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# FIXED POINTS AND FIBRE MAPS

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**Let $\mathscr{F} = (E, p, B)$ be a (Hurewicz) fibre space and let $\lambda$ be a lifting function for $\mathscr{F}$. For $W$ a subset of $B$, a map $f: p^{-1}(W) \to E$ is called a fibre map if $p(e) = p(e')$ implies $p(f(e)) = p(f(e'))$. Define $\bar{f}: W \to B$ to be the map such that $\bar{f}p = pf$. If $[W \cup \bar{f}(W)] \subseteqq V \subseteqq B$ where $V$ is pathwise connected, define $f_b^V: p^{-1}(b) \to p^{-1}(b)$, for $b \in W$, by $f_b^V(e) = \lambda(f(e), \omega)(1)$ where $\omega: I \to V$ is a path such that $\omega(0) = \bar{f}(b)$ and $\omega(1) = b$. Let $i$ be a fixed point index defined on the category of compact ANR's and let $Q$ denote the rationals. The main result of this paper is:**

**THEOREM 1. Let $\mathscr{F} = (E, p, B)$ be a fibre space such that $E, B$, and all the fibres are compact ANR's. Let $f: E \to E$ be a fibre map. If $U$ is an open subset of $B$ such that $\bar{f}(b) \neq b$ for all $b \in bd(U)$ and $\text{cl}\,[U \cup \bar{f}(U)] \subseteqq V \subseteqq \dot{B}$ where $V$ is open and pathwise connected and $\mathscr{F} \mid V = (p^{-1}(V), p, V)$ is $Q$-orientable, then**

$$i(f, p^{-1}(V)) = i(\bar{f}, U). L(f_b^V)$$

**where $L(f_b^V)$ is the Lefschetz number of $f_b^V$ for any $b \in U$.**

Independence of $L(f_b^V)$. For $\mathscr{F} = (E, p, B)$ a Hurewicz fibre space with lifting function $\lambda$ [7] and $\omega$ a loop in $B$ based at $b$, define $\varphi: p^{-1}(b) \to p^{-1}(b)$ by $\varphi(e) = \lambda(e, \omega)(1)$. The fibre space $\mathscr{F}$ is called *$Q$-orientable* if

$$\varphi^*: H^*(p^{-1}(b); Q) \to H^*(p^{-1}(b); Q)$$

is the identity isomorphism for all pairs $(b, \omega)$ where $b \in B$ and $\omega$ is a loop in $B$ based at $b$.

LEMMA. *Let $\mathscr{F} = (E, p, B)$ be a $Q$-orientable fibre space and let $\omega_i: I \to B$, $i = 1, 2$, be paths such that $\omega_i(0) = b$ and $\omega_i(1) = b'$. Define $\varphi_i: p^{-1}(b) \to p^{-1}(b')$ by $\varphi_i(e) = \lambda(e, \omega_i)(1)$, then*

$$\varphi_1^* = \varphi_2^*: H^*(p^{-1}(b'); Q) \xrightarrow[\cong]{} H^*(p^{-1}(b); Q) .$$

*Proof.* By Proposition 2 of [4], each $\varphi_i$ is a homotopy equivalence with homotopy inverse $\psi_i: p^{-1}(b') \to p^{-1}(b)$ given by $\psi_i(e') = \lambda(e', \bar{\omega}_i)(1)$ where $\bar{\omega}_i(s) = \omega_i(1 - s)$. Therefore, $\varphi_i^*: H^*(p^{-1}(b'); Q) \to H^*(p^{-1}(b); Q)$ is an isomorphism and $\psi_i^* = (\varphi_i^*)^{-1}$. Consider $\omega: I \to B$ defined by

$$\omega(s) = \begin{cases} \omega_1(2s) & 0 \leqq s \leqq 1/2 \\ \bar{\omega}_2(1 - 2s) & 1/2 \leqq s \leqq 1 \end{cases}$$

then $\omega$ is a loop in $B$ based at $b$ and since $\mathscr{F}$ is $Q$-orientable, for $\varphi(e) = \lambda(e, \omega)(1)$, $\varphi^*$ is the identity isomorphism. It follows from [4] that $\varphi$ is homotopic to $\psi_2\varphi_1$ so $\varphi^* = \varphi_1^*\psi_2^*$ and $\psi_2^* = (\varphi_1^*)^{-1}$. Hence $\psi_2^* = \psi_1^*$ and $\varphi_2^* = \varphi_1^*$.

