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POLYNOMIALS IN LINEAR RELATIONS

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We point out an error in a paper on linear relations by R. Arens. We also re-establish, in a weakened form, a result of Arens on the polynomials in a closed linear relation which was placed in jeopardy by this error.

The error in question is in the proposition [1, 3.52], restated below as 3.1. It was detected by Professor P. R. Masani, who also constructed a counter example. We take the liberty of presenting his example below.

The theorem on closed linear relations, viz. [1, 3.8], which depended on this proposition 3.1 was to be a generalization of the theorem of A. E. Taylor [7, 6.1] who proved that if T is a closed operator with a nonempty resolvent in a Banach space, then each polynomial in T is also a closed operator.

Our weakened form, viz. Theorem 3.16 below, of Arens' purely algebraic proposition involves topological linear space concepts, and is still a generalization of Taylor's theorem [7, 6.1].

2. **Linear relations.** Let X_1, X_2, and X_3 be linear spaces. A *relation* between X_1 and X_2 is a subset of $X_1 \oplus X_2$. A *linear relation* between X_1 and X_2 is a linear subspace of $X_1 \oplus X_2$ These concepts generalize the notions of function and linear operator respectively. If T is a relation, the definitions of the domain and range of $T, D(T)$ and $R(T)$ respectively, are obvious.

If S and T are relations between X_1 and X_2,

$$S + T = \{(x_1, x_2 + x_3) : (x_1, x_2) \in S \text{ and } (x_1, x_3) \in T\}$$

is a relation between X_1 and X_2. If S is a relation between X_1 and X_2, and T is a relation between X_2 and X_3, $T \circ S = \{(x_1, x_3) : (x_1, x_2) \in S$ and $(x_2, x_3) \in T$ for some $x_2 \in X_2\}$ is a relation between X_1 and X_3. For any relation T, $T^{-1} = \{(p, q): (q, p) \in T\}$. If T is linear, T^{-1} is also. If S and T are linear, $S + T$ and $S \circ T$ are also. If λ is an element of the field, we may consider λ as the linear relation between X_i and X_i, namely $\{(x, \lambda x) : x \in X_i\}$. We write λT for $\lambda \circ T$.

If X_i and Y_i, $i = 1, 2$, form a dual pair, i.e., there exists a canonical bilinear form $\langle , \rangle : X_i \oplus Y_i \to F$ (where F is the field) such that $\langle x, y_0 \rangle = 0$ for every $x \in X_i$ implies $y_0 = 0$ and $\langle x_0, y \rangle = 0$ for every $y \in Y_i$ implies $x_0 = 0$, and T is a relation between X_1 and X_2, we may define the adjoint of T, $T^* = \{(y_2, y_1)$: for every

$$(x_1, x_2) \in T, \langle x_1 y_1 \rangle = \langle x_2, y_2 \rangle\}$$

which is a relation between Y_2 and Y_1. Again if T is linear, T^* is also. We can also consider $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ to form a dual pair with $\langle (x_1, x_2), (y_1 y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$. Thus we may speak of $T^\perp$ for T a relation. From [1, 3. 3] it is obvious that $T^{**} = T^{\perp\perp}$. A relation is called *regularly closed* if and only if $T = T^{\perp\perp} = T^{**}$.

If X and Y form a dual pair over $\mathbf{C}$, we consider X as a class of linear functionals on Y. An obvious topology of importance on X is that of pointwise convergence on Y. This topology is denoted $\sigma(X, Y)$. It can be shown that the dual of $(X, \sigma(X, Y))$ is Y. The interested reader is referred to Schaefer [6], page 122 et seq. A locally convex vector topology on X, Ω, such that the dual of (X, Ω) is Y is called a consistent topology for the dual pair $\langle X, Y \rangle$. It was shown by Mackey [5] that there exists a finest such topology. Arens [2] identified it with the topology of uniform convergence on absolutely convex $\sigma(Y, X)$ compact sets. This topology is denoted $\tau(X, Y)$. Thus a locally convex vector topology on X is consistent if and only if it is coarser than $\tau(X, Y)$ and finer than $\sigma(X, Y)$.

It was first shown by Banach [3] that if T is a linear subspace, and thus in particular, a linear relation, it is regularly closed if and only if it is closed in the Cartesian product of any two consistent topologies. Thus the question of closed linear relations can be attacked from either viewpoint. Granted that the algebraic approach does not depend on the field, we will see that the topological approach can lead to stronger results in special cases.

3. Closed products of linear relations. In the course of investigating when the product $S \circ T$ of two closed linear relations is closed, Arens [1, 3. 52] states the following proposition.

