PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 3, 1967

SUPERATOMIC BOOLEAN ALGEBRAS

GEORGE W. DAY

A Boolean algebra B can be imbedded in a complete Boolean algebra C in such a way that all homomorphisms of B into complete Boolean algebras can be extended to complete homomorphisms on C if and only if B is superatomic, that is, every homomorphic image of B is atomic. This paper is a study of the structure of superatomic Boolean algebras, through the development of techniques of construction and classification, and through the representation of these algebras as compact Hausdorff clairsemé spaces.

It is shown that the weak direct product of any family of superatomic Boolean algebras is superatomic, and that any Boolean algebra generated by the union of a finite family of superatomic subalgebras is superatomic. The number of nonisomorphic superatomic Boolean algebras of infinite cardinality $\aleph$ is shown to be greater than $\aleph$. A complete algebraic and topological description of the countable superatomic Boolean algebras is given.

The concept of a superatomic Boolean algebra, that is, a Boolean algebra each of whose homomorphs is atomic, was first studied by Mostowski and Tarski in their work on Boolean algebras with ordered bases, [5]. Their results on these algebras are summarized in conditions (a)-(b′) of Theorem 1. The superatomic Boolean algebras again arose in the study of free extensions of Boolean algebras. It was proven by F. M. Yaqub ([8]) that, if $\alpha \geqq 2^{\aleph_0}$, then the free α-extension of a Boolean algebra B is α-representable if and only if B is superatomic. In considering the question of the existence of free complete extensions of Boolean algebras, it was found ([1]) that a Boolean algebra B has a free complete extension if and only if B is superatomic. Other results are summarized in the following theorem:

THEOREM 1. *If B is a Boolean algebra, then the following conditions are equivalent*:

(a) *B is superatomic; i.e., every homomorph of B is atomic.*

(a′) *No homomorph of B is atomless.*

(b) *Every subalgebra of B is atomic.*

(b′) *No subalgebra of B is atomless.*

(c) *No subalgebra of B is an infinite free Boolean algebra.*

(d) *B has no chain of elements order-isomorphic to the chain of rational numbers.*

(e) *$\mathscr{S}(B)$, the Stone space of B, is clairsemé; that is, every nonempty subspace of $\mathscr{S}(B)$ has at least one isolated point.*

Conditions (c)-(e) are developed in [1]. Mostowski and Tarski also showed that all homomorphic images and subalgebras of superatomic Boolean algebras are superatomic.

The purpose of this paper is the development of several notions of construction and classification of superatomic Boolean algebras. In § 2, it is shown that the class of these algebras is closed under the operation of weak direct product. The weak direct product is shown to be interpreted topologically as the one-point compactification of the topological sum of the Stone spaces of the factors. Also, we prove that a Boolean algebra generated by the union of finitely many superatomic subalgebras is superatomic, although the corresponding result does not hold for the concept "atomic". From this result, it follows that the direct product and free product of a set of superatomic Boolean algebras are superatomic if and only if the set is finite.

In § 3, the cardinal sequence of a superatomic Boolean algebra is defined, as a natural classification device for these algebras. Restrictions on the possible nature of these sequences are developed, and several examples are given. The number of nonisomorphic superatomic Boolean algebras of a given infinite cardinality is shown to be greater than that cardinal; the assumption of the Generalized Continuum Hypothesis then implies that there are as many nonisomorphic superatomic Boolean algebras of a given cardinality as there are nonisomorphic Boolean algebras of that cardinality. Section 4 yields a complete algebraic and topological description of the countable superatomic Boolean algebras, and shows that the cardinal sequence of such an algebra completely characterizes the structure of the algebra.

Boolean concepts which are used without discussion in this paper are defined in [2] and [7]. Our notation will coincide with that used in [2]. In the body of this paper, the symbol sBa will represent the phrase "superatomic Boolean algebra".

2. Constructions with sBas. The structure of sBas may be studied in several ways, including the examination and elaboration of relatively simple examples, and the direct construction of clairsemé Stone spaces. In this section, we will consider some of the usual ways of combining Boolean algebras and describe conditions under which the result can be superatomic. Observing that a characteristic of these combining methods is that the basic algebras all appear as subalgebras of the resulting algebra, and noting Theorem 1, we restrict ourselves to consideration of combinations of sBas.