THEOREM 2. *Let* $\mathscr{F} = (E, p, B)$ *be a* $Q$*-orientable fibre space where* $B$ *is pathwise connected and* $H^*(p^{-1}(b); Q)$ *is finitely generated for* $b \in B$. *For* $W \subseteq B$, *let* $f\colon p^{-1}(W) \to E$ *be a fibre map, then* $L(f_b) = L(f_{b'})$ *for all* $b, b' \in W$, *where* $f_b$ *means* $f_b^B$.

*Proof.* Since $f_b = \varphi_i(f \mid p^{-1}(b))$, the lemma implies that

$$f_b^*\colon H^*(p^{-1}(b); Q) \to H^*(p^{-1}(b); Q)$$

is independent of the choice of the path $\omega_i$ from $\bar{f}(b)$ to $b$. Let $\omega_0, \omega_1\colon I \to B$ such that $\omega_0(0) = \bar{f}(b)$, $\omega_0(1) = \omega_1(0) = b$, and $\omega_1(1) = b'$. Define $\omega_2\colon I \to B$ by

$$\omega_2(s) = \begin{cases} \overline{\bar{f}\omega_1}(2s) & 0 \leqq s \leqq 1/2 \\ \omega_0(2s-1) & 1/2 \leqq s \leqq 1\,. \end{cases}$$

We first show that diagram (1) is homotopy commutative, where $\varphi_i(e) = \lambda(e, \omega_i)(1)$, $i = 0, 1, 2$.

$$(1) \qquad \begin{tikzcd} p^{-1}(b) \arrow[r, "(f \mid p^{-1}(b))"] \arrow[d, "\varphi_1"'] & p^{-1}(\bar{f}(b)) \arrow[r, "\varphi_0"] & p^{-1}(b) \\ p^{-1}(b') \arrow[rr, "(f \mid p^{-1}(b'))"'] & & p^{-1}(\bar{f}(b')) \arrow[u, "\varphi_2"'] \end{tikzcd}$$

Define the homotopy $H\colon p^{-1}(b) \times I \to p^{-1}(b)$ by

$$H(e, t) = \lambda[f(\lambda(e, \omega_1)(1-t)), \omega^t](1)$$

where

$$\omega^t(s) = \begin{cases} \bar{f}(\bar{\omega}_1(2s+t)) & 0 \leqq s \leqq \dfrac{1-t}{2} \\ \omega_0\left(\dfrac{2s+t-1}{t+1}\right) & \dfrac{1-t}{2} \leqq s \leqq 1\,. \end{cases}$$

Then $H(e, 0) = \varphi_2 f \varphi_1(e)$ and $H(e, 1) = \varphi_0 f(e)$ as required. By the lemma and [4], $(f_{b'})^* = (\varphi_1\varphi_2(f \mid p^{-1}(b')))^*$. Furthermore,

$$\begin{aligned} (\psi_1 f_{b'} \varphi_1)^* &= (\psi_1\varphi_1\varphi_2(f \mid p^{-1}(b'))\varphi_1)^* \\ &= (\varphi_2(f \mid p^{-1}(b'))\varphi_1)^* = (\varphi_0(f \mid p^{-1}(b)))^* = f_b^*\,. \end{aligned}$$

Since $Q$ is a field, $H^*(p^{-1}(b); Q)$ and $H^*(p^{-1}(b'); Q)$ are finite dimensional vector spaces and $\varphi_1^*, f_{b'}^*, \psi_1^*$ are linear transformations. Pick bases for $H^k(p^{-1}(b); Q)$ and $H^k(p^{-1}(b'); Q)$and let $\Phi$, $F'$, and $\Psi$ be the matrices with respect to these bases representing $\varphi_1^{*,k}$, $f_b^{*,k}$, and $\psi_1^{*,k}$ respectively. Since $\psi_1^* = (\varphi_1^*)^{-1}$, $\Psi\Phi = E_n$, the $n \times n$ identity matrix, where $n$ is the dimension of $H^k(p^{-1}(b); Q)$. Therefore, trace $(\Phi F'\Psi) =$ trace $(F')$ which implies that $L(f_{b'}) = L(\psi_1 f_{b'} \varphi_1)$. The theorem now follows because $(\psi_1 f_{b'} \varphi_1)^* = f_b^*$ implies $L(\psi_1 f_{b'} \varphi_1) = L(f_b)$.

