ITERATES OF BERNSTEIN POLYNOMIALS

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$B_n(f)$ transforms each function defined on $[0,1]$ into its
Bernstein polynomial of degree $n$. In this paper we study
the convergence of the iterates $B_n^{(k)}(f)$ as $k \to \infty$ both in the
case that $k$ is independent of $n$ and (for polynomial $f$) when
$k$ is a function of $n$.

To each $f(x)$ defined on $I: 0 \leq x \leq 1$ there is associated its Bernstein
polynomial of degree $n$ defined by

\begin{equation}
B_n(f; x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k}.
\end{equation}

It is well known that if $f$ is continuous on $I$, then

\begin{equation}
\lim_{n \to \infty} B_n(f; x) = f(x)
\end{equation}

uniformly on $I$. (Cf., Lorentz [2] for this and other properties of the
Bernstein polynomials used here.) Let $B_n(f)$ denote the (polynomial)
function defined by (1.1), then for $k > 1$, $B_n^{(k)}(f; x) = B_n(B_n^{(k-1)}(f); x)$
defines, by mathematical induction, a sequence of iterates of the
Bernstein polynomials. Our purpose is to study the convergence
behavior of this sequence as $k \to \infty$, both in the case that $k$ is inde-
pendent of $n$ and when it is a nonconstant function of $n$.

We show in §2 that $B_n^{(k)}(f; x)$ converges (uniformly) for fixed $n$, to the line segment joining $(0, f(0))$ to $(1, f(1))$, and in § 3 that the
sequence $B_n^{(g(n))}(x^s; x)$ with appropriate assumptions on $g(n)$, also con-
verges, for each $s = 0, 1, 2, \cdots$ to a polynomial of degree $s$ whose
coefficients we determine explicitly. Finally, in § 4 arbitrary iterates
are defined as a natural generalization of the positive integral iterates.

When (1.1) is rewritten in conventional polynomial form, it becomes

\begin{equation}
B_n(f; x) = \sum_{q=0}^{s} \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k} x^q
\end{equation}

which reveals that if $f$ is a polynomial of degree $m$, then $B_n(f)$ is a
polynomial whose degree is at most $\min(m, n)$. Let $s$ be a fixed
positive integer satisfying $s \leq n$. (There is no loss of generality in
this restriction on $s$ for $k > 1$, since for $s > n$, $B_n^{(k)}(x^s) = B_n^{(k-1)}(B_n(x^s))$
and $B_n(x^s)$ is of degree at most $n$.) We consider $f(x) = x^j, j = 1, \cdots, s$.
(1.3) implies that
\begin{equation}
B_n(x^n) = a_1x + a_2x^2 + \cdots + a_jx^j = \sum_{q=1}^{j} \pi_q \sigma_j^q \frac{1}{n^{j-q}} x^j,
\end{equation}

where $\sigma_j^q$ are the Stirling numbers of the second kind (Cf., Jordan [1, pp. 168-173]) defined by

\begin{equation}
\sigma_j^q = \frac{(-1)^q}{q!} \sum_{k=1}^{q} k^j (q \choose k) (-1)^k,
\end{equation}

and

\begin{equation}
\begin{cases}
\pi_q = \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{q - 1}{n}\right), & q = 2, \ldots, s \\
\pi_1 = 1.
\end{cases}
\end{equation}

2. Limit of the iterates. The study of the iterates of $B_n(f; x)$ for $f(x) = x^n$ is considerably simplified if we use the language of linear algebra. There is no loss of generality in this choice of $f(x)$ since $B_n$ replaces $f$ by a polynomial.

Let $A$ denote the $s \times s$ upper triangular matrix whose entries $a_{ij}$ are defined in (1.4), i.e.,

\begin{equation}
a_{ij} = \begin{cases}
\pi_i \sigma_j^{n^i-j}, & i \leq j \\
0 & i > j.
\end{cases}
\end{equation}

Let $e_s$ be the column vector of $s$ components, the first $s - 1$ components being zero and the last one. Then

**Lemma 1.** If $A^k e_s = (\alpha_{1}^{(k)}, \ldots, \alpha_{s}^{(k)})^T$, then

\begin{equation}
B_n(x^n)^k = \alpha_{1}^{(k)} x + \alpha_{2}^{(k)} x^2 + \cdots + \alpha_{s}^{(k)} x^s, \quad k = 1, 2, \ldots.
\end{equation}

**Proof.** If $p(x) = c_1x + c_2x^2 + \cdots + c_sx^s$ (for example, $p(x) = B^{(j)}(x^n)$) and

\[ B_n(p) = d_1x + d_2x^2 + \cdots + d_sx^s = \sum_{j=1}^{s} c_j (a_{1j}x + \cdots + a_{sj}x^s) \]

\[ = \sum_{i=1}^{s} \sum_{j=1}^{s} c_j a_{ij} x^i, \]

then $(d_1, \ldots, d_s)^T = A(e_1, \ldots, e_s)^T$. The lemma now follows by mathematical induction on $k$.