THEOREM 2. *If the Boolean algebra B is generated by $B_1 \cup B_2 \cup \cdots \cup B_n$, where $B_1, B_2, \cdots, B_n$ are superatomic subalgebras of B, then B is superatomic.*

Proof. Clearly, we need only prove that this result holds for $n = 2$. Moreover, since every homomorphic image of B is generated by the union of homomorphic images of $B_1, B_2, \cdots, B_n$, and these homomorphic images are superatomic, condition (a′) of Theorem 1 assures us that it will suffice to prove that if the Boolean algebra B is generated by $B_1 \cup B_2$, where B_1 and B_2 are superatomic subalgebras of B, then B has at least one atom.

Suppose, then, that, B, B_1, and B_2 satisfy these hypotheses. Let a be an atom of B_1 and note that the mapping h defined by $h(x) = ax$ is a homomorphism of B_2 into the Boolean algebra that is the principal ideal of B generated by a. Since B_2 is superatomic, so is $h[B_2]$; thus, there is an element b of B_2 such that $ab \neq 0$, and, for every $x \in B_2$, $(ax)(ab) = 0$ or ab. For every $u \in B_1$ and $x \in B_2$, we then have that $(ux)(ab) = ((ua)x)(ab) = 0$ or ab. Hence, since every element of B is a finite sum of finite products of elements from B_1 and B_2, ab is an atom of B.

In spite of the "naturalness" of the statement of the theorem, the similar statement, "If a Boolean algebra is generated by the union of finitely many atomic subalgebras, then it is atomic" is false. Consider the following:

EXAMPLE. Let B be the complete atomic Boolean algebra with $\aleph_0$ atoms, and let $\{x_1, x_2, \cdots\}$ be a countable independent set of elements of B. Let B_1 denote the subalgebra of B that is generated by $\{x_1, x_2, \cdots\}$ and the atoms of B. Let B_2 be the free Boolean algebra with countable free generating set $\{y_1, y_2, \cdots\}$.

Let A denote the direct product of B_1 and B_2, and define A_1 to be the subalgebra of A with elements $(0, 0), (0, 1), (1, 0), (1, 1)$ and A_2 to be the subalgebra of A generated by all elements of the form $(z, 0)$, where z is an atom of B, and the elements $(x_i, y_i), i = 1, 2, \cdots$. It is easy to see that both A_1 and A_2 are atomic, and that their union generates the nonatomic Boolean algebra A.

COROLLARY. *Suppose that $\mathscr{B}$ is a set of sBas. The following conditions are equivalent:*

(a) *$\mathscr{B}$ is finite.*

(b) *The direct product of the elements of $\mathscr{B}$ is superatomic.*

(c) *The free product of the elements of $\mathscr{B}$ is superatomic.*

Proof. That (a) implies both (b) and (c) follows from the fact that both the direct product and the free product of a finite set of Boolean algebras are generated by the union of a set of subalgebra isomorphic to the factor algebras. If, on the other hand, $\mathscr{B}$ is infinite, then the direct product of the elements of $\mathscr{B}$ contains an infinite complete subalgebra, which in turn must have an infinite free subalgebra; likewise, if $\mathscr{B}$ is infinite, the free product will have an infinite free subalgebra.

This corollary greatly limits the usefulness of the free and direct products as tools in the description of sBas. A device more useful, and more related to the nature of sBas is the weak direct product.

DEFINITION. If $\mathscr{B}$ is a nonempty family of Boolean algebras, let $\times\mathscr{B}$ denote the direct product of the elements of $\mathscr{B}$. The weak direct product of $\mathscr{B}$ is that subalgebra of $\times\mathscr{B}$ generated by the elements of $\times\mathscr{B}$ having only finitely many nonzero coordinates. The weak direct product of $\mathscr{B}$ is denoted by $w \times \mathscr{B}$.

THEOREM 3. *If $\mathscr{B} = \{B_\gamma : \gamma \in \Gamma\}$ is a set of sBas, then $w \times \mathscr{B}$ is superatomic.*

Proof. We shall show that every homomorphic image of $w \times \mathscr{B}$ contains an atom. For convenience, let (B_γ) denote the principal ideal of $w \times \mathscr{B}$ naturally isomorphic to B_γ.

