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# EQUIVALENT DECOMPOSITION OF $R^3$

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**If $G$ is any monotone decomposition of $R^3$, let $H_G$ denote the union of the nondegenerate elements of $G$, and let $P_G$ denote the projection map from $R^3$ onto the decomposition space $R^3/G$ associated with $G$. Suppose that $F$ and $G$ are monotone decompositions of $R^3$ such that each of $\mathrm{Cl}(P_F[H_F])$ and $\mathrm{Cl}(P_G[H_G])$ is compact and 0-dimensional. Then $F$ and $G$ are *equivalent* decompositions of $R^3$ if and only if there is a homeomorphism $h$ from $R^3/F$ onto $R^3/G$ such that**

$$h[\mathrm{Cl}(P_F[H_F])] = \mathrm{Cl}(P_G[H_G]) .$$

**A necessary and sufficient condition for two decompositions to be equivalent is given. It is shown that there is a decomposition with only a countable number of nondegenerate elements which is equivalent to the dogbone decomposition, and several related results are obtained.**

By introducing the idea of equivalent decompositions of $R^3$, we are able to analyze in a precise way, a process that seems quite natural in the study of monotone decompositions of $R^3$ of the type we are considering. If $F$ is a monotone decomposition of $R^3$, the stipulation that $\mathrm{Cl}\, P_F[H_F]$ be a compact 0-dimensional set is equivalent to the following condition: There is a sequence $M_1, M_2, M_3, \cdots$ of compact 3-manifolds-with-boundary in $R^3$ such that for each positive integer $j$, $M_{j+1} \subset \mathrm{Int}\, M_j$ and $g$ is a nondegenerate element of $F$ if and only if $g$ is a nondegenerate component of $\bigcap_{j=1}^{\infty} M_j$.

A process one finds useful in certain situations is one that involves a sequence $f_1, f_2, f_3, \cdots$ of homeomorphisms from $R^3$ onto $R^3$ such that (1) $f_1$ shrinks or stretches $M_1$, (2) $f_2$ agrees with $f_1$ on $R^3 - M_1$ and shrinks or stretches $M_2$, (3) $f_3$ agrees with $f_2$ on $R^3 - M_2$ and shrinks or stretches $M_3$, and so on. The "new" decomposition has as its nondegenerate elements the nondegenerate components of

$$f_1[M_1] \cap f_2[M_2] \cap f_3[M_3] \cap \cdots .$$

We are able to show that under fairly mild restrictions, there exists such a sequence of homeomorphisms if and only if the original decomposition and the "new" one are equivalent in the sense of this paper.

We indicate some examples that illustrate these concepts. The first two examples give instances of previous applications of the ideas of this paper. The remaining ones are described in detail in the present paper.

EXAMPLE 1. Meyer proved [10] that if $C$ is a 3-cell in $R^3$ such that Bd $C$ is locally polyhedral except at points of an arc $\alpha$ on Bd $C$, then $R^3/C$ is homeomorphic to $R^3/\alpha$.

EXAMPLE 2. Bing described [6] a 2-sphere $S$ in $R^3$ such that $S$ is locally wild at each point of $S$ and $S$ bounds a 3-cell $B$ in $R^3$. Armentrout proved [1] that there is a 3-cell $B'$ in $R^3$ such that Bd $B'$ is locally polyhedral except on a Cantor set on Bd $B'$ and $R^3/B$ is homeomorphic to $R^3/B'$.

EXAMPLE 3. Suppose $G$ is a monotone decomposition of $R^3$ such that there is a sequence $M_1, M_2, M_3, \cdots$ of compact 3-manifolds-with-boundary as described above. Suppose further that each component of each $M_i$ is a 3-cell-with-handles. Then $G$ is equivalent to a decomposition into 1-dimensional continua and one-point sets; see §7.

EXAMPLE 4. Bing's dogbone decomposition [5] is equivalent to a decomposition into one-point sets and at most countably many nondegenerate continua; see §4.