PROPOSITION 3.1. Let X_i and Y_i, $i = 1, 2, 3$ be dual pairs. Let T be a linear relation between X_1 and X_2 and S a linear relation between X_2 and X_3. If $D(S^*) = X_3$ or $R(T^*) = X_1$, then $(S \circ T)^* = T^* \circ S^*$. The following counterexample is due to Masani:

3.11. Let $X_i = Y_i =$ Hilbert space $i = 1, 2, 3$, with its own natural duality. Let M be a closed subspace of X such that the dimension of $M = \infty$. Let P_M be the projection onto M and $T_1 = P_M^{-1} \circ P_M$. Let S_1 be any closed one-to-one operator such that $D(S_1) \perp M$ and $R(S_1)$ is not closed. S_1 may even be chosen to be continuous. We shall first establish that $S_1 \circ T_1$ is not closed. Let $z \in \overline{R(S_1)} - R(S_1)$ and z_n a sequence such that $z_n \rightarrow z$ and $z_n \in R(S_1)$. Let $y_n = S_1^{-1}(z_n)$. Certainly the sequence y_n does not converge (since $z \notin R(S_1)$). However,

$y_n \in D(S_1) \subseteqq N(T_1)$. Since $T_1 = T_1^{-1}$, $(0, y_n) \in T_1$. Since $(y_n, z_n) \in S_1$, we have $(0, z_n) \in S_1 \circ T_1$. Now since $z \notin R(S_1) \subseteqq R(S_1 \circ T_1)$, $z \notin R(S_1 \circ T_1)$. Thus $(0, z) \notin S_1 \circ T_1$. However $(0, z_n) \to (0, z)$ and $(0, z_n) \in S_1 \circ T_1$. Thus $S_1 \circ T_1$ is not closed.

Now let $S = T_1^*$ and $T = S_1^*$. Then $D(T_1) = D(S^*) = X$. But $T^* \circ S^* = S_1^{**} \circ T_1^{**} = S_1 \circ T_1$ which we have shown to be not closed. But $(S \circ T)^*$ is necessarily closed since all adjoints are; hence $(S \circ T)^* \neq T^* \circ S^*$.

The reader should note that the above counter example also disproves Arens [1, 3. 7] on the closedness of $S \circ T$.

As Masani has suggested, there is a weakened form of 3.1 which is true and we present it next (3.12). We remark that 3.12 cannot be used in place of 3.1 in [1] to validate the proposed generalization [7, 6. 1].

PROPOSITION 3.12. *If* $T \subseteqq X_1 \oplus X_2$, $S \subseteqq X_2 \oplus X_3$ *and either* $R(T^*) = Y_1$ *and* $D(S) \subseteqq R(T)$ *or* $D(S^*) = Y_3$ *and* $R(T) \subseteqq D(S)$, *then* $(S \circ T)^* = T^* \circ S^*$.

Proof. First note that if one hypothesis is sufficient for the intended result, the other is also. This is an easy consequence of the fact that $(S \circ T)^* = T^* \circ S^*$ if and only if

$$(T^{-1} \circ S^{-1})^* = (S^{-1})^* \circ (T^{-1})^* .$$

Thus let us suppose $R(T^*) = Y_1$ and $D(S) \subseteqq R(T)$. It is trivial that $T^* \circ S^* \subseteqq (S \circ T)^*$. Thus we are reduced to showing that

$$(S \circ T)^* \subseteqq T^* \circ S^* .$$

Let $(y_3, y_1) \in (S \circ T)^*$. Certainly $(y_2, y_1) \in T^*$ for some $y_2 \in Y_2$ since $R(T^*) = Y_1$. If we can show $(y_3, y_2) \in S^*$, we are finished. To do this we must show for every $(x_2, x_3) \in S$, $\langle x_2, y_2 \rangle = \langle x_3, y_3 \rangle$. Since $D(S) \subseteqq R(T)$, for each $(x_2, x_3) \in S$, there exists $x_1 \in X$, such that $(x_1, x_2) \in T$. Thus $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$ since $(y_2, y_1) \in T^*$. Next note that $(x_1, x_3) \in S \circ T$ and thus $\langle x_1, y_1 \rangle = \langle x_3, y_3 \rangle$. Thus $\langle x_2, y_2 \rangle = \langle x_3, y_3 \rangle$ as was to be shown.

This proposition can be used to establish the closedness of some, but not very many, products.

PROPOSITION 3.13. *If S is a closed relation between X_1 and X_2, T a closed relation between X_2 and X_3, and either $R(T) = X_3$ and $D(S^*) \subseteqq R(T^*)$ or $D(S) = X_1$ and $R(T^*) \subseteqq D(S^*)$, then $T \circ S$ is a closed relation between X_1 and X_3.*

Proof. Certainly $S^* \circ T^* \subseteqq (T \circ S)^*$. Hence $(T \circ S)^{**} \subseteqq (S^* \circ T^*)^*$. If we can show that $(S^* \circ T^*)^* \subseteqq T^{**} \circ S^{**} = T \circ S$, we are finished, for then $(T \circ S)^{**} \subseteqq T \circ S$. But direct application of 3.12, substituting S^* for S and T^* for T, asserts that $(S^* \circ T^*)^* \subseteqq T^{**} \circ S^{**}$.

This proposition is true for any field, the proof given above being valid in such generality.

However, we now shift to the employment of properly topological methods to obtain a result for topological linear spaces which is stronger than 3.13.

PROPOSITION 3.14. Let X_i and Y_i be dual pairs over $\boldsymbol{C}$. If S is a closed relation between X_1 and X_2, $T \subseteqq X_2 \oplus X_3$ and either $R(T)$ is a closed subspace of X_3 and $D(S^*) \subseteqq R(T^*)$ or $D(S^*)$ is a closed subspace of Y_2 and $R(T^*) \subseteqq D(S^*)$ then $T \circ S$ is closed.

Proof. We shall only consider the first set of hypotheses. The other set is handled as in 3.12. Suppose (d_γ, c_γ) is a net in $T \circ S$ and $(d_\gamma, c_\gamma) \rightarrow (d, c)$ in $\sigma(X, Y) \oplus \sigma(X, Y)$. We wish to show $(d, c) \in T \circ S$. Since $R(T)$ is closed, there exists $b \in X_2$ such that $(b, c) \in T$. If $(d, b) \in S$, we are finished. Since S is closed, $(d, b) \in S$ if and only if for every $(x, y) \in S^*$, $\langle d, y\rangle = \langle b, x\rangle$. Since $(d_\gamma, c_\gamma) \in T \circ S$ there exists $b_\gamma \in X_2$ such that $(d_\gamma, b_\gamma) \in S$ and $(b_\gamma, c_\gamma) \in T$. Since $D(S^*) \subseteqq R(T^*)$, there exists w such that $(w, x) \in T^*$. Thus $\langle d_\gamma, y\rangle = \langle b_\gamma, x\rangle$. But $\langle b_\gamma, x\rangle = \langle c_\gamma, w\rangle$. Thus since $\langle d_\gamma, y\rangle \rightarrow \langle d, y\rangle$ and $\langle c_\gamma, w\rangle \rightarrow \langle c, w\rangle$, $\langle d, y\rangle = \langle c, w\rangle$. However $\langle b, x\rangle = \langle c, w\rangle$ since $(b, c) \in T$ and $(w, x) \in T^*$. Hence $\langle d, y\rangle = \langle b, x\rangle$ as was to be shown.

PROPOSITION 3.15. Let X_i and Y_i be dual pairs over $\boldsymbol{C}$. If S is a closed relation between X_1 and X_2 and $T \subseteqq X_2 \oplus X_3$ such that $R(T)$ is closed and T^{-1} is singlevalued and continuous from

$$(R(T), \sigma(X, Y)|_{R(T)}) \rightarrow (X, \sigma(X, Y)) ,$$

then $T \circ S$ is closed.

Proof. First note that the hypotheses on T necessarily imply that T is closed. Let $(d_\gamma c_\gamma)$ be a net in $T \circ S$ such that $(d_\gamma, c_\gamma) \rightarrow (d, c)$ in $\sigma(X, Y) \oplus \sigma(X, Y)$. Since $c_\gamma \rightarrow c$ in $\sigma(X, Y)$, $T^{-1}(c_\gamma) \rightarrow T^{-1}(c)$. Since T is closed, $(T^{-1}(c), c) \in T$ and since S is closed, $(d, T^{-1}(c)) \in S$. Hence $(d, c) \in T \circ S$.

We now use 3.15 to prove a generalization of Taylor's theorem concerning polynomials in linear operators.

THEOREM 3.16. *Let X and Y be a dual pair over $\boldsymbol{C}$. If T is a closed linear relation on X such that for some $\lambda \in \boldsymbol{C}$, $R(T - \lambda)$ is* *a closed subspace of* X *and* $(T - \lambda)^{-1}$ *is singlevalued and continuous from* $(R(T), \sigma(X, Y)|_{R(T)})$ *to* $(X, \sigma(X, Y))$, *then for every polynomial* P, $P(T)$ *is a closed linear relation on* X.

Proof. We will use induction and the fact that if S is closed, $S + \alpha$ is closed for every $\alpha \in \boldsymbol{C}$. Thus we may assume that for every polynomial Q of degree $\leqq n$, $Q(T)$ is closed. Let $P(z)$ be an $(n + 1)$ degree polynomial. Then there exists a polynomial Q of degree n such that for some $\alpha \in \boldsymbol{C}$, $P(z) - \alpha = (z - \lambda) \circ Q(z)$, $z \in \boldsymbol{C}$. Hence by [1, 2. 3], $P(T) - \alpha = (T - \lambda) \circ Q(T)$. Now apply 3.15 with T replaced by $T - \lambda$ and S by $Q(T)$. Then $P(T) - \alpha$ is closed and hence $P(T)$ is closed.

4. Further investigation. In our doctoral dissertation [4], we have treated this theorem and related topics in greater detail.

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