Suppose that I is a proper ideal of $w \times \mathscr{B}$. If for every $\gamma \in \Gamma$, $(B_\gamma) \subseteqq I$, then I is a prime ideal and $w \times \mathscr{B}/I$ is atomic. If, on the other hand, γ is an element of Γ such that $(B_\gamma) \nsubseteq I$, then, since B_γ is superatomic, $(B_\gamma)/[I \cap (B_\gamma)]$ has atoms. If x is an element of (B_γ) whose image in $(B_\gamma)/[I \cap (B_\gamma)]$ is an atom, then the image of x in $w \times \mathscr{B}/I$ is also an atom.

We now give a topological characterization of the weak direct product of any infinite set of Boolean algebras. Together with the above theorem, this topological approach will prove useful in the examination of cardinal sequences of sBas. It will also suggest an alternate proof of Theorem 3, namely, the demonstration that the one-point compactification of the topological sum of an infinite set of compact Hausdorff clairsemé spaces is also clairsemé. The following lemma, an immediate consequence of the definition of the weak direct product, will be needed.

LEMMA. *Suppose that $\mathscr{B} = \{B_\gamma : \gamma \in \Gamma\}$ is a set of Boolean algebras. If C is a Boolean algebra containing a maximal disjointed set of elements $D = \{b_\gamma : \gamma \in \Gamma\}$ such that*

(a) *for each $\gamma \in \Gamma$ the principal ideal (b_γ) in C is isomorphic to B_γ, and*

(b) *C is generated by the set of elements* $\bigcup\{(b_\gamma): \gamma \in \Gamma\}$, *then C is isomorphic to* $w \times \mathscr{B}$.

THEOREM 4. *Let* $\mathscr{B} = \{B_\gamma: \gamma \in \Gamma\}$ *be an infinite family of Boolean algebras.* $\mathscr{S}(w \times \mathscr{B})$, *the Stone space of* $w \times \mathscr{B}$, *is homeomorphic to the one-point compactification of the topological sum of the set of Stone spaces* $\{\mathscr{S}(B_\gamma): \gamma \in \Gamma\}$.

Proof. The topological sum of a set of disjoint topological spaces is a topological space having as point set the union of the point sets of the summands, and having the open sets of the summands as a basis for its topology. Recall also that the one-point compactification of a locally compact, noncompact space is the space obtained by adjoining one point to the original space, and taking as a basis the open sets of the original space and the complements (in the augmented space) of the compact sets of the original space.

The following argument, suggested by Ph. Dwinger, is based on §13 of [2]: The Stone space of $w \times \mathscr{B}$ is clearly a compactification of the topological sum of $\{\mathscr{S}(\beta_\gamma): \gamma \in \Gamma\}$, and is a continuous image of every other compactification of that topological sum. Since the zero dimensional Hausdorff compactifications of a zero dimensional Hausdorff space are partially ordered by the continuous mappings that leave the original space fixed, and the one-point compactification is the least compactification under this ordering, our theorem follows.

3. Cardinal sequences of sBas.

If B is a Boolean algebra, it is convenient to define a sequence of ideals, $I_\beta(B)$, for ordinal β such that $|\beta| < 2^{|B|}$, as follows:

DEFINITION. Let $I_0(B)$ be the zero ideal of B. If β is an ordinal such that $I_{\beta-1}(B)$ is defined, let $I_\beta(B)$ be the preimage in B of the ideal generated by the atoms of $B/I_{\beta-1}(B)$. If β is a limit ordinal, and $I_\alpha(B)$ is defined for all $\alpha < \beta$, let $I_\beta(B)$ be the ideal in B generated by $\bigcup\{I_\alpha(B): \alpha < \beta\}$.

It follows immediately that if $\alpha > \beta$, then $I_\alpha(B)$ is contained in $I_\beta(B)$; thus, if β is a limit ordinal, then $I_\beta(B) = \bigcup\{I_\alpha(B): \alpha < \beta\}$.

PROPOSITION. *B is superatomic if and only if for some ordinal β, $I_\beta(B) = B$.*

Proof. If B is superatomic, and $I_\beta(B) \neq B$, then $|I_\beta(B)| \geqq |\beta|$. Thus, for some β such that $|\beta| < 2^{|B|}$, $I_\beta(B) = B$.

Conversely, suppose that $I_\beta(B) = B$ and that I is a proper ideal of B. Let α' be the least ordinal in the set $\{\alpha: I_\alpha(B) \nsubseteq I\}$. α' is not a limit ordinal; let x be the preimage of an atom of $B/I_{\alpha'-1}(B)$ such that $x \notin I$. Since $I_{\alpha'-1}(B) \subseteqq I$, it follows that the image of x in B/I is an atom.