EXAMPLE 5. In §3 of [7], Bing described a point-like decomposition $G$ of $R^3$ with only countably many nondegenerate elements such that $R^3/G$ is not homeomorphic to $R^3$. There exists a decomposition $F$ of $R^3$ such that $F$ is equivalent to $G$ and $F$ has uncountably many nondegenerate elements; see §5.

**2. Notation and terminology.** The statement that $G$ is a *monotone* decomposition of $R^3$ means that $G$ is an upper semi-continuous decomposition of $R^3$ into compact continua. A compact continuum $K$ in $R^3$ is *point-like* if and only if $R^3 - K$ is homeomorphic to the complement, in $R^3$, of a one-point set. A set $M$ in $R^3$ is *cellular* if and only if there is a sequence $C_1, C_2, C_3, \cdots$ of 3-cells in $R^3$ such that for each $i$, $C_{i+1} \subset \operatorname{Int} C_i$ and $M = \bigcap_{i=1}^{\infty} C_i$. For compact continua in $R^3$, "point-like" and "cellular" are equivalent [12]. The statement that $G$ is a *point-like* decomposition of $R^3$ means that $G$ is a monotone decomposition of $R^3$ into point-like sets.

We shal use the notation and terminology introduced in the introduction.

If $M$ is a 3-manifold-with-boundary, $M$ need not be connected, and Bd $M$ and Int $M$ denote the boundary and interior, respectively, of $M$.

The statement that the subset $K$ of $R^3$ is a *3-cell-with-handles* means that there is a finite collection $C, C_1, C_2, \cdots$, and $C_n$ of 3-cells such that if $i = 1, 2, \cdots$, or $n$, $C_i \cap C$ is the union of two disjoint discs, and $C_i \cap C = (\text{Bd } C_i) \cap (\text{Bd } C)$, and if $i$ and $j$ are distinct, $C_i$ and $C_j$ are disjoint. Such a collection $C, C_1, C_2, \cdots$, and $C_n$ of 3-cells will be called a *standard decomposition* of $K$.

We shall use Cl to denote topological closure. If $X$ is a subset of $R^3$ and $\varepsilon$ is a positive number, then $V(X, \varepsilon)$ denotes the $\varepsilon$-neighborhood of $X$ in $R^3$.

Suppose $G$ is a monotone decomposition of $R^3$. Then $M_1, M_2, M_3, \cdots$ is a *defining sequence for* $G$ if and only if $M_1, M_2, M_3, \cdots$ is a sequence such that (1) for each positive integer $i$, $M_i$ is a compact 3-manifold-with-boundary such that $M_{i+1} \subset \text{Int } M_i$ and (2) $g$ is a nondegenerate element of $G$ if and only if $g$ is a nondegenerate component of $\bigcap_{i=1}^{\infty} M_i$. $G$ has a defining sequence if and only if $\text{Cl } P_G[H_G]$ is a compact 0-dimensional set. $G$ is *definable by 3-cells-with-handles* if and only if $G$ has a defining sequence $M_1, M_2, M_3, \cdots$ such that for each positive integer $i$, each component of $M_i$ is a 3-cell-with-handles. $G$ is a *toroidal* decomposition of $R^3$ if and only if $G$ has a defining sequence $M_1, M_2, M_3, \cdots$ such that for each positive integer $i$, each component of $M_i$ is a solid torus (3-cell with one handle).

**3. The existence of sequences of homeomorphisms.** In this section we establish, under fairly weak conditions on the decompositions involved, the equivalence of two decompositions with the existence of a sequence of homeomorphisms $h_1, h_2, h_3, \cdots$ from $R^3$ to $R^3$ as indicated in the introduction.