**Lemma 2.** The eigenvalues of $A$ are $\pi_1, \pi_2, \ldots, \pi_s$.

**Proof.** $a_{ii} = \pi_i, i = 1, \ldots, s$, and $a_{ij} = 0$ if $i > j$. 

Let $A$ denote the $s \times s$ matrix with the eigenvalues of $A, \pi_1, \ldots, \pi_s$ on the main diagonal and zeros everywhere else. Let $V$ denote the matrix of eigenvectors of $A$, normalized so that the entries on its main diagonal are all 1. $V$ is upper triangular and its entries are, in general, functions of $n$. Since $AV = VA$ we conclude that
\begin{equation}
A^k = VA^kV^{-1}.
\end{equation}
Essentially, the following arguments rest on the observation that $A^k$ is known to us and $V$ and its inverse are independent of $k$.

**Lemma 3.** If $V^{-1} = (\bar{v}_{ij})$ then $\bar{v}_{ij} = 1, j = 1, \ldots, s$.

**Proof.** Let $U$ be the eigenmatrix of $A^T$, i.e.,
\[ A^T U = U A. \]
Let $U$ (which is lower triangular) be normalized so that the entries on its main diagonal are all 1. Since $B_n(x; 1) = 1$ the column sums of $A$ are all 1 and hence the row sums of $A^T$ are all 1. The first column of $U$ is the eigenvector associated with the eigenvalue $\pi_1 = 1$, and hence consists of all entries 1. Due to the way we have normalized $V$ and $U$ we know that $U^T = V^{-1}$ and the lemma is proved.

**Lemma 4.** If $n$ is fixed
\[ \lim_{k \to \infty} A^k e_s = (1, 0, 0, \ldots, 0)^T. \]

**Proof.** The entries on the main diagonal of $A^k$ are $\pi_1^k, \ldots, \pi_s^k$ and
\[ \lim_{k \to \infty} \pi_j^k = 0, \quad j = 2, \ldots, s \]
\[ \lim_{k \to \infty} \pi_1^k = 1. \]
Thus, as $k \to \infty$, $V A^k V^{-1}$ approaches a matrix whose first row consists of all 1's, by Lemma 3, and the rest of whose elements are all 0. Clearly,
\[ (1, 0, 0, \ldots, 0)^T = \left( \lim_{k \to \infty} A^k \right) e_s = \lim_{k \to \infty} (A^k e_s). \]

**Theorem 1.** If $n$ is fixed then
\begin{equation}
\lim_{j \to \infty} B_n^{(j)}(f; x) = f(0) + (f(1) - f(0))x, \quad 0 \leq x \leq 1.
\end{equation}

**Proof.** Let $B_n(f; x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_s x^n$, then
\[ B_n^{(j)}(f; x) = \alpha_0 + \alpha_1 B_n^{(j-1)}(x; x) + \alpha_2 B_n^{(j-1)}(x^2; x) + \cdots + \alpha_s B_n^{(j-1)}(x^n; x); \]
hence, in view of Lemma 1 and Lemma 4, with \( s = 1, 2, \cdots, n \),

\[
\lim_{j \to \infty} B_n^{(j)}(f; x) = \alpha_0 + (\alpha_1 + \cdots + \alpha_n)x \\
= f(0) + (f(1) - f(0))x.
\]

**Remark.** The convergence in (2.4) is uniform since we have a sequence of polynomials of fixed degree approaching a fixed polynomial of the same degree for all \( x \) on a bounded interval. Also we have used the obvious fact that \( B_n(1) = 1 \), all \( n \).

It is a curious fact that the matrix \( V \) has the property that \( v_{ij} \) is independent of \( n \), for \( j = 1, 2, 3 \). We have, when \( s = 3 \),

\[
V = \begin{pmatrix}
1 & -1 & 1/2 \\
0 & 1 & -3/2 \\
0 & 0 & 1
\end{pmatrix}.
\]

Let \( p_3(x) = -x + x^3 \) and \( p_5(x) = (1/2)x - (3/2)x^2 + x^3 \), then we conclude that,

\[
B_n^{(j)}(p_3) = \left(1 - \frac{1}{n}\right)^j p_3, \quad j = 0, 1, 2, \cdots
\]

\[
B_n^{(j)}(p_5) = \left[\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\right]^j p_5.
\]

These results should be contrasted to the well-known remark (Cf., Schoenberg [3]) that the Bernstein operators are "poor reproducers", in that they never reproduce polynomials of degree greater than 1.