DEFINITION. If B is a sBa, let $\delta(B)$ denote the least ordinal in the set $\{\beta: |\beta| < 2^{|B|}, I_\beta(B) = B\}$.

It is important to note that, since the union of a chain of proper ideals of a Boolean algebra is a proper ideal, $\delta(B)$ cannot be a limit ordinal.

DEFINITION. The cardinal sequence of a sBa B is the sequence of order type $\delta(B)$ whose β-term, for $\beta < \delta(B)$, is the cardinality of the set of atoms of $B/I_\beta(B)$.

Several elementary properties of the cardinal sequence of a sBa may be mentioned. Its $\delta(B) - 1$ term is necessarily finite, since the preimage in B of the ideal of atoms of $B/I_{\delta(B)-1}(B)$ is all of B; also, all other terms of the sequence are infinite. From cardinality considerations, we can see that if $\aleph$ is the cardinality of the set of atoms of B, then no term of the cardinal sequence of B is greater than $2^{\aleph}$, and $|\delta(B)| \leqq 2^{\aleph}$. Also, it is clear that if B_1 and B_2 have distinct cardinal sequences, then they are not isomorphic.

EXAMPLES. 1. It is known (see [6]) that there is a family $\mathscr{F}$ of subsets of the set of integers such that each element of $\mathscr{F}$ is infinite, such that any two elements of $\mathscr{F}$ have finite intersection, and such that $|\mathscr{F}| > \aleph_0$. If B is the field of subsets of the set of integers generated by the finite sets of integers and the elements of $\mathscr{F}$, then B is a sBa and has cardinal sequence $\aleph_0, |\mathscr{F}|, 1$.

2. Let $\aleph_1$ and $\aleph_2$ be infinite cardinals such that $\aleph_1 < \aleph_2$, and B_1 and B_2 be the Boolean algebras of finite and cofinite subsets of sets of cardinalities $\aleph_1$ and $\aleph_2$, respectively. Consider the direct products of B_1 with B_2, and of B_2 with itself. Clearly, both of these algebras have cardinal sequence $\aleph_2, 2$; we note, however, that they are not isomorphic.

PROPOSITION. If B is an infinite sBa, then the cardinality of B is the sum of the terms of its cardinal sequence.

Proof. Let S be a set of elements of B consisting of one preimage in B of each of the atoms of each of the algebras, $B/I_\beta(B)$, for $\beta < \delta(B)$. Let B' denote the subalgebra of B generated by S. Clearly, B' contains

$I_0(B)$. Also, every element of $I_{\beta+1}(B) \sim I_\beta(B)$ is congruent modulo $I_\beta(B)$ to a finite sum of elements of S; therefore, if $I_\beta(B)$ is contained in B', so is $I_{\beta+1}(B)$. It follows that $B' = B$; thus the cardinality of the infinite Boolean algebra B is the cardinality of its generating set S, which is simply the sum of the terms of the cardinal sequence of B.

The topological interpretation of the cardinal sequence will be helpful in our further discussion. Recall that the derived set of a Hausdorff space $\mathscr{S}$ is the set of nonisolated points of $\mathscr{S}$, and that one defines the β-derived set of $\mathscr{S}$, denoted $D_\beta(\mathscr{S})$, by: $D_0(\mathscr{S}) = \mathscr{S}$; if β is an ordinal such that $D_\beta(\mathscr{S})$ is defined, then $D_{\beta+1}(\mathscr{S})$ is the derived set of $D_\beta(\mathscr{S})$; if β is a limit ordinal, then

$$D_\beta(\mathscr{S}) = \bigcap \{D_\alpha(\mathscr{S}) : \alpha < \beta\} .$$

Note that each derived set of $\mathscr{S}$ is closed in $\mathscr{S}$.

PROPOSITION. Under the natural correspondence between the structures of a Boolean algebra B and its Stone space $\mathscr{S}(\mathscr{B})$, the ideal $I_\beta(B)$ corresponds to the open set $\mathscr{S}(B) \sim D_\beta(\mathscr{S}(B))$. If B is a sBa, then the β-term of the cardinal sequence of B is the cardinality of the set of isolated pointed of $D_\beta(\mathscr{S}(B))$.

