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EXISTENCE OF LEVI FACTORS IN CERTAIN ALGEBRAIC GROUPS

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If G is a connected algebraic linear group with unipotent radical U, Borel and Tits define a Levi factor of G to be any connected reductive subgroup L of G such that $G = L.U$ (semi-direct product in the sense of algebraic groups). This differs from the usual notion of Levi decomposition in Lie theory but leads to equivalent results at characteristic 0. The existence of Levi factors at characteristic p is problematic, in view of an example of a group having no Levi factor constructed by Chevalley (unpublished). In this note sufficient conditions are given for a Levi factor to exist, based on the structure of the Lie algebra of G.

THEOREM. *Let G be a connected algebraic linear group defined over a field of characteristic $p > 2$, with unipotent radical U. Denote by $\mathfrak{G}$, $\mathfrak{U}$ the respective Lie algebras, and suppose the following conditions are satisfied:*

(a) $\mathfrak{G} = \mathfrak{L} + \mathfrak{U}$, *where $\mathfrak{L}$ is a reductive subalgebra (definition below).*

(b) *If T is a maximal torus of G whose Lie algebra is included in $\mathfrak{L}$, then* Ad T *stabilizes $\mathfrak{L}$ (where* Ad: $G \rightarrow$ Aut $(\mathfrak{G})$ *is the adjoint representation of G).*

(c) *Distinct maximal tori of G have distinct Lie algebras. Then G has a Levi factor L, whose Lie algebra is $\mathfrak{L}$.*

It should be observed that, in the presence of (c), conditions (a) and (b) are *necessary* for the existence of a Levi factor [3, § 11]. Condition (c) is far from necessary, as easy examples show, but it is satisfied in certain cases of interest. In fact (c) is equivalent to the requirement that the Lie algebra of a Cartan subgroup of G be a Cartan subalgebra of $\mathfrak{G}$.

We begin by summarizing some facts [3, § 9, 11] about the Lie algebra $\mathfrak{G}$ of a connected algebraic linear group G defined over a field of characteristic $p > 2$. This restriction on p will be assumed throughout the paper. Using a Jordan decomposition theorem of Borel and Springer [1, Prop. 1.3] we define $\mathfrak{G}$ (or a subalgebra of $\mathfrak{G}$) to be *reductive* if it has no nontrivial nil ideal (= ideal consisting of nilpotent elements). A *maximal torus* of $\mathfrak{G}$ is a subalgebra of maximal dimension consisting of commuting semisimple elements. Then:

(1) The Lie algebra $\mathfrak{U}$ of U is the largest nil ideal of $\mathfrak{G}$. In particular, G is reductive if and only if $\mathfrak{G}$ is reductive.

(2) If L is a reductive subgroup of G such that $G = L.U$ is a semi-direct product of abstract groups, then the product is semi-direct in the sense of algebraic groups, i.e. $\mathfrak{G} = \mathfrak{L} + \mathfrak{U}$ is a semi-direct sum, where $\mathfrak{L}$ is the Lie algebra of L.

(3) If G is reductive, maximal tori of G and $\mathfrak{G}$ correspond one-to-one. In any case, a maximal torus of $\mathfrak{G}$ is the Lie algebra of at least one maximal torus of G.

(4) If G is reductive, maximal tori of $\mathfrak{G}$ are Cartan subalgebras.

LEMMA. *Let G be a connected reductive algebraic group, with Lie algebra $\mathfrak{G}$. Then G (resp. $\mathfrak{G}$)* is generated by its maximal tori.

Proof. The assertion for G follows from the fact that the semisimple elements of G form a dense subset (see [5, 2.14]: this extends at once from semisimple to reductive groups). The assertion for $\mathfrak{G}$ is immediate if $\mathfrak{G}$ is a three-dimensional simple algebra and follows in the general case because $\mathfrak{G}$ is generated by any maximal torus along with certain three-dimensional simple subalgebras [3, 3.9 and 11.9].

Proof of theorem. We proceed by induction on dim G, the one-dimensional case being trivial. In view of (1) and (2) above, it will suffice to find a connected reductive subgroup L of G whose Lie algebra is $\mathfrak{L}$ (for L is then automatically a Levi factor). For purposes of induction, observe that: (*) if H is a closed connected subgroup of G whose Lie algebra includes $\mathfrak{L}$, then H satisfies conditions (a)-(c). For let $f: G \rightarrow \bar{G} = G/U$ be the canonical map; the differential df maps $\mathfrak{L}$ isomorphically onto the Lie algebra $\bar{\mathfrak{G}}$ of $\bar{G}$. Since in any case $df(\mathfrak{L})$ is included in the Lie algebra of $f(H)$, it follows that $f(H) = \bar{G}$. Now $H \cap U$ lies in the unipotent radical of H, so by comparison of dimensions we have the decomposition of the Lie algebra $\mathfrak{H}$ of H required for (a): $\mathfrak{H} = \mathfrak{L} + \mathfrak{U}'$, $\mathfrak{U}'$ the Lie algebra of $H \cap U$. Since H has maximal rank in G, (b) and (c) are obviously satisfied and (*) is valid.

Next, let H be the subgroup of G generated by all maximal tori whose Lie algebras lie in $\mathfrak{L}$. Then H is a closed connected subgroup (all tori are connected). The Lemma along with (3) above implies that $\mathfrak{L}$ lies in the Lie algebra of H, so (*) applies and we can appeal to the induction hypothesis if $H \neq G$.