3. Limit of the coupled iterates. Suppose \( f(x) = x^r \). Theorem 1 tells us that for fixed \( n \), \( B_n^{(j)}(x^r) \to x \) as \( j \to \infty \), while according to (1.2), \( B_n(x^r) \to x^r \) as \( n \to \infty \). Thus, it is of interest to "play-off" the upper and lower subscripts in \( B_n^{(j)}(x^r) \), by considering \( j = g(n) \). To this end we must examine the behavior of the eigenmatrix, \( V \), as \( n \to \infty \).

Let the elements of \( V \) be \( v_{ij}(=v_{ij}(n)) \). For \( j = 1, \cdots, s \) we have

\[
A(v_{1j}, \cdots, v_{sj})^r = \pi_j(v_{1j}, \cdots, v_{sj})^r.
\]

We examine these linear equations more closely. Since \( V \) is upper triangular,

\[
v_{ij} = 0, \quad i = j + 1, \cdots, s,
\]

and because of the way we have normalized \( V \)

\[
v_{jj} = 1.
\]

It remains, then, to determine the behavior of \( v_{ij}(n), i < j \), as \( n \to \infty \).
We consider the relevant linear equations from (3.1) (and write \( v_i \) in place of \( v_{i,j} \) for simplicity)

\[
\begin{align*}
\alpha_{i-1,j-1}v_{j-1} + \alpha_{j-1,j} &= \pi_j v_{j-1} \\
\alpha_{j-2,j-2}v_{j-2} + \alpha_{j-2,j}v_{j-1} + \alpha_{j-1,j} &= \pi_j v_{j-2} \\
&\vdots \\
a_{i}v_{1} + a_{i2}v_{2} + \cdots + a_{i,j-1}v_{j-1} + a_{i,j} &= \pi_j v_1 .
\end{align*}
\]

(3.4)

Define \( \pi_{ij} = \pi_i - \pi_j \), let \( P \) denote the determinant \( |p| \) such that

\[
p_{ij} = \begin{cases} 
\alpha_{ij} & i < j \\
\pi_{ij} & i = j \\
0 & i > j 
\end{cases}
\]

then

\[
P = \prod_{k=1}^{j-1} \pi_{kj} .
\]

Let \( P^{(i)} \) denote the determinant identical to \( P \) except that the \( i \)-th column of \( P \) is replaced by \((-a_{i,j}, -a_{i,j}, \cdots, -a_{i,j-1})\). Then, if we solve (3.4) for \( v_i (= v_{i,j}) \) by Cramer's rule, we obtain

\[
v_i = \frac{P^{(i)}}{P} .
\]

(3.5)

If we denote by \( P^{(i)}_{pj} \) the minor of \(-a_{pj}\) in \( P^{(i)}\), then \( P^{(i)}_{pj} \) is upper triangular and

\[
(-1)^{i+p}P^{(i)}_{pj} = \begin{cases} 
0 & p < i \\
\frac{P_{ij}}{\pi_{ij}} & p = i \\
a_{i,i+1}a_{i+1,i+2} \cdots a_{p-1,p}P/\prod_{k=i}^{p} \pi_{kj} & p > i 
\end{cases} .
\]

Now,

\[
(-1)^{i+p+v}a_{pj}P^{(i)}_{pj}/P = \begin{cases} 
\frac{-a_{ij}/\pi_{ij}}{\pi_{ij}} & p = i \\
(-1)^{i+p+v}a_{pj}a_{i,i+1} \cdots a_{p-1,p}/\prod_{k=i}^{p} \pi_{kj} & p > i ,
\end{cases}
\]

(3.6)

and for \( q < j \),

\[
\pi_{qj} = \pi_q \left[ 1 - \left( 1 - q/n \right) \cdots \left( 1 - \frac{j-1}{n} \right) \right] \\
= \pi_q \left\{ \frac{1}{n} \left[ q + (q + 1) + \cdots + (j - 1) \right] + O(n^{-1}) \right\} .
\]