A compact continuum $M$ in $R^3$ is *semi-cellular* if and only if for each open set $U$ in $R^3$ containing $M$, there is an open set $V$ lying in $U$ and containing $M$ and such that each simple closed curve in $V$ is null-homotopic in $U$. Every point-like compact continuum in $R^3$ is semi-cellular, since each such set is cellular. Each compact absolute retract in $R^3$ is semi-cellular. Since there exist noncellular arcs in $R^3$, the two categories above are not identical. An example of a semi-cellular compact continuum in $R^3$ neither cellular nor an absolute retract may be obtained as follows: Let $T_1, T_2, T_3, \cdots$ be a sequence of solid tori (3-cells with one handle) in $R^3$ such that for each $i$, $T_{i+1} \subset \text{Int } T_i$, $T_2$ lies in $T_1$ as shown in Figure 1, $T_3$ lies in $T_2$ as $T_2$ lies in $T_1$, and for each $i$, $T_{i+1}$ lies in $T_i$ as $T_i$ lies in $T_{i-1}$. Then $\bigcap_{i=1}^{\infty} T_i$ is a continuum with the desired properties.

LEMMA 1. *Suppose that $F$ and $G$ are monotone decompositions of $R^3$ such that $\text{Cl } P_F[H_F]$ and $\text{Cl } P_G[H_G]$ are compact 0-dimensional*

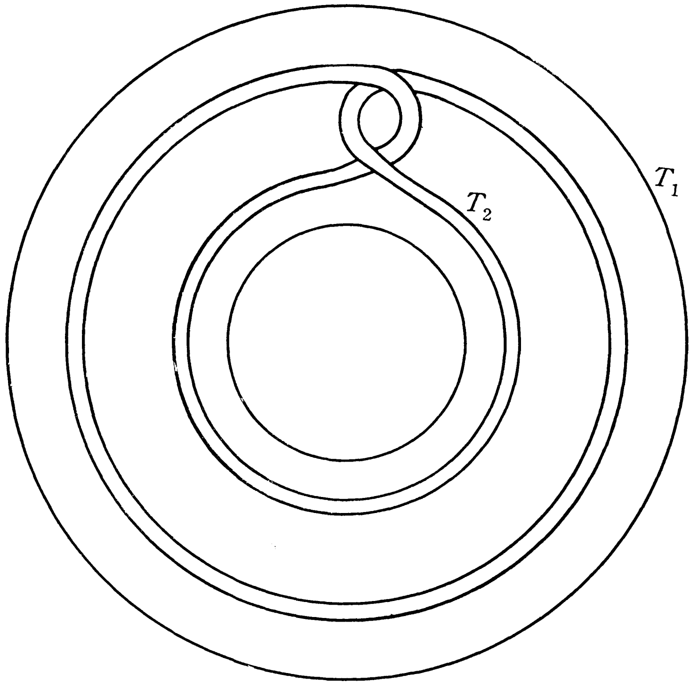


FIGURE 1.

*sets. Suppose that $M$ is a compact polyhedral 3-manifold-with-boundary, each component of which is a 3-cell-with-handles, such that* $\mathrm{Cl}\, H_F \subset \mathrm{Int}\, M$. *Suppose that each element of $G$ is semi-cellular. Suppose that there is a homeomorphism $h$ from $R^3/F$ onto $R^3/G$ such that* $h[\mathrm{Cl}\, P_F[H_F]] = \mathrm{Cl}\, P_G[H_G]$. *Let $\varphi$ be the function from* $R^3 - \mathrm{Cl}\, H_F$ *onto* $R^3 - \mathrm{Cl}\, H_G$ *such that if* $x \in (R^3 - \mathrm{Cl}\, H_F)$, $\varphi(x) = P_G^{-1}hP_F(x)$. *Then there is a homeomorphism $f$ from $R^3$ onto $R^3$ such that*

(1) *if* $x \in R^3 - \mathrm{Int}\, M$, $f(x) = \varphi(x)$ *and*

(2) $f[M] = P_G^{-1}hP_F[M]$, *and each component of $f[M]$ is a 3-cell-with-handles.*

