PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 3, 1967

A THEOREM ON RANDOM FOURIER SERIES ON NONCOMMUTATIVE GROUPS

ALESSANDRO FIGÀ-TALAMANCA AND DANIEL RIDER

Let G be a compact group. For $x \in G$ we shall consider a formal Fourier series (*) $\sum d_i Tr(U_i A_i D_i(x))$ where the D_i are distinct (non equivalent) irreducible representations of G of degree d_i, U_i are arbitrary unitary operators and A_i fixed linear transformations on the Hilbert space of dimension d_i and Tr denotes the ordinary trace. We shall prove that $\sum d_i Tr(A_i A_i^*) < \infty$, provided that (*) represents a function in $L^1(G)$ for all $U = \{U_i\}$ belonging to a set M which has positive Haar measure in the group $\mathfrak{G} = \prod \mathscr{U}(d_i)$, where $\mathscr{U}(d_i)$ is the group of all unitary operators on the d_i-dimensional space. If we think of $\mathfrak{G}$ as a probability space, with respect to its Haar measure, then (*) is a Fourier series with "random coefficients" and the result can be stated in the following way: if (*) represents, with positive probability, a function in $L^1(G)$ then $\sum d_i Tr(A_i A_i^*) < \infty$. An earlier result of the authors implies then that, under the same hypothesis, (*) is, with probability one, the Fourier series of a function belonging to $L^p(G)$ for every $p < \infty$.

This result is a generalization of a classical result for the unit circle (cf. e.g. [6, 8.14 p. 215]). With the stronger hypothesis that (*) represents an integrable function for every choice of $U = \{U_i\} \in \mathfrak{G}$, the theorem was proved by Helgason [5]. His proof, as the proof of [2, Th. 4], exploited the "lacunary" properties of a subset of the irreducible representations of $\mathfrak{G}$. Or, from another point of view, it was based on the fact that certain functions defined on $\mathfrak{G}$ share some of the properties of Rademacher series (the reader should compare [6, 8.4, p. 213] with [5, (4.12), p. 279] and [2, Lemma 3]). In effect, to obtain the main result of this paper we prove first that yet another property of Rademacher series [6, 8.3, p. 213] is shared by their noncommutative analogue (cf. Lemma 1, below). To conclude the proof it is then necessary to apply some recent results of Edwards and Hewitt [1] on methods of pointwise summability for arbitrary compact groups.

1. Preliminaries. Let $\mathfrak{G} = \prod_{i \in I} \mathscr{U}(d_i)$. The projection $D_i(V)$ of $V \in \mathfrak{G}$ into $\mathscr{U}(d_i)$ is clearly an irreducible unitary representation. $\bar{D}_i$ will denote the representation conjugate to D_i. We shall consider functions of $L^2(\mathfrak{G})$ of the form

$$F(V) = \sum_{i \in I} d_i Tr(A_i D_i(V))$$

where A_i is a $d_i \times d_i$ matrix. The element of Haar measure on $\mathfrak{G}$ will be denoted by dV. The Schur-Peter-Weyl formula yields

$$\int |F(V)|^2 dV = \sum_i d_i\, Tr(A_i A_i^*) . \tag{1.1}$$

LEMMA 1. *Given a set $M \subset \mathfrak{G}$ of positive Haar measure $m(M)$ and $\varepsilon > 0$, there exists a finite set $I_0 \subset I$ (depending on M and ε) such that if*

$$F(V) = \sum_{i \notin I_0} d_i Tr(A_i D_i(V)) \in L^2(\mathfrak{G})$$

then

$$m(M)\int |F(V)|^2 dV \leqq (1 + \varepsilon)\int_M |F(V)|^2 dV .$$

Proof. We first make the following observations.

(a) If $d_i \geqq 2$ then $D_i \otimes \bar{D}_i$ decomposes into two irreducible components. One is the identity; the other will be denoted by $D_{i,i}$.

(b) If $i \neq j$ then $D_i \otimes \bar{D}_j = D_{i,j}$ is irreducible.

(c) $D_{i,j}$ and $D_{m,n}$ are equivalent if and only if $i = m$ and $j = n$.

(a) and (b) follow directly from the remarks of Helgason [4, p. 788]. He notes that, for $d_i \geqq 2$, $D_i \otimes D_i$ decomposes into two irreducible components and that, for $i \neq j$, $D_i \otimes D_j$ is irreducible. But the number of components of $D_i \otimes D_j$ is

$$\int_{\mathfrak{G}} | Tr(D_i(V)) Tr(D_j(V)) |^2 dV$$

which is also the number of components of $D_i \otimes \bar{D}_j$. Since D_i is irreducible the identity appears once as a component of $D_i \otimes \bar{D}_i$.