**2. Extension of a theorem of Leray.** Let $B$ and $F$ be topological spaces and let $(B \times F, \pi^1, B)$ be the trivial fibre space. Suppose $W$ is a subset of $B$ and $f\colon W \times F \to B \times F$ is a fibre map. Define $f_b\colon F \to F$ by $f_b = \pi^2 f j_b$ where $j_b\colon F \to W \times F$ is given by $j_b(x) = (b, x)$ and $\pi^2\colon B \times F \to F$ is projection. Theorem 3 is a restatement of Theorem 27 of [9] in the somewhat specialized form in which we shall use it.

THEOREM 3 (*Leray*). *Let $(B \times F, \pi^1, B)$ be the trivial fibre space where $B$ and $F$ are finite polyhedra. For $U$ an open connected subset of $B$, let $f\colon \mathrm{cl}\,(U) \times F \to B \times F$ be a fibre map.*[1] *If $\bar{f}(b) \neq b$ for all $b \in bd(U)$, then*

$$\bar{i}(f, U \times F) = \bar{i}(\bar{f}, U)\cdot L(f_b)$$

*for all $b \in U$, where $\bar{i}$ denotes the Leray fixed point index.*

By Theorem 22 and Corollary 26–27 of [9], the Leray index [9, p. 208] satisfies the O'Neill axioms [10, p. 500]. (We will use the formulation of the axioms and the terminology of [1]). Therefore, an index $i$ for the category of compact ANR's, satisfying the O'Neill axioms, may be obtained from the index $\bar{i}$ in the following manner [2, p. 20]. Let $X$ be a compact ANR and let $\alpha$ be a finite open cover of $X$, then there exists a finite polyhedron $K$ and maps $\varphi\colon X \to K$, $\psi\colon K \to X$ such that $\psi\varphi$ is $\alpha$-homotopic to the identity map on $X$, i.e. there exists a map $H\colon X \times I \to X$ such that $H(x, 0) = x$, $H(x, 1) = \psi\varphi(x)$, and for each $x \in X$, the set $\{H(x, t) \mid t \in I\}$ lies in a single element of $\alpha$ [5, Theorem 6.1]. Write $\psi\varphi \sim_\alpha 1_x$. For $U$ an open subset of $X$ and $f\colon X \to X$ a map such that $f(x) \neq x$ for all $x \in bd(U)$, let

$$i_\alpha(f, U) = \bar{i}(\varphi f \psi, \psi^{-1}(U))\ .$$

Browder [2, Theorem 2, p. 20] showed that there exists a finite open cover $\kappa_f(U)$ of $X$ such that if $\alpha$ is a refinement of $\kappa_f(U)$, then $i_\alpha(f, U)$ is well-defined and independent of $\alpha$, $\varphi$, and $\psi$. Write $i_\alpha = i$ for all

[1] The notation $\mathrm{cl}\,(U)$ denotes the closure of $U$. We use $\mathrm{bd}(U)$ for the boundary of $U$.

such $\alpha$.