Proof. The second statement follows from the first and the observation that if I is a proper ideal of B, then the Stone space of B/I is homeomorphic to the complement in $\mathscr{S}(B)$ of the open set corresponding to I.

If the first statement is true for a fixed ordinal β, then $D_\beta(\mathscr{S}(B))$ is homeomorphic to $\mathscr{S}(B/I_\beta(B))$ and the ideal of atoms of $B/I_\beta(B)$ corresponds to the set of isolated points of $D_\beta(\mathscr{S}(B))$. It follows that $I_{\beta+1}(B)$ corresponds to the union of $\mathscr{S}(B) \sim D_\beta(\mathscr{S}(B))$ and the set of isolated points of $D_\beta(\mathscr{S}(B))$; that is, $I_{\beta+1}(B)$ corresponds to the open set $\mathscr{S}(B) \sim D_{\beta+1}(\mathscr{S}(B))$.

If α is a limit ordinal and the statement of the proposition is true for all ordinals β such that $\beta < \alpha$, then $I_\alpha(B) = \cup \{I_\beta(B) : \beta < \alpha\}$ corresponds to the open set

$$\cup \{\mathscr{S}(B) \sim D_\beta(\mathscr{S}(B)) : \beta < \alpha\} = \mathscr{S}(B) \sim D_\alpha(\mathscr{S}(B)) .$$

We now connect the concepts of cardinal sequence and weak direct product. By the sum of a set of cardinal sequences, we shall mean that sequence of cardinals whose β-term is the sum of the β-terms of those summands whose order type is greater than β.

THEOREM 5. *Let $\mathscr{B} = \{B_\gamma : \gamma \in \Gamma\}$ be a set of* sBas, *and let $B = w \times \mathscr{B}$. Let θ be the sum of the cardinal sequences of the elements of $\mathscr{B}$ and* *let θ_β denote the β-term of θ. Let β^* be the order type of θ, and let β_0 be the least ordinal in the set $\{\beta: \{\gamma: \beta < \delta(B_\gamma)\}$ is finite$\}$. The cardinal sequence of B is determined as follows:*

(a) *If Γ is infinite and $\beta_0 = \beta^*$, then the cardinal sequence of B is of order type $\beta^* + 1$, has β^*-term 1, and for $\beta < \beta^*$ has β-term θ_β.*

(b) *If Γ is infinite and $\beta_0 = \beta^* - 1$, then the cardinal sequence of B is of order type β^*, has $(\beta^* - 1)$-term $(\theta_{\beta^*-1}) + 1$, and for $\beta < \beta^* - 1$ has β-term θ_β.*

(c) If Γ is finite, or if Γ is infinite and $\beta_0 < \beta^*$ and $\beta_0 \neq \beta^* - 1$, then the cardinal sequence of B is θ.

Proof. We need two observations concerning derived sets. First, if X is a Hausdorff space with subspace Y, then the β-derived set of Y is contained in the β-derived set of X. From this it follows that if X is locally compact, and p is an isolated point in the β-derived set of X, then p is an isolated point in the β-derived set of X^*, the one-point compactification of X. We can also conclude that if $\mathscr{X}$ is a set of Hausdorff spaces and X is their topological sum, then the β-derived set of X (respectively, the set of isolated points of that set) is the union of the β-derived sets of the elements of $\mathscr{X}$ (respectively, of the sets of isolated points of those sets).

If Γ is finite, the Stone space of B is simply the topological sum of the Stone spaces $\mathscr{S}(B_\gamma), \gamma \in \Gamma$. From the remarks above, and our previous topological observations, it is clear that the cardinal sequence of B is θ.

In case Γ is infinite, $\mathscr{S}(B)$ is the one-point compactification of the topological sum of the spaces $\mathscr{S}(B_\gamma), \gamma \in \Gamma$. Let p denote the adjoined point of this compactification. We see that the β-derived set of $\mathscr{S}(B)$ is the union of the β-derived sets of those $\mathscr{S}(B_\gamma)$ for which $\beta < \delta(B_\gamma)$, with the possible inclusion of p. Thus, to determine the cardinal sequence of B, it will suffice to determine the ordinal β such that p is an isolated point of $D_\beta(\mathscr{S}(B))$.