*Proof.* Let $M_1, M_2, \cdots$, and $M_n$ be the components of $M$. If $i = 1, 2, \cdots$, or $n$, $\varphi \mid \mathrm{Bd}\, M_i$ is a homeomorphism, and thus $\varphi[\mathrm{Bd}\, M_i]$ is a compact tame 2-manifold-with-boundary and $\varphi[\mathrm{Bd}\, M_i]$ bounds a compact 3-manifold-with-boundary $N_i$ in $R^3$. Since $\bigcup_{i=1}^n \mathrm{Bd}\, N_i$ is the boundary of the connected 3-manifold-with-boundary $\varphi[R^3 - \mathrm{Int}\, M]$, the sets $N_1, N_2, \cdots$, and $N_n$ are mutually disjoint. Let $N$ denote $\bigcup_{i=1}^n N_i$. It is not hard to see that $N$ contains $\mathrm{Cl}\, H_G$, $\varphi$ takes $R^3 - \mathrm{Int}\, M$ homeomorphically onto $R^3 - \mathrm{Int}\, N$, and $\varphi[\mathrm{Bd}\, M] = \mathrm{Bd}\, N$. Therefore, in order to describe $f$, it is sufficient to construct, for each $i$, an extension of $\varphi \mid \mathrm{Bd}\, M_i$ to $M_i$.

Suppose then that $i = 1, 2, \cdots$, or $n$. Since $M_i$ is a 3-cell-with-handles, there is a finite set $\{D_{i1}, D_{i2}, \cdots, D_{im_i}\}$ of mutually disjoint polyhedral discs such that (1) if $j = 1, 2, \cdots,$ or $m_i$, $\text{Bd}\, D_{ij} \subset \text{Bd}\, M_i$ and $\text{Int}\, D_{ij} \subset \text{Int}\, M_i$ and (2) the closures of the components of

$$M - \bigcup_{i=1}^{m_i} D_{ij}, \qquad C_{i1}, C_{i2}, \cdots, C_{ik_i} ,$$

are polyhedral 3-cells such that if $t = 1, 2, \cdots,$ or $k_i$, $\text{Bd}\, C_{it}$ is the union of a punctured disc $A_{it}$ and certain ones of the discs $D_{i1}, D_{i2}, \cdots,$ and $D_{im_i}$ such that if $D_{ij}$ and $A_{it}$ intersect, $D_{ij} \cap A_{it} = \text{Bd}\, D_{ij}$ and is also a boundary curve of $A_{it}$. If $t = 1, 2, \cdots,$ or $k_i$, let $S_{it}$ denote $\text{Bd}\, C_{it}$.

If $j = 1, 2, \cdots,$ or $m_i$, there is a polyhedral subdisc $D'_{ij}$ of $D_{ij}$ such that $D'_{ij} \subset \text{Int}\, D_{ij}$ and $(\text{Cl}\, H_F) \cap D_{ij} \subset \text{Int}\, D'_{ij}$. Let $B_{ij}$ denote the annulus $D_{ij} - \text{Int}\, D'_{ij}$.

Now $P_G^{-1}hP_F[(\text{Cl}\, H_F) \cap \text{D}_{ij}]$ is compact and lies in $\text{Int}\, N_i$. Since each element of $G$ is semi-cellular, there exists a finite collection $\{(U_1, V_1), (U_2, V_2), \cdots, (U_r, V_r)\}$ of pairs of open sets in $R^3$ such that

(1) if $t = 1, 2, \cdots,$ or $r$, $V_t \subset U_t$, $U_t \subset \text{Int}\, N_i$, each simple closed curve in $V_t$ is null-homotopic in $U_t$, and $V_t$ is a union of elements of $G$, and

(2) each element of $G$ that intersects $P_G^{-1}hP_F[D_{ij}]$ lies in some one of $V_1, V_2, \cdots,$ and $V_r$.

There is a triangulation $T$ of $D'_{ij}$ such that if $\sigma$ is any 2-simplex of $T$, then for some $t$, $P_G^{-1}hP_F[\sigma] \subset V_t$. Let $\sigma_1, \sigma_2, \cdots,$ and $\sigma_q$ denote the 2-simplexes of $T$.