Now $Tr(D_{i,j}(V)) = Tr(D_i(V))\overline{Tr(D_j(V))} - \delta_{ij}$ where δ_{ij} is the Kronecker delta. It follows that if $D_{i,j}$ and $D_{m,n}$ are equivalent then

$$\begin{aligned} 1 &= \int Tr(D_{i,j}(V))\overline{Tr(D_{m,n}(V))} dV \\ &= \int Tr(D_i(V))\overline{Tr(D_j(V))}\,\overline{Tr(D_m(V))}\, Tr(D_n(V)) dV - \delta_{ij}\delta_{mn} . \end{aligned}$$

This is possible only if the second integral is not zero. But, by the invariance of dV, this implies $i = j$ and $m = n$ or $i = m$ and $j = n$. Now if $i = j$, $m = n$, but $i \neq m$ then (by (b)) $D_i \otimes \bar{D}_m = D_j \otimes \bar{D}_n$ is irreducible, and the second integral is one. But since $\delta_{ij}\delta_{mn} = 1$ this is not possible. Thus $i = m$ and $j = n$ so that (c) is proved.

Since $Tr(A_i D_i(V))$ and $\overline{Tr(A_j D_j(V))}$ lie in the invariant subspaces generated by $Tr(D_i(V))$ and $Tr(\bar{D}_j(V))$ it follows from (a), (b) and

(1.1) that

$$Tr(A_iD_i(V))\overline{Tr(A_jD_j(V))} = \frac{\delta_{ij}}{d_i} Tr(A_iA_i^*) + Tr(A_{i,j}D_{i,j}(V)) \tag{1.2}$$

where $d_{i,j}$ is the degree of $D_{i,j}$ and $A_{i,j}$ is a $d_{i,j} \times d_{i,j}$ matrix.

If M is a subset of $\mathfrak{G}$ of positive measure then its characteristic function, φ_M, has an expansion in $L^2(\mathfrak{G})$

$$\varphi_M(V) = \sum_{i,j} d_{i,j} Tr(B_{i,j}D_{i,j}(V)) + \sum_\alpha d(\alpha) Tr(B_\alpha D_\alpha(V)) \tag{1.3}$$

the second sum is over the representations of $\mathfrak{G}$ which are not equivalent to any $D_{i,j}$.

From the Schur-Peter-Weyl formula we obtain

$$\sum_{i,j} d_{i,j} Tr(B_{i,j}B_{i,j}^*) \leqq m(M) .$$

Given $\varepsilon > 0$ it follows from the above and (c) that there is a finite set $I_0 \subset I$ such that

$$\sum_{i,j \notin I_0} d_{i,j} Tr(B_{i,j}B_{i,j}^*) < \varepsilon^2 . \tag{1.4}$$

Suppose $F(V) = \sum_{i \notin I_0} d_i Tr(A_iD_i(V)) \in L^2(\mathfrak{G})$. From

(1.2) and (1.3) it follows that

$$\begin{aligned} \int_M |F(V)|^2 dV &= \sum_{i \notin I_0} d_i Tr(A_iA_i^*) m(M) \\ &+ \sum_{i,j \notin I_0} d_i d_j d_{i,j} \int Tr(A_{i,j}D_{i,j}(V)) Tr(B_{j,i}D_{j,i}(V)) . \end{aligned} \tag{1.5}$$

From (1.2) and Holder's inequality it follows that the integrals in the second sum of (1.5) are bounded by

$$\left[\int |Tr(B_{j,i}D_{j,i}(V))|^2\right]^{1/2} \left[\int |Tr(A_iD_i(V))|^4 \cdot \int |Tr(A_jD_j(V))|^4\right]^{1/4} .$$

But by [2, Lemma 1] there is a finite constant B such that

$$\int |Tr(A_iD_i(V))|^4 \leqq \frac{B^2}{d_i^2} [Tr(A_iA_i^*)]^2 .$$

Hence the second summand of (1.5) is majorized by

$$\begin{aligned} B \sum_{i,j \notin I_0} &[d_{i,j} Tr(B_{j,i}B_{j,i}^*) d_i Tr(A_iA_i^*) d_j Tr(A_jA_j^*)]^{1/2} \\ &\leqq B\left[\sum_{i,j \notin I_0} d_{i,j} Tr(B_{i,j}B_{i,j}^*)\right]^{1/2} \sum_{i \notin I_0} d_i Tr(A_iA_i^*) \end{aligned}$$

which by (1.4) is bounded by

$$B\varepsilon\int |F(V)|^2 dV .$$

Hence we have

$$\left|\int |F(V)|^2 dV - \int |F(V)|^2 dV \cdot m(M)\right| \leqq B\varepsilon\int |F(V)|^2 dV$$

which proves the lemma.