From the definition of the one-point compactification, it follows that if $\mathscr{S}'$ is a subspace of $\mathscr{S}(B)$ and $p \in \mathscr{S}'$, then p is an isolated point of $\mathscr{S}'$ if and only if $\mathscr{S}'$ has empty intersection with all but finitely many of the subspaces $\mathscr{S}(B_\gamma)$. If $\beta < \beta_0$, then $D_\beta(\mathscr{S}(B))$ has non-empty intersection with infinitely many of the spaces $\mathscr{S}(B_\gamma)$; since $D_\beta(\mathscr{S}(B))$ is closed, p must be contained in it as a non-isolated point. Consequently, $p \in D_{\beta_0}(\mathscr{S}(B))$, and is an isolated point of the that set.

The conclusions of (a) and (b) follow immediately. In the remaining case, given that $\beta_0 < \beta^*$ and $\beta_0 \neq \beta^* - 1$, it follows that β^* is a non-limit ordinal and for some $\gamma \in \Gamma$, $\delta(B_\gamma) > \beta_0 + 1$. Hence, $D_{\beta 0}(\mathscr{S}(B_\gamma))$ is infinite, $D_{\beta 0}(\mathscr{S}(B))$ is infinite, and the presence of p in this set does not affect its cardinality.

The next proposition suggests the structure of an extensive subclass of the sBas, and enables us to put a lower bound on the number of non-isomorphic sBas of a given cardinality.

THEOREM 6. *Suppose that θ is a sequence of nonzero cardinals, that θ has nonlimit ordinal type β_0, and that θ_β denotes the β-term of θ. If θ has the following properties*:

(a) *θ_β is finite if and only if $\beta = \beta_0 - 1$,*

(b) *if $\alpha < \beta < \beta_0$, then $\theta_\alpha \geqq \theta_\beta$,*

(c) *if $\beta < \beta_0$, then $|\{\alpha : \beta < \alpha < \beta_0\}| \leqq \theta_\beta$,*

then θ is the cardinal sequence of a sBa.

Proof. From Theorem 5(c), we may conclude that if this true under the added restriction that the last term of θ is 1, then it is true in general. Suppose, then, that β' is a nonlimit ordinal, that the theorem is true for all ordinals β_0 such that $\beta_0 < \beta'$, and that θ is a sequence of type β' that satisfies the hypotheses of the theorem for $\beta_0 = \beta'$ and has $(\beta' - 1)$-term 1.

First consider the case that $\beta' - 1$ is a nonlimit ordinal. Let θ' be the sequence of order type $\beta' - 1$ with $(\beta' - 2)$-term 1, and such that if $\beta < \beta' - 2$, then $\theta'_\beta = \theta_\beta$. Let B be the weak direct product of a set of cardinality $\theta_{\beta'-2}$ of sBas each having cardinal sequence θ'. Using Theorem 5(a) with $\beta^* = \beta' - 1$ and condition (b) of this theorem, we find that B has cardinal sequence θ.

Now suppose that $\beta' - 1$ is a limit ordinal. For each ordinal β less than $\beta' - 1$, let B_β be a sBa with cardinal sequence of order type $\beta + 1$, with α-term θ_α for all $\alpha < \beta$ and β-term 1. Using Theorem 5(a) with $\beta^* = \beta' - 1$ again, and condition (c) of this theorem, we find that $w \times \{B_\beta : \beta < \beta' - 1\}$ has cardinal sequence θ.

COROLLARY. *If $\aleph$ is an infinite cardinal, then there are more than $\aleph$ nonisomorphic* sBas *of cardinality $\aleph$.*

Proof. Let β^* be the least ordinal of cardinality greater than $\aleph$. If β is a non limit ordinal less than β^*, then there is a sBa whose cardinal sequence is of type β, has $(\beta - 1)$-term 1, and has α-term $\aleph$ for every $\alpha < \beta - 1$. By a previous proposition, such a Boolean algebra will have cardinality $\aleph$.

Since every Boolean algebra of cardinality $\aleph$ is a homomorphic image of the free Boolean algebra on $\aleph$ generators, and since that algebra can have at most $2^\aleph$ ideals, the set of isomorphism classes of sBas of cardinality $\aleph$ has cardinality greater than $\aleph$ and not greater than $2^{\aleph}$. The assumption of the Generalized Continuum Hypothesis thus implies that there are exactly $2^{\aleph}$ isomorphism classes of sBas of cardinality $\aleph$.