(3.7)
as \( n \to \infty \). Since \( \pi_i \to 1 \) as \( n \to \infty \), we obtain, in view of (3.6), (3.7), and (2.1),

\[
\lim_{n \to \infty} \frac{a_{p,j} P^{(i)}_j}{P} = 0, \quad p < j - 1,
\]

while

\[
\lim_{n \to \infty} \frac{a_{j-1,i,j} P^{(i)}_{j-1,j}}{P} = \left\{ \prod_{t=1}^{i-1} \left( \frac{j - t}{2} \right) (j + t - 1) \right\}^{-1} \sigma_{i+1}^t.
\]

Thus, we obtain, finally, that

\[
(3.8) \quad \lim_{n \to \infty} v_{ij} = v_{ij}^n = (-1)^{i+j} 2^{-i} \prod_{t=1}^{i-1} \left( \frac{t + 1}{2} \right) \frac{(j - i)!}{(j - 1)!} \]

where we have used the fact that (Cf., Jordan [1])

\[
\sigma_{i+1}^t = \left( \begin{array}{c} t + 1 \\ 2 \end{array} \right).
\]

(3.2), (3.3), and (3.8) give the limit of \( V \) as \( n \to \infty \). In an entirely analogous fashion, with \( A^r \) in place of \( A \), we may obtain the limit of \( V^{-1} \) as \( n \to \infty \). We suppress the details, but the result is

\[
(3.9) \quad \lim_{n \to \infty} \bar{v}_{ij} = \bar{v}_{ij}^n = \begin{cases} 0, & i > j \\ 1, & i = j \\ \frac{2^{-i} \prod_{t=1}^{i-1} i + 1}{(j - i)! \left( \frac{j + 1}{j - 1} \right)} & i < j. \end{cases}
\]

Let us put

\[
(3.10) \quad E_j = \exp \left[ -\left( \frac{j}{2} \right) \right] = \lim_{n \to \infty} \pi_j^n.
\]

**Theorem 2.** Suppose \( g(n) \) is a nonnegative integer for each \( n \), and

\[
(3.11) \quad \lim_{n \to \infty} \frac{g(n)}{n} = \alpha,
\]

then we have
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\[
\lim_{n \to \infty} B_n^{(\varepsilon(n))}(x) = \sum_{i=1}^{s} b_i x^i
\]

where

\[
b_i = \frac{i}{s} \binom{s}{i}^2 \sum_{j=1}^{s} \binom{s - i}{j - i} \frac{(-1)^{j+i} \binom{j + s - 1}{j - i}}{(j - i)^2(2j - 2)(j + s - 1)} E_j^s,
\]

\(i = 1, \ldots, s\) (where, when \(\alpha = \infty\) in (3.11), we have \(E_i^s = 1\) and \(E_j^s = 0, j > 1\) in (3.13)).

**Proof.** \(A^{\varepsilon(n)} = V A^{\varepsilon(n)} V^{-1}\). Now

\[
\lim_{n \to \infty} A^{\varepsilon(n)} = A^s
\]

where \(A^s\) is a diagonal matrix with entries \(E_j^s, j = 1, \ldots, s\) on its main diagonal.

Let

\[
\lim_{n \to \infty} V = V^s
\]

and

\[
\lim_{n \to \infty} V^{-1} = (V^{-1})^* = (V^s)^{-1}.
\]

The entries in \(V^s\) and \((V^s)^{-1}\) are given by (3.2), (3.3), (3.8), and (3.9). Thus, we may conclude that

\[
V^s A^s (V^s)^{-1} e_s = \left( \lim_{n \to \infty} A^{\varepsilon(n)} \right) e_s = \lim_{n \to \infty} (A^{\varepsilon(n)} e_s)
\]

and the existence of the limit in (3.12) is established. In order to verify (3.13), we need only note that

\[
(b_1, \ldots, b_s)^T = V^s A^s (V^s)^{-1} e_s,
\]

so that

\[
b_i = \sum_{j=1}^{s} v_{ij}^s \bar{v}_{is}^s E_j^s,
\]

\(i = 1, \ldots, s\).