THEOREM 4. *Let $(B \times F, \pi^1, B)$ be the trivial fibre space where $B$ is a finite polyhedron and $F$ is a compact* ANR. *For $U$ a connected open subset of $B$, let $f\colon \mathrm{cl}\,(U) \times F \to B \times F$ be a fibre map. If $\bar{f}(b) \neq b$ for all $b \in bd(U)$, then*

$$i(f, U \times F) = \bar{i}(\bar{f}, U) \cdot L(f_b)$$

*for all $b \in U$.*

*Proof.* Let $F$ be dominated by a finite polyhedron $K$ by means of maps $\varphi\colon F \to K$ and $\psi\colon K \to F$. Define $f^\#\colon B \times K \to B \times K$ by $f^\#(b, k) = (\bar{f}(b), \varphi f_b \psi(k))$ then $f^\#$ is a fibre map with respect to $(B \times K, \pi^1, B)$ and $\bar{f}^\# = \bar{f}$. Since $\psi\varphi$ is homotopic to the identity map on $F$, $L(f_b^\#) = L(f_b)$ (see the proof that $L(f_{b'}) = L(\psi_1 f_{b'} \varphi_1)$ in Theorem 2). Let $\alpha$ be a finite open cover of $B$ which refines $\kappa_{\bar{f}}(U)$, then $\tau = \{(\pi^1)^{-1}(A) \mid A \in \alpha\}$ refines $\kappa_f(p^{-1}(U))$. Since $f^\# = (1_B \times \varphi) f (1_B \times \psi)$ and, trivially,

$$(1_B \times \psi)(1_B \times \varphi) \sim_\tau 1_B \times 1_F\ ,$$

then $i(f, U \times F) = \bar{i}(f^\#, U \times K)$. Therefore, by Theorem 3,

$$i(f, U \times F) = \bar{i}(\bar{f}, U) \cdot L(f_b)\ .$$

3. **Proof of Theorem 1.** We first assume that $B$ is a finite polyhedron. By a theorem of Hopf [6, Theorem 5], given $\varepsilon > 0$, there exists a map $\bar{g}\colon B \to B$ homotopic to $\bar{f}$ by a homotopy $h\colon B \times I \to B$ such that $h(b, 0) = \bar{f}(b)$, $h(b, 1) = \bar{g}(b)$ and $\rho[h(b, t), h(b, t')] < \varepsilon$ for $b \in B$, $t, t' \in I$, where $\rho$ is the metric of $B$. The map $\bar{g}$ has a finite number of fixed points $b_1, \cdots, b_s$ where, with respect to some barycentric subdivision of $B$, each $b_j$ lies in the interior of a different simplex $\sigma_j$ of $B$, where $\sigma_j$ is not a face of any other simplex of $B$. Since $\bar{f}$ has no fixed points on $bd(U)$, $\inf\{\rho(b, f(b)) \mid b \in bd(U)\} = \varepsilon_1 > 0$. Let $\varepsilon_2 > 0$ be the distance from $\mathrm{cl}\,[U \cup \bar{f}(U)]$ to $B - V$ (if $V = B$, take $\varepsilon_2 = \infty$). Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ then $h(b, t) \neq b$ for all $b \in bd(U)$. Hence $i(\bar{f}, U) = i(\bar{g}, U)$ by the homotopy axiom. Furthermore, $\mathrm{cl}\,[U \cup \bar{g}(U)] \subseteqq V$. The homotopy $h$ induces $h'\colon B \to B^I$. Let $\lambda$ be regular lifting function for $\mathscr{F}$ and define $H'\colon E \to E^I$ by

$$H'(e)(t) = \lambda(f(e), h'(p(e)))(t)\ .$$

Define $g\colon E \to E$ by $g(e) = H'(e)(1)$ then $g$ is a fibre map homotopic to $f$ by a homotopy without fixed points on $bd(p^{-1}(U))$ so $i(f, p^{-1}(U)) = i(g, p^{-1}(U))$. Furthermore, $pg = \bar{g}p$. Since $f_{b_j}$ is precisely $g_{b_j}$ if we use the path $h'(b_j)$ to define $f_{b_j}$ and the constant path to define $g_{b_j}$, then $L(f^V_{b_j}) = L(g^V_{b_j})$. We have shown that when $B$ is a finite polyhedron, it is sufficient to verify the conclusion for the map $g$.