We now introduce some terminology which will be used in the rest of the paper and state the result of Hewitt and Edwards which will be used in the proof of the main theorem. Let G be an arbitrary compact group and Γ the set of equivalence classes of irreducible unitary representations of G. If $\gamma \in \Gamma$ we let D_γ be a representative of the class γ and d_γ be the degree of γ. For $f \in L^1(G)$ we let

$$\hat{f}(D_\gamma) = \int_G f(x)D_\gamma(x^{-1})dx$$

so that the Fourier series of f is written as $\sum_{\gamma\in\Gamma} d_\gamma Tr(\hat{f}(D_\gamma)D_\gamma(x))$.

LEMMA 2. (*Edwards and Hewitt*). *Let G be a compact group and $Y = \{\gamma_j\}_{j=1}^\infty$ be a countable subset of Γ. Let D_j be a representative of the class γ_j. Then there exist complex numbers $\alpha_{m,n,j}$ such that*

(i) *for fixed m and n, $\alpha_{m,n,j} = 0$ except for finitely many j's.*

(ii) *if $f \in L^1(G)$ and $\hat{f}(D_\gamma) = 0$ for $\gamma \notin Y$*

$$\lim_m \lim_n \sum_j \alpha_{m,n,j} Tr(f(D_j)D_j(x)) = f(x)$$

almost everywhere with respect to the Haar measure on G.

Proof. [1, 5.11, p. 216 and 3.5, p. 199]. It should be noted that the lemma implies that $\lim_m \lim_n \alpha_{m,n,j} = 1$ for each j.

2. The main theorem. We consider now the formal Fourier series

$$\sum d_\gamma Tr(U_\gamma A_\gamma D_\gamma(x)) \tag{2.1}$$

and we prove:

THEOREM 3. *Suppose that there exists a set M of positive Haar measure in $\mathfrak{G} = \prod_{\gamma\in\Gamma} \mathscr{U}(d_\gamma)$ such that (2.1) is the Fourier series of an integrable function for $\{U_\gamma\} = U \in M$, then $\sum d_\gamma Tr(A_\gamma A_\gamma^*) < \infty$.*

Proof. Since for some choice of $\{U_\gamma\}$ (2.1) represents a function of $L^1(G)$, $A_\gamma = 0$ except for γ belonging to a countable set $Y = \{\gamma_j\}$.

Therefore we can rewrite (2.1) as $\sum_1^\infty d_j Tr(U_j A_j D_j(x))$. We define for $U \in M$ and $x \in G$, $f(x, U) = \sum d_j Tr(U_j A_j D_j(x))$. Then for every $U \in M$, $\int_G |f(x, U)|\, dx < \infty$. Therefore there exists a set of positive measure $M_1 \subset M$ and a number B such that $\int_G |f(x, U)|\, dx < B$ for $U \in M_1$. Thus $\int_{M_1}\int_G |f(x, U)|\, dx dU < \infty$ and $f(x, U)$ is an integrable function on $G \times M_1$.

Let $\alpha_{m,n,j}$ be as in Lemma 2. Define

$$f_{m,n}(x, U) = \sum d_j \alpha_{m,n,j} Tr(U_j A_j D_j(x))$$

and

$$f_m(x, U) = \lim_n f_{m,n}(x, U) .$$

Lemma 2 implies that $f_m(x, U)$ exists almost everywhere in $G \times M$ and $\lim_m f_m(x, U) = f(x, U)$ almost everywhere in $G \times M_1$. Now there exists a set of positive measure $P \subset G \times M_1$ such that

$$\sup_{(x,U)\in P} |f(x, U)| < \infty,\ \lim_n \sup_{(x,U)\in P} |f_{m,n}(x, U) - f_m(x, U)| = 0$$

and

$$\lim_m \sup_{(x,U)\in P} |f_m(x, U) - f(x, U)| = 0 .$$

Indeed as $f(x, U)$ is integrable, it is bounded on a subset of positive measure of $G \times M_1$. Furthermore, given $\delta > 0$, Egoroff's theorem [2, p. 88] implies that $\lim_n f_{m,n}(x, U) = f_m(x, U)$ uniformly for (x, U) outside a set of measure less that $\delta/2^n$ and $\lim_n f_n(x, U) = f(x, U)$ uniformly outside a set of measure δ. As δ can be arbitrarily small we can find a set P of positive measure satisfying our requirements.