Let $\langle x_{10}x_{11}x_{12}\rangle$ denote the 2-simplex $\sigma_1$. Let $y_{10}, y_{11},$ and $y_{12}$ be points of $P_G^{-1}hP_F(x_{10})$, $P_G^{-1}hP_F(x_{11})$, and $P_G^{-1}hP_F(x_{12})$, respectively. Since $G$ is monotone, $P_G^{-1}hP_F[\langle x_{10}x_{11}\rangle]$ is a compact continuum and near it we can choose a polygonal arc $\langle y_{10}y_{11}\rangle$ such that if $\sigma$ is any 2-simplex of $T$ having $\langle x_{10}x_{11}\rangle$ as an edge and $P_G^{-1}hP_F[\langle x_{10}x_{11}\rangle] \subset V_s$, then $\langle y_{10}y_{11}\rangle \subset V_s$. It is to be true that if $\langle x_{10}x_{11}\rangle$ misses $\text{Cl}\, H_F$ then $\langle y_{10}y_{11}\rangle = P_G^{-1}hP_F[\langle x_{10}x_{11}\rangle]$. In a similar manner we choose polygonal arcs $\langle y_{11}y_{12}\rangle$ and $\langle y_{10}y_{12}\rangle$. We adjust these slightly near $\text{Cl}\, H_G$ so that if $\gamma_1 = \langle y_{10}y_{11}\rangle \cup \langle y_{11}y_{12}\rangle \cup \langle y_{10}y_{12}\rangle$, then $\gamma_1$ is a simple closed curve. Now for some $t_1$, $P_G^{-1}hP_F[\sigma_1] \subset V_{t_1}$ and by construction $\gamma_1 \subset V_{t_1}$. Hence there is a polygonal singular disc $\tau_1$ in $U_{t_1}$ and bounded by $\gamma_1$.

Corresponding to $\sigma_2$, we construct $\gamma_2$ and $\tau_2$ such that for some $t_1$, $\tau_2$ is a polyhedral singular disc in $U_{t_1}$. It is to be the case that if a vertex of $\sigma_2$ belongs to $\sigma_1$, we make the same choice for that vertex of $\sigma_2$ as was made for $\sigma_1$, and similarly if an edge of $\sigma_2$ lies in $\sigma_1$. In addition, if either a vertex or edge of $\sigma_2$ misses $\text{Cl}\, H_F$, then for the corresponding set in $\gamma_2$, we use its image under $\varphi$ and do not move it in adjusting to obtain $\gamma_2$.

Continue this process. There result polyhedral singular discs

$\tau_1, \tau_2, \cdots$, and $\tau_q$ in $\operatorname{Int} N_i$ such that $\bigcup_{t=1}^{q} \tau_t$ is a singular disc whose boundary is $\varphi[\operatorname{Bd} D'_{ij}]$ and which lies in $\operatorname{Int} N_i$. By applying Dehn's lemma [10] to the polyhedral singular disc $\varphi[B_{ij}] \cup (\bigcup_{t=1}^{q} \tau_t)$, we see that there is a disc $\Delta'_{ij}$ such that $\operatorname{Bd} \Delta'_{ij} = \varphi[\operatorname{Bd} D_{ij}]$ and $\operatorname{Int} \Delta'_{ij} \subset \operatorname{Int} N_i$.

By well-known techniques it may be shown that there exist mutually disjoint discs $\Delta_{i1}, \Delta_{i2}, \cdots$, and $\Delta_{im_1}$ such that for each $j$, $\operatorname{Bd} \Delta_{ij} = \varphi[\operatorname{Bd} D_{ij}]$ and $\operatorname{Int} \Delta_{ij} \subset \operatorname{Int} N_i$.

Recall that if $t = 1, 2, \cdots$, or $k_i$, $C_{it}$ is a 3-cell contained in $M_i$, $S_{it} = \operatorname{Bd} C_{it}$, and $A_{it}$ is the punctured disc $S_{it} - \bigcup_{j=1}^{m_i} \operatorname{Int} D_{ij}$. It is clear that if $D_{ij_1}, D_{ij_2}, \cdots$, and $D_{ij_{w_t}}$ are those discs of $D_{i1}, D_{i2}, \cdots$, and $D_{im_i}$ whose boundaries are contained in $A_{it}$, then $\varphi[A_{it}] \cup (\bigcup_{p=1}^{w_t} \Delta_{ij_p})$ is a tame 2-sphere $S'_{it}$.

We can easily show that if $s$ and $t$ are distinct, then $\operatorname{int} S'_{it}$ and $\operatorname{int} S'_{is}$ are disjoint, where "int" denotes the interior, in $E^3$, of a 2-sphere. Both $\operatorname{int} S'_{it}$ and $\operatorname{int} S'_{is}$ are contained in $\operatorname{Int} N_i$. If $S'_{it}$ and $\operatorname{int} S'_{it}$ intersect, then some point of either $\varphi[A_{it}]$ or $\varphi[A_{is}]$ lies in $\operatorname{Int} N_i$. This is a contradiction, so $\operatorname{int} S'_{it}$ and $\operatorname{int} S'_{is}$ are disjoint.

There is a homeomorphism $\theta_{i1}$ from $S_{i1}$ onto $S'_{i1}$ such that (1) $\theta_{i1} \mid A_{i1} = \varphi \mid A_{i1}$ and (2) if $D_{ij} \subset S_{i1}$, then $\theta_{i1}[D_{ij}] = \Delta_{ij}$. There is a homeomorphism $\theta_{i2}$ from $S_{i2}$ onto $S'_{i2}$ such that (1) $\theta_{i2} \mid A_{i2} = \varphi \mid A_{i2}$, (2) if $D_{ij} \subset S_{i1} \cap S_{i2}$, then $\theta_{i2} \mid D_{ij} = \theta_{i1} \mid D_{ij}$, and (3) if $D_{ij} \subset S_{i2}$, $\theta_{i2}[D_{ij}] = \Delta_{ij}$. If $t = 3, 4, \cdots$, or $k_i$, there is a homeomorphism $\theta_{it}$ from $S_{it}$ onto $S'_{it}$ such that (1) $\theta_{it} \mid A_{it} = \varphi \mid A_{it}$, (2) if $s = 1, 2, \cdots$, or $(t-1)$ and $\Delta_{ij} \subset S_{it} \cap S_{is}$, then $\theta_{it} \mid \Delta_{ij} = \theta_{ij} \mid \Delta_{ij}$, and (3) if $D_{ij} \subset S_{it}$, $\theta_{it}[D_{ij}] = \Delta_{ij}$.

If $t = 1, 2, \cdots$, or $k_i$, there is a homeomorphism $\theta^*_{it}$ from $C_{it}$ onto $(S'_{ij} \cup \operatorname{int} S'_{ij})$ such that $\theta^*_{it} \mid S_{it} = \theta_{it}$. Now let $\varphi_i$ be the function from $M_i$ onto $N_i$ defined as follows: If $x \in M_i$, let $t$ be an integer such that $x \in C_{it}$, and let $\varphi_i(x)$ be $\theta^*_{it}(x)$. The function $\varphi_i$ is well-defined because if $x \in C_{it} \cap C_{is}$, then $\theta^*_{it}(x) = \theta^*_{is}(x)$. It is easy to see that $\varphi_i$ is a homeomorphism from $M_i$ onto $N_i$ and that $\varphi_i \mid \operatorname{Bd} M_i = \varphi \mid \operatorname{Bd} M_i$.

Now we are ready to define $f$. If $x \in R^3 - \operatorname{Int} M$, then define $f(x)$ to be $\varphi(x)$. If $x \in M$, let $i$ be the integer such that $x \in M_i$. Then define $f(x)$ to be $\varphi_i(x)$. It is easily seen that $f$ is a homeomorphism from $R^3$ onto $R^3$ satisfying the conclusion of Lemma 1.

THEOREM 1. *Suppose that $F$ and $G$ are monotone decompositions of $E^3$ such that $\operatorname{Cl} P_F[H_F]$ and $\operatorname{Cl} P_G[H_G]$ are compact 0-dimensional sets. Suppose that $F$ is definable by 3-cells-with-handles $M_1, M_2, \cdots$ Suppose each element of $G$ is semi-cellular. Then if $F$ and $G$ are equivalent decompositions, there exists a sequence $f_1, f_2, f_3, \cdots$ of homeomorphisms from $R^3$ onto $R^3$ such that* (1) *for each*

$$i, f_{i+1} \mid (R^3 - \operatorname{Int} M_i) = f_i \mid (R^3 - \operatorname{Int} M_i)\ ,$$

*and* (2) $f_1[M_1], f_2[M_2], f_3[M_3], \cdots$ *is a defining sequence for* $G$.