Let $U_j$ be a $\delta$-neighborhood of $b_j$ where $\delta$ is chosen small enough so that $[\mathrm{cl}\,(U_j) \cup \bar{g}(\mathrm{cl}\,(U_j))] \subseteqq \sigma_j$. We may contract $\sigma_j$ to $b_j$ so that $b_j$ stays fixed throughout the contraction and such that the restriction to $\mathrm{cl}\,(U_j)$ contracts $\mathrm{cl}\,(U_j)$ through itself to $b_j$. The contraction induces fibre homotopy equivalences

$$\alpha\colon p^{-1}(\sigma_j) \rightleftarrows \sigma_j \times F\colon \beta$$
$$\alpha'\colon p^{-1}(\mathrm{cl}\,(U_j)) \rightleftarrows \mathrm{cl}\,(U_j) \times F\colon \beta'$$

where the primes denote restriction and $F = p^{-1}(b_j)$ [4, Proposition 4]. Consider the diagram

$$\begin{tikzcd}
\mathrm{cl}\,(U_j) \times F \arrow[r, shift right, "\beta'"'] \arrow[rrr, bend left, "g'"] \arrow[dr, "\text{proj.}"'] & p^{-1}(\mathrm{cl}\,(U_j)) \arrow[l, shift right, "\alpha'"'] \arrow[r, "g"] \arrow[d, "p"] & p^{-1}(\sigma_j) \arrow[r, shift left, "\alpha"] \arrow[d, "p"'] & \sigma_j \times F \arrow[l, shift left, "\beta"] \arrow[dl, "\text{proj.}"] \\
 & \mathrm{cl}\,(U_j) \arrow[r, "\bar{g}"'] & \sigma_j &
\end{tikzcd} \tag{2}$$

where $g' = \alpha g \beta'$. By Theorem 4,

$$i(g', U_j \times F) = \bar{i}(\bar{g}, U)\cdot L(g'_{b_j}) .$$

If we use the constant path to define $g_{b_j}$, then $g_{b_j} = g'_{b_j}$, so $L(g^V_b) = L(g'_{b_j})$. Let $\mu = g\beta'\colon p^{-1}(\mathrm{cl}\,(U_j)) \to \sigma_j \times F$, then by the commutativity axiom

$$i(\alpha\mu, U_j \times F) = i(\mu\alpha', p^{-1}(U_j)) .$$

Now $i(\alpha\mu, U_j \times F) = i(g', U_j \times F)$ by definition. On the other hand, $\mu\alpha' = g\beta'\alpha'$ is homotopic to $g$ by a homotopy which has no fixed points on $bd(p^{-1}(U_j))$ since $\bar{g}$ has no fixed points on $bd(U_j)$ and the homotopy between $\beta'\alpha'$ and the identity is fibre-preserving, so by the homotopy axiom $i(\mu\alpha', p^{-1}(U_j)) = i(g, p^{-1}(U_j))$. Therefore

$$i(g, p^{-1}(U_j)) = \bar{i}(\bar{g}, U_j)\cdot L(g^V_b) .$$

Renumber the fixed points of $\bar{g}$ so that $b_1, \cdots, b_q$ are the fixed points which lie in $U$. Since $g(e) = e$ implies $p(e) = b_j$ for some $j = 1, \cdots, s$, $g$ has no fixed points on $[p^{-1}(\mathrm{cl}\,(U)) - \bigcup_{j=1}^{q} p^{-1}(U_j)]$. Hence by the additivity axiom,

$$i(g, p^{-1}(U)) = \sum_{j=1}^{q} i(g, p^{-1}(U_j))$$
$$= \sum_{j=1}^{q} \bar{i}(\bar{g}, U_j)L(g_b^V) = \bar{i}(\bar{g}, U)\cdot L(g_b^V) .$$

Now suppose that $B$ is a compact ANR, let $K$ be a finite polyhedron and let $\varphi\colon B \to K$, $\psi\colon K \to B$ be maps such that $\psi\varphi \sim_\alpha 1_B$ where $\alpha$ refines $\kappa_{\bar f}(U)$ and $\alpha(\bar f(U))$, the union of all $A \in \alpha$ such that $A \cap \bar f(U) \neq \varnothing$, is contained in $V$. Let $\psi^\#(\mathscr{F}) = (\psi^\#(E), p^\#, K)$ where