Now let C be such that $|f_m(x, U)| \leqq C$ and $|f(x, U)| \leqq C$ for $(x, U) \in P$. We let $\alpha_{mj} = \lim_n \alpha_{m,n,j}$ and we define a set of positive integers n_m such that for $j = 1, \cdots, m$, $|\alpha_{m,n_m,j} - \alpha_{mj}| < (1/m)$ and $|f_m(x, U) - f_{m,n_m}(x, U)| < 1$ for $(x, U) \in P$. We let $\beta_{mj} = \alpha_{m,n_m,j}$ and $g_m(x, U) = f_{m,n_m}(x, U)$. Then $\lim \beta_{mj} = 1$ for each j and $|g_m(x, U)| \leqq C + 1 = C'$. We notice that $g_m(x, U) = \sum_j d_j \beta_{mj} Tr(U_j A_j D_j(x))$ where the sum only extends over a finite number of j's. Since the measure of P is positive, Fubini's theorem implies that for some $x \in G$ the set $P_x = \{U \in \mathfrak{G} : (x, U) \in P\}$ has positive measure. We fix such an x and consider the functions $g_m(x, U)$ as functions defined on $\mathfrak{G}$.

With reference to the subset P_x of $\mathfrak{G}$ and $\varepsilon = 1$ we can find a finite subset $F \subset \Gamma$ which satisfies the conclusion of Lemma 1. If we let $g'_m(x, U) = \sum_{\gamma_j \notin F} d_j \beta_{mj} Tr(U_j A_j D_j(x))$ then $|g'_m(x, U)| \leqq C''$ for $U \in P_x$ and an application of Lemma 1 yields

$$\begin{aligned}\sum d_j \mid \beta_{mj} \mid^2 Tr(A_j A_j^*) &= \sum_{\gamma_j \in F} d_j \mid \beta_{mj} \mid^2 Tr(A_j A_j^*) \\ &\quad + \sum_{\gamma_j \notin F} d_j \mid \beta_{mj} \mid^2 Tr(A_j A_j^*) \\ &\leqq \sum_{\gamma_j \in F} d_j \mid \beta_{mj} \mid^2 Tr(A_j A_j^*) \\ &\quad + \frac{2}{m(P_x)} \int_{P_x} \mid g'_m(x, U) \mid^2 dU \\ &\leqq \sum_{\gamma_j \in F} d_j \mid \beta_{mj} \mid^2 Tr(A_j A_j^*) + 2(C'')^2 < \infty .\end{aligned}$$

Taking limits as $m \rightarrow \infty$ one finds that

$$\sum d_j Tr(A_j A_j^*) \leqq \sum_{\gamma_j \in F} d_j Tr(A_j A_j^*) + 2(C'')^2 < \infty .$$

COROLLARY 4. *If the formal Fourier series* (2.1) *satisfies the hypothesis of Theorem* 3, *then for almost every* $U \in \mathfrak{G}$ *it is the Fourier series of a function in* $\bigcap_{p<\infty} L^p(G)$.

Proof. Since $\sum d_j Tr(A_j A_j^*) < \infty$, [2, Th. 4] implies that

$$\sum d_j Tr(U_j A_j D_j(x)) \in L^p$$

except for $U \in N_p$ with $m(N_p) = 0$. Letting $p = 1, 2, \cdots$ and $N = \bigcup_{p=1}^{\infty} N_p$, one has $m(N) = 0$ and the conclusion follows.

REMARK. To obtain the conclusion of Theorem 3 it is enough to assume that (2.1) represents a Fourier-Stieltjes series for $U \in M$, $m(M) > 0$. Indeed by Theorem 3, one has under this hypothesis a bounded regular measure μ, with $\mu(D_\gamma) = U_\gamma A_\gamma$, satisfying $f * \mu \in L^2(G)$ for every $f \in L^1(G)$. The theorem of Helgason [5, Th. A] implies then that $d\mu = fdx$ with $f \in L^2(G)$.

REFERENCES

1. R. E. Edwards and E. Hewitt, *Pointwise limits for sequences of convolution operators*, Acta Math. **133** (1965), 181-217.

2. A. Figà-Talamanca and D. Rider, *A theorem of Littlewood and lacunary series for compact groups*, Pacific J. Math. **16** (1966), 505-514.

3. P. R. Halmos, *Measure Theory*, Van Nostrand, Princeton, New Jersey, 1950.

4. S. Helgason, *Lacunary Fourier series on noncommutative groups*, Proc. Amer. Math. Soc. **9** (1958), 782-790.

5. ———, *Topologies of group algebras and a theorem of Littlewood*, Trans. Amer. Math. Soc. **86** (1957), 269-283.

6. A. Zygmund, *Trigonometric Series*, Vol. I, Cambridge University Press, Cambridge, 1959.

Received March 15, 1966. Research sponsored in part by the Air Force Office of Scientific Research, Grant A-AFOSR 335-63.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY