; \text{ i.e. } \varphi_p(H; f)$$

is a form of degree p so that for every real number λ we have

$$\varphi_p(\lambda H; f) = \lambda^p \varphi_p(H; f) .$$

These forms also have the property that if $X \in E_n$ and if

$$f(X) = \sum_{r_1=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} K(R)(x_1 - a_1)^{r_1} \cdots (x_n - a_n)^{r_n}$$

converges absolutely for X in some neighborhood of A, then in such a neighborhood

$$f(X) = \sum_{p=0}^{\infty} \varphi_p(X - A; f) .$$

For $n = 2$, Mann [4] has shown that if f is harmonic in a neighborhood of A, then each form $\varphi_p(H; f)$ is harmonic and, unless identically zero, it is an indefinite form. Here we generalize this result to more variables and to more general differential equations.

3. A lemma.

Let $\mathscr{L}_X$ be an arbitrary homogeneous linear differential operator of order q with constant coefficients $B(R)$:

$$\mathscr{L}_X = \sum_{|R|=q} B(R) D_X^{|R|} . \tag{3}$$

Let $F(X) = \mathscr{L}_X[f(X)]$ and $\Phi_p(H) = \mathscr{L}_H[\varphi_p(H; f)]$. We have the following result.

Lemma. *If $f \in C^k(\Omega)$, $A \in \Omega$ and $k \geqq q$ then*

$$\Phi_p(H) = \begin{cases} \varphi_{p-q}(H; F) & \text{if } p \geqq q \\ 0 & \text{if } p \leqq q - 1 . \end{cases} \tag{4}$$

Proof. The second line of (4) is clear since $\mathscr{L}_H$ is of order

$$q \geqq p + 1$$

whereas $\varphi_p(H; f)$ is a polynomial of degree p. The results is also obvious if $q = 0$.

Now let $1 \leqq q \leqq p \leqq k$. Applying the special operator $\partial/\partial h_1$ to (2) we obtain

$$\frac{\partial}{\partial h_1} \varphi_p(H; f) = \sum_{\substack{|R|=p \\ r_1 \geqq 1}} \frac{r_1}{R!} h_1^{r_1-1} h_2^{r_2} \cdots h_n^{r_n} (D_X^{|R|} f)_{X=A} .$$

Putting $t_1 = r_1 - 1$ but $t_2 = r_2, \cdots, t_n = r_n$, we see that all $t_j \geqq 0$ and

$$\frac{\partial}{\partial h_1}\varphi_p(H;f) = \sum_{|T|=p-1}\frac{1}{T!}h_1^{t_1}h_2^{t_2}\cdots h_n^{t_n}\left(D_X^{|T|}\frac{\partial}{\partial x_1}f\right)_{X=A}$$
$$= \varphi_{p-1}\left(H;\frac{\partial}{\partial x_1}f\right).$$

Iteration immediately gives

$$\frac{\partial^{r_1}}{\partial h_1^{r_1}}\varphi_p(H;f) = \varphi_{p-r_1}\left(H;\frac{\partial^{r_1}}{\partial x_1^{r_1}}f\right)$$

if $r_1 \leqq k$. Hence, if $|R| = q$ we get from (1)

$$D_H^{|R|}\varphi_p(H;f) = \varphi_{p-|R|}(H;D_X^{|R|}f) = \varphi_{p-q}(H;D_X^{|R|}f).$$

Multiplying by $B(R)$ and summing over all R such that $|R| = q$, we obtain (4) after applying (3) and the definitions of $\Phi_p(H)$ and $F(X)$.

COROLLARY. *Let $\mathscr{L}_X$ be a homogeneous linear differential operator of order q with constant coefficients. Let $f \in C^k(\Omega)$, $A \in \Omega$ and $k \geqq q$. If f satisfies $\mathscr{L}_X[f(X)] = 0$ in Ω, then $\mathscr{L}_H[\varphi_p(H;f)] = 0$ for all $H \in E_n$ and all $p = 0, 1, \cdots$.*

Proof. By hypothesis $F \in C^{k-q}(\Omega)$ is identically zero in Ω. From (2) it follows that $\varphi_j(H;F) \equiv 0$ for all $j \geqq 0$ so that (4) gives

$$\Phi_p(H) \equiv 0$$

for all $p \geqq q$. The same conclusion also holds if $p \leqq q-1$ by (4) and the result is proved.