This leaves the case $H = G$. We will produce the desired Levi factor, which in view of (c) will contain all maximal tori whose Lie algebras lie in $\mathfrak{L}$ (so in this case G itself will turn out to be reductive). By (b) we have that Ad T stabilizes $\mathfrak{L}$ for each maximal torus T with Lie algebra contained in $\mathfrak{L}$; since these tori generate G, Ad G stabilizes $\mathfrak{L}$. As is well known, this implies $[\mathfrak{G}, \mathfrak{L}] \subseteq \mathfrak{L}$. In particular, $[\mathfrak{U}, \mathfrak{L}] =$

0 since $\mathfrak{U}$ is also an ideal. Let x be any semisimple element of $\mathfrak{L}$, $\mathfrak{T}$ some maximal torus of $\mathfrak{L}$ (hence of $\mathfrak{G}$) containing x. By (3) above, $\mathfrak{T}$ is the Lie algebra of a maximal torus T of G. Applying a result of Borel and Springer [1, Prop. 1.5] to the group $T.U$ and its Lie algebra $\mathfrak{T} + \mathfrak{U}$, we see that the group in question centralizes x because the algebra does. In view of the lemma, U centralizes all of $\mathfrak{L}$.

Let $u \in U$, and let T be any maximal torus of G with Lie algebra $\mathfrak{T}$ included in $\mathfrak{L}$. Then it is easy to see that $u^{-1}Tu$ has Lie algebra $u^{-1}\mathfrak{T}u = \mathfrak{T}$. By assumption (c), $u^{-1}Tu = T$. But T normalizes U, so it follows that T and U commute elementwise. In particular, U centralizes a set of generators of G and thus U is central in G.

In order to apply a theorem of Steinberg on central extensions of finite type, a further reduction is needed. Let $f\colon G \to \bar{G} = G/U$ be the canonical map. $\bar{G}$ is reductive and can be written as the product of its derived group $\bar{G}'$ and a central torus Z. Let $S = f^{-1}(Z)$, so S is the ordinary (solvable) radical of G. Since U is central, S is clearly nilpotent and contains a unique maximal torus S_1. If $s \in S_1$ and $x \in G$, we have $f(s^{-1}xs) = f(x)$, so $s^{-1}xsx^{-1} \in \operatorname{Ker}(f) = U$. But then $xsx^{-1} = su$, with the left side semisimple and the right side a product of commuting semisimple and unipotent elements. Thus $u = 1$, and S is central in G because $S = S_1.U$. If S_1 is nontrivial, observe that $H_0 = f^{-1}(\bar{G}')$ and its Lie algebra satisfy conditions (a)–(c). The verification is straightforward for (a) and (b), using the Cartan decomposition of $\mathfrak{L}$ along with facts summarized above. For (c) observe that the centrality of U implies that maximal tori of H_0 correspond one-to-one with maximal tori of $\bar{G}'$ under f. But now $\dim H_0 < \dim G$, so the induction hypothesis yields a semisimple subgroup R of H_0 with Lie algebra included in $\mathfrak{L}$, such that the product $L = R.S_1$ is the desired Levi factor of G.

We can therefore assume without loss of generality that $\bar{G}$ is semisimple. Thus $\bar{G}$ is a product of certain normal subgroups of simple type. By imitating the procedure of the preceding paragraph (with recourse to the Lemma) one reduces readily to the case where $\bar{G}$ itself is of simple type. (Alternatively one can extend Steinberg's results discussed below from the simple to the semisimple case.) Since $\bar{G}$ is now its own derived group, and maximal tori of G correspond to maximal tori of $\bar{G}$ under f, it follows (since U is central) that all maximal tori of G lie in the derived group G'. But these tori generate G, so $G = G'$.

In [4] Steinberg constructs a simply connected covering group Γ for each simple type (by a uniform procedure involving generators and relations). In particular, there is a canonical epimorphism $\pi : \Gamma \to \bar{G}$. We will apply Theorem 5.2 of [4] to the situation

$$\begin{array}{ccccccccc} & & & & \Gamma & & & & \\ & & & & & \searrow^{\pi} & & & \\ 1 & \longrightarrow & U & \xrightarrow[i]{} & G & \xrightarrow[f]{} & \bar{G} & \longrightarrow & 1\,. \end{array}$$

It must be observed that the indicated extension of $\bar{G}$ is central and of *finite type*, i.e. $u^n = 1$ for some fixed n and all $u \in U$. This is easy to check, using a composition series for U with factors isomorphic to the additive group of the universal domain. Therefore a homomorphism (of abstract groups) $\pi_0 : \Gamma \rightarrow G$ exists, with $\pi = f\pi_0$. Let $L = \pi_0(\Gamma)$. Since Γ has finite center [4, 3.2] and f is surjective, the relation $\pi = f\pi_0$ implies G is generated by L and U, while $L \cap U$ is finite. But U is central, so $G = G' = L'$ and $U \subseteq L$. Thus U is finite (and connected), hence trivial. We conclude in this case that G is already reductive, which completes the proof.

REMARKS. (1) In case G is a subgroup of maximal rank in some reductive group H, condition (c) holds for G because of statement (3) above. Thus our theorem may be viewed as a partial generalization of [2, 3.14].

(2) The theorem reduces in some measure the problem of Levi decompositions in algebraic groups to the corresponding problem in algebraic Lie algebras. The latter problem may be simpler to handle, but little seems to be known. The "classical" notion of Levi decomposition (semisimple plus solvable) fails for rather unimportant reasons in such cases as the Lie algebra of $SL\ (p,K)$, and it may be hoped that the notion developed by Borel and Tits will prove to be more manageable when applied to Lie algebras. (Cf. the paper by H. E. Campbell, *Pacific J. Math.* 7(1957), 1325-1331.)

(3) An example is constructed in [2, 3.15] to show that Levi factors need not be conjugate. It can be verified that this example satisfies our conditions (a)-(c), which suggests that conjugacy may be a rather delicate question in prime characteristic.

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