**Remark.** If \(\alpha = 0\), then \(A^s = I\) and we conclude from (3.14) that \((b_1, \ldots, b_s)^T = e_s\), or \(b_j = 0, j = 1, \ldots, s - 1, b_s = 1\). In particular, then, if \(g(n) \equiv 0\), we have proved (1.2) for the case \(f(x) = x^s\). As a curiosity we also note that we have established the seemingly nontrivial identities
With some simplification (3.16) may be written in the equivalent form (3.17) which holds for odd $t$ and $n$ positive

\begin{equation}
\sum_{k=0}^{n} (-1)^k \binom{t+k}{t} \binom{2n+t}{n-k} \frac{2k+t}{k+t} = 0.
\end{equation}

Additionally, since

\[ \sum_{i=1}^{s} a_{ij} = 1, \quad j = 1, \ldots, s \]

and

\[ \sum_{j=1}^{s} a_{ij} v_{jk} = \pi_k v_{ik}, \quad i = 1, \ldots, s; \quad k = 1, \ldots, s, \]

we obtain, after summing on $i$ on both sides of (3.18) and interchanging the order of summation on the left

\[ \sum_{j=1}^{s} v_{jk} = \pi_k \sum_{i=1}^{s} v_{ik}, \]

from which we conclude that, if $\delta_{ik}$ is a Kronecker delta.

\[ \sum_{i=1}^{s} v_{ik} = \delta_{ik} \]

and hence also

\[ \sum_{i=1}^{s} v_{ik}^* = \delta_{ik}. \]

We thus have the seemingly nontrivial identities:

\begin{equation}
1 + \sum_{i=1}^{s} (-1)^{i+1} 2^{i-1} \frac{t+1}{(j-i)!} \binom{2j-2}{j-i} = 0, \quad j = 2, \ldots, s,
\end{equation}

or, equivalently, if $n \geq 1$,

\begin{equation}
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{n}{k} \frac{1}{k+1} = 0.
\end{equation}

4. Iterates of all orders. If $t$ is any real number, $-\infty < t < \infty$, we are now in a position to define $B_{k}^{(t)}(f)$, in a manner consistent
with our definition when \( t \) is a nonnegative integer. We define

\[
B_n^{(t)}(x^t) = b_1(t)x + b_2(t)x^2 + \cdots + b_k(t)x^k, \quad k = 1, 2, \ldots,
\]

where

\[
(b_1(t), \ldots, b_k(t))^T = V A^t V^{-1} e_k.
\]

In (4.2), \( A^t \) is defined to be the diagonal \( k \times k \) matrix whose entries on the main diagonal are \( \pi_1^t, \pi_2^t, \ldots, \pi_k^t \). It now follows that, since \( e_i, \ldots, e_s \) is a basis in \( E^s(s \leq n) \), if

\[
p = \alpha_0 x + \alpha_1 x^2 + \cdots + \alpha_s x^s,
\]

then

\[
B_n^{(t)}(p) = \sum_{i=1}^s \alpha_i B_n^{(t)}(x^i).
\]

Moreover, if we define

\[
B_n^{(t)}(c) = c
\]

and

\[
B_n^{(t)}(c + p) = c + B_n^{(t)}(p)
\]

where \( c \) is a constant and \( p \) is given by (4.3), then we obtain

\[
B_n^{(t)}(p) = \sum_{i=0}^s \alpha_i B_n^{(t)}(x^i)
\]

when

\[
p = \alpha_0 + \alpha_1 x + \cdots + \alpha_s x^s.
\]

We observe further that if \(-\infty < u < \infty\), then

\[
A^{u+t} = A^u A^t
\]

and so it is easy to see that

\[
B_n^{(t+u)}(x^t) = B_n^{(t)}(B_n^{(u)}(x^t)) = B_n^{(u)}(B_n^{(t)}(x^t))
\]

and hence

\[
B_n^{(t+u)}(p) = B_n^{(t)}(B_n^{(u)}(p)) = B_n^{(u)}(B_n^{(t)}(p))
\]

for any polynomial \( p \) of degree at most \( n \).

If \( f \) is bounded on \([0,1]\), we can now define

\[
B_n^{(t)}(f) = B_n^{(t-1)}(B_n(f))
\]

This definition focuses attention on the case \( t = 0 \). The polynomial
of degree at most \( n \)

\[
B_n^*(f) = B_n^{(s)}(f) = B_n^{s}(B_n f)
\]

is a kind of surrogate \( f \). How is this polynomial related to \( f \)? It is clear that if \( f = p \), a polynomial of degree at most \( n \), then

\[
B_n^* p = p .
\]

In particular, let \( p = L_n(f) \) be the unique polynomial of degree at most \( n \) which agrees with \( f(x) \) at \( x = j/n, j = 0, \ldots, n \). Then \( B_n(f) = B_n(L_n(f)) \) and so

\[
B_n^*(f) = B_n^*(L_n(f)) = L_n(f) .
\]

Of course, this result could have been obtained without the apparatus of this paper, but it comes out of our discussion quite naturally.

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REFERENCES


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