$$\psi^\#(E) = \{(k, e) \in K \times E \mid \psi(k) = p(e)\}$$

and $p^\#(k, e) = k$, then $\psi^\#(\mathscr{F})$ is a fibre space with lifting function $\lambda^\#$ given by

$$\lambda^\#((k, e), \omega)(t) = (\omega(t), \lambda(e, \psi\omega)(t))$$

where $\lambda$ is the lifting function of $\mathscr{F}$. Let $h\colon B \times I \to B$ be the $\alpha$-homotopy such that $h(b, 0) = b$, $h(b, 1) = \psi\varphi(b)$, then $h$ induces $h'\colon B \to B^I$. Define $\varphi'\colon E \to \psi^\#(E)$ by

$$\varphi'(e) = (\varphi p(e), \lambda(e, h'(p(e)))(1))$$

Consider

(3)
$$\begin{tikzcd}
\psi^\#(E) \arrow[ddd, "p^\#"'] & & & E \arrow[lll, "\varphi'"'] \arrow[ddd, "p"] \\
& \psi^\#(E) \arrow[ul, "f^\#"'] \arrow[r, "\psi'"] \arrow[d, "p^\#"'] & E \arrow[ur, "f"] \arrow[d, "p"] & \\
& K \arrow[r, "\psi"'] \arrow[dl, "\bar f^\#"] & B \arrow[dr, "\bar f"'] & \\
K & & & B \arrow[lll, "\varphi"]
\end{tikzcd}$$

where $\psi'(k, e) = e$ and $f^\# = \varphi' f \psi'$. Since $\bar f^\# = \varphi \bar f \psi$ and $\psi\varphi \sim_\alpha 1_B$, then $i(\bar f, U) = \bar i(\bar f^\#, \psi^{-1}(U))$. We let $\nu = \varphi' f\colon E \to \psi^\#(E)$, then by the commutativity axiom,

$$i(\psi'\nu, p^{-1}(U)) = i(\nu\psi', \psi'^{-1}p^{-1}(U)) .$$

Define $H\colon E \times I \to E$ by $H(e, t) = \lambda(e, h'(p(e)))(t)$. If $H(f(e), t) = e$ for any $e \in bd(p^{-1}(U))$, $t \in I$, then $h(\bar f(p(e)), t) = p(e)$ which is impossible since $\alpha$ refines $\kappa_{\bar f}(U)$ [2, p. 20], so $\psi'\nu = \psi'\varphi' f$ is homotopic to $f$ by a homotopy without fixed points on $bd(p^{-1}(U))$ and by the homotopy axiom

$$i(\psi'\nu, p^{-1}(U)) = i(f, p^{-1}(U)) .$$

On the other hand, $i(\nu\psi', \psi'^{-1}p^{-1}(U)) = i(f^{\#}, p^{\#-1}(\psi^{-1}(U)))$. If $k \in \psi^{-1}(U)$, then $\bar{f}^{\#}(k) \in \psi^{-1}(V) = W$ since $\alpha(\bar{f}(U)) \subseteqq V$. Let $\omega: I \to W$ be a path such that $\omega(0) = \bar{f}^{\#}(k)$ and $\omega(1) = k$. Define $\omega': I \to V$ by

$$\omega'(s) = \begin{cases} h'(\bar{f}\psi(k))(2s) & 0 \leqq s \leqq 1/2 \\ \psi\omega(2s-1) & 1/2 \leqq s \leqq 1 \end{cases}$$

and let $f_{\psi(k)}$ be given by $f_{\psi(k)}(e) = \lambda(f(e), \omega')(1)$. Define $f'_{\psi(k)}: p^{-1}(\psi(k)) \to p^{-1}(\psi(k))$ by

$$f'_{\psi(k)}(e) = \lambda[\lambda(f(e), h'(\bar{f}\psi(k)))(1), \psi\omega](1) ,$$

then by [4], $f'_{\psi(k)}$ is homotopic to $f_{\psi(k)}$. But $f^{\#}_k(k, e) = \lambda^{\#}((k, e), \omega)(1) = (k, f'_{\psi(k)}(e))$. Therefore $L(f^{\#W}_k)$ is equal to $L(f^V_b)$ and is independent of $k$ and $\omega$. Applying the first part of the proof to the fibre space $\psi^{\#}(\mathscr{F})$, the map $f^{\#}$, and the open set $\psi^{-1}(U) \subseteqq K$, we get

$$i(f^{\#}, p^{\#-1}(\psi^{-1}(U))) = \bar{i}(\bar{f}^{\#}, \psi^{-1}(U)) \cdot L(f^{\#W}_k) .$$

Therefore,

$$i(f, p^{-1}(U)) = i(\bar{f}, U) \cdot L(f^V_b)$$

which completes the proof of Theorem 1.