4. **The main results.** In order to formulate the first of our results, we need to recall the idea of a hypoelliptic linear differential operator. If

$$\mathscr{P} = \sum_{|R|\leqq s} K(R)D_X^{|R|}$$

is a linear differential operator of order not exceeding s with constant coefficients, where $D_X^{|R|}$ is defined by (1), then we can associate with $\mathscr{P}$ the polynomial P, of degree not exceeding s, defined by

$$P(W) = \sum_{|R|\leqq s} K(R)i^{|R|}w_1^{r_1}\cdots w_n^{r_n},\ W = (w_1, \cdots, w_n) \in E_n;$$

this polynomial results from the formal replacement in $\mathscr{P}$ of each differentiation operator $\partial/\partial x_k$ by iw_k where $i^2 = -1$. If

$$||X|| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

is the usual Euclidean length of the vector X, we also associate with

$\mathscr{P}$ the function d defined on E_n by

$$d(Y) = g.\,l.\,b.\,\{\| Y - U\|^2 + \| V\|^2\}^{1/2}$$

where the $g.\,l.\,b.$ is extended over all U, $V \in E_n$ such that

$$P(U + iV) = 0\ .$$

Finally, we say that $\mathscr{P}$ is hypoelliptic if $d(Y) \to \infty$ as

$$\min \{| y_1 |, \cdots, | y_n |\} \to \infty\ .$$

For other equivalent definitions, see Hörmander [3, p. 100]. Both elliptic [3, p. 102] and parabolic [3, p. 152] operators are hypoelliptic.

THEOREM 1. *Let $\mathscr{L}_X$ be a homogeneous linear operator (i.e. $|R| = s$ for some s) with constant coefficients which is hypoelliptic. If $A \in \Omega$ and f is a Lebesgue integrable solution of $\mathscr{L}_X[f(X)] = 0$ in Ω, then $\mathscr{L}_H[\varphi_p(H; f)] = 0$ for all $H \in E_n$ and all $p = 0, 1, \cdots$*

Proof. Since f is integrable on Ω, the expression $\int_\Omega f(X)\psi(X)\,dX$ defines a distribution; here ψ, a test function, is in $C^\infty(\Omega)$ and vanishes outside a compact set. (See Hörmander [3, pp. 2–5].) It follows from the last part of Theorem 7.4.1 on p. 176 of Hörmander that $f \in C^\infty(\Omega)$. Since k may be taken arbitrarily large in the Corollary, its conclusion yields the conclusion of the present theorem.

THEOREM 2. *Let $\mathscr{L}_X$ be a homogeneous elliptic operator of the second order with constant coefficients. If $A \in \Omega$ and f is a Lebesgue integrable solution of $\mathscr{L}_X[f(X)] = 0$ in Ω, then for each $p \geqq 1$ the form $\varphi_p(H;f)$ is either indefinite or is identically zero.*

Proof. By the preceding result $\mathscr{L}_H[\varphi_p(H; f)] = 0$ for all $H \in E_n$ and all $p \geqq 0$. Suppose that for some p $\varphi_p(H; f)$ is not indefinite; then it is semi-definite and, without loss of generality, we may assume that it is negative semi-definite. Then for all $H \in E_n$ we have

$$\varphi_p(H; f) \leqq 0 = \varphi_p(\theta; f)$$

where $\theta = (0, 0, \cdots 0)$. However, by the strong form of the maximum principle, see Courant-Hilbert [2, v. 2, p. 326], for solutions of homogeneous elliptic equations of the second order, it follows that $\varphi_p(H; f)$ is constant in E_n. This constant is 0 since $\varphi_p(\theta; f) = 0$.

For odd p a much simpler proof results from

$$\varphi_p(-H; f) = (-1)^p \varphi_p(H; f) = -\varphi_p(H; f);$$

hence, if $\varphi_p(H; f)$ is not identically 0, it takes both positive and negative values and is therefore indefinite.

It may be remarked that there is a result connected with this which is independent of differential operators. This result asserts that if $f \in C^k(\Omega)$, $A \in \Omega$ and the $\varphi_r(H; f)$ are identically zero for

$$r = 1, 2, \cdots, p - 1 ,$$

where $1 \leqq p \leqq k$, but $\varphi_p(H; f)$ is an indefinite form, then in each neighborhood of A the function f assumes values which are both greater than $f(A)$ and less than $f(A)$. This result is proved by expanding f about A in a finite Taylor series and using the continuity of the partial derivatives of order p. For $p = 2$, the result is particularly well-known and may be found, for example in Apostol, [1, pp. 149–152].

It is a consequence of this result that if one could prove Theorem 2 without an appeal to the maximum principle, then one would have an independent proof of this principle. In fact, when $n = 2$ and $\mathscr{L}_X$ is the Laplacian, Mann [4] does this and it is not unreasonable to suppose that there are other cases of second order elliptic equations for which this can be done.

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