It is clear that the cardinal sequence of a sBa must satisfy part of the hypotheses of Theorem 6; certainly, such a sequence must de of nonlimit order type, and have only its last term finite. However, we displayed above a sBa whose cardinal sequence did not satisfy condition (b) of the theorem. Our last example shows that the cardinal sequence of a sBa need satisfy neither (b) nor (c). (It is not known whether the cardinal sequence of a sBa may satisfy (b) but not (c).)

EXAMPLE. Let S be a set of cardinality $\aleph_0$. There is a sequence of type ω_1 (the first uncountable ordinal) of infinite subsets of S such that if $\alpha < \beta < \omega_1$ and U and V are the α- and β-terms of the sequence, then $U \sim V$ is finite and $V \sim U$ is infinite (see [6]). Let $\mathscr{S}$ be such a sequence, and let B be the field of subsets of S generated the finite subsets of S and the terms of $\mathscr{S}$. B is atomic, and I, the ideal of atoms of B, is simply the set of finite subsets of S. B/I is generated by the images of the elements of $\mathscr{S}$. These image form a well-ordered chain of type ω_1. From [3], Theorem 3.3, it follows that B/I is superatomic; consequently, B is superatomic. The atoms of B/I are the images of elements of B of the form $V \sim U$, where U V are successive terms of $\mathscr{S}$. The image of B/I modulo its ideal of atoms is generated by the images of the limit terms of $\mathscr{S}$; thus, this algebra is also generated by a well-ordered chain of type ω_1. We can conclude that the cardinal sequence of B is of type $\omega_1 + 1$, that it has 0-term $\aleph_0$, and that all other terms except the last are $\aleph_1$. Thus, the cardinal sequence of B satisfies neither conditions (b) nor (c) of Theorem 6. It is also of some interest to note that for $0 < \alpha < \beta < \omega_1$, $B/I_\alpha(B)$ is isomorphic to $B/I_\beta(B)$.

4. Countable sBas.

We first note that Theorem 6 assures us of the following:

PROPOSITION. A sequence of cardinals is the cardinal sequence of a countable sBa if and only if it is of nonlimit order type $\beta < \omega_1$, has finite last term, and has all other terms equal to $\aleph_0$.

The topological interpretation of a countable sBa proves to be extremely simple. We have seen that if B is an infinite sBa, then the cardinality of $\mathscr{S}(B)$, the sum of the cardinal sequence of B, and the cardinality of B itself, are all equal. It is well-known that every compact countable Hausdorff space is totally disconnected and contains an isolated point. Since the Stone space of any homomorphic image of a Boolean algebra B is homeomorphic to a subspace of $\mathscr{S}(B)$, the next proposition is immediate.

PROPOSITION. The topological space X is homeomorphic to the Stone space of a countable sBa if and only if X is compact, Hausdorff, and countable.

We can also observe that every countable, compact Hausdorff space is homeomorphic to a compact subspace of the real line, Euclidean 1-space. Mazurkiewicz and Sierpiński ([4]) have proven that if X_1 and X_2 are bounded, closed, countable sets in Euclidean m-space with non empty, finite α-derived sets of equal size, then X_1 and X_2 are homeomorphic. Our final result follows from that observation.

PROPOSITION. Two countable sBas are isomorphic if and only if they have the same cardinal sequence.

REFERENCES

1. G. W. Day, *Free complete extensions of Boolean algebras,* Pacific J. Math. **15** (1965), 1145-1151.
2. Ph. Dwinger, *Introduction to Boolean Algebras,* Wurzburg, 1961.
3. R. D. Mayer and R. S. Pierce, *Boolean algebras with ordered bases,* Pacific J. Math. **10** (1960), 925-942.
4. S. Mazurkiewicz and W. Sierpinski, *Contributions a la topologie des ensembles denombrables,* Fund. Math. **1** (1920), 17-27.
5. A. Mostowski and A. Tarski, *Boolesche Ringe mit geordneter Basis,* Fund. Math. **32** (1939), 69-86.
6. W. Sierpinski, *Cardinal and Ordinal Numbers,* Warsaw, 1958.
7. R. Sikorski, *Boolean Algebras,* Berlin, 1960.
8. F. M. Yaqub, *Free extensions of Boolean algebras,* Pacific J. Math. **13** (1963), 761-771.

Received January 27, 1967.

THE UNIVERSITY OF WYOMING
LARAMIE, WYOMING