**4. The index of a fixed point class.** Let $X$ be a compact ANR and let $f: X \to X$ be a map. Denote the fixed point classes of $f$ by $\boldsymbol{F}_1, \cdots, \boldsymbol{F}_r$. Let $(\tilde{X}, \tilde{p}, X)$ be the universal covering space of $X$, then by [2, pp. 43–44] there is a map $\tilde{f}^j: \tilde{X} \to \tilde{X}$ such that $\tilde{p}\tilde{f}^j = f\tilde{p}$ which has the following properties: (1) if $\tilde{f}^j(e) = e$, then $p(e) \in \boldsymbol{F}_j$, (2) for each $b \in \boldsymbol{F}_j$ there exists $e \in \tilde{p}^{-1}(b)$ such that $\tilde{f}^j(e) = e$. We say that $\tilde{f}^j$ *covers* $\boldsymbol{F}_j$. There is an open set $U_j$ in $X$ containing $\boldsymbol{F}_j$ such that $\mathrm{cl}\,(U_j) \cap \boldsymbol{F}_k = \varnothing$ for $k \neq j$. The *index* of $\boldsymbol{F}_j$ is defined by $i(\boldsymbol{F}_j) = i(f, U_j)$ and is independent of the choice of $U_j$.

THEOREM 5. *Let $X$ be a compact* ANR *with finite fundamental group. Let $f: X \to X$ be a map, let $\boldsymbol{F}$ be a fixed point class of $f$, and let $\tilde{f}: \tilde{X} \to \tilde{X}$ cover $\boldsymbol{F}$. If there exists an open subset $U$ of $X$ such that for $x \in U$, $f(x) = x$ if, and only if, $x \in \boldsymbol{F}$, $f(x) \neq x$ for $x \in bd(U)$, and $\mathrm{cl}\,[U \cup f(U)] \subseteqq V$, where $V$ is an open connected simply-connected subset of $X$, then*

$$i(\boldsymbol{F}) = L(\tilde{f})/L(\tilde{f}^V_x)$$

*for $x \in U$.*

*Proof.* We first observe that $L(\tilde{f}^V_x) \neq 0$. Take $x \in \boldsymbol{F}$, then since the fibre is discrete $L(\tilde{f}^V_x)$ is just the number of fixed points of $\tilde{f}$

restricted to $\tilde{p}^{-1}(x)$ which, since $\tilde{f}$ covers $\boldsymbol{F}$, must be greater than zero. Since $\pi_1(X)$ is finite, $\tilde{X}$ is compact and we can apply Theorem 1 to obtain

$$i(f, U) = i(\tilde{f}, \tilde{p}^{-1}(U))/L(\tilde{f}_x^V) .$$

Since $\tilde{f}$ has no fixed points outside of $\tilde{p}^{-1}(U)$, $i(\tilde{f}, \tilde{p}^{-1}(U)) = L(\tilde{f})$.

The existence of the simply-connected set $V$ in the hypotheses of Theorem 5 is not as severe a restriction as it may appear. For example, if $X$ is a finite polyhedron, (or a compact topological manifold, with or without boundary) $f$ is homotopic to a map $g$ which has only isolated fixed points [6, Theorem 5] [3, Theorem 2] and the homotopy carries $\boldsymbol{F}$ to a fixed point class $\boldsymbol{F}'$ of $g$ of the same index [2, Theorem 3, p. 36]. Hence we can apply Theorem 5 to $g$ and $\boldsymbol{F}'$ to compute $i(\boldsymbol{F})$ (compare Theorem 5.2 of [8]).

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