*Proof.* Since $F$ and $G$ are equivalent, there is a homeomorphism $h$ from $R^3/F$ onto $R^3/G$ such that $h[\mathrm{Cl}\, P_F[H_F]] = \mathrm{Cl}\, P_G[H_G]$. Let $\varphi$ be the function from $R^3 - \mathrm{Cl}\, H_F$ onto $R^3 - \mathrm{Cl}\, H_G$ such that if

$$x \in (R^3 - \mathrm{Cl}\, H_F), \qquad \varphi(x) = P_G^{-1} h P_F(x) .$$

Since $F$ is definable by 3-cells-with-handles, there exists a defining sequence $M_1, M_2, M_3, \cdots$ for $F$ such that for each positive integer $i$, each component of $M_i$ is a 3-cell-with-handles. By Lemma 1, if $i$ is any positive integer, there is a homeomorphism $f_i$ from $R^3$ onto $R^3$ such that if $x \in E^3 - \mathrm{Int}\, M_i$, $f_i(x) = \varphi(x)$. We will show that the sequence $f_1, f_2, f_3, \cdots$ satisfies the conclusion of Theorem 1.

Suppose $i$ is any positive integer. Then $M_{i+1} \subset \mathrm{Int}\, M_i$ since $M_1, M_2, M_3, \cdots$ is a defining sequence for $H_F$. Since

$$f_{i+1} \mid R^3 - \mathrm{Int}\, M_{i+1}) = \varphi \mid (R^3 - \mathrm{Int}\, M_{i+1}) ,$$

then

$$f_{i+1} \mid (R^3 - \mathrm{Int}\, M_i) = \varphi \mid (R^3 - \mathrm{Int}\, M_i) .$$

Since $f_i \mid (R^3 - M_i) = \varphi \mid (R^3 - \mathrm{Int}\, M_i)$, it follows that

$$f_{i+1} \mid (R^3 - \mathrm{Int}\, M_i) = \quad f_i \mid (R^3 - \mathrm{Int}\, M_i) .$$

Suppose $U$ is an open set in $R^3$ containing $\mathrm{Cl}\, H_G$. Then $P_F^{-1} h^{-1} P_G[U]$ is open in $R^3$ and contains $\mathrm{Cl}\, H_F$. Hence there is a positive integer $n$ such that $M_n \subset P_F^{-1} h^{-1} P_G[U]$, and it follows that $P_G^{-1} h P_F[M_n] \subset U$. Since $f_n[M_n] = P_G^{-1} h P_F[M_n]$, $f_n[M_n] \subset U$. It is clear that for any $i$, $(\mathrm{Cl}\, H_G) \subset f_i[M_i]$. Consequently, $f_1[M_i], f_2[M_2], f_3[M_3], \cdots$ is a defining sequence for $G$. Hence Theorem 1 holds.

COROLLARY. 1. *If $F$ and $G$ satisfy the hypothesis of Theorem* 1, *then $G$ is definable by* 3-*cells-with-handles. If $F$ is toroidal, so is $G$.*

*Proof.* We use the notation of Theorem 1. By Theorem 1, $f_1[M_1], f_2[M_2], f_3[M_3], \cdots$ is a defining sequence for $G$. By Lemma 1, for each positive integer $i$, each component of $f_i[M_i]$ is a 3-cell-with-handles. Hence $G$ is definable by 3-cells-with-handles. It is clear that if for each positive integer $i$, $M_i$ is a solid torus, so is $f_i[M_i]$. Therefore, if $F$ is toroidal, so is $G$.

THEOREM 2. *Suppose that $F$ and $G$ are monotone decompositions of $R^3$ such that $\mathrm{Cl}\, P_F[H_F]$ and $\mathrm{Cl}\, P_G[H_G]$ are compact* 0-*dimensional sets. Suppose that $F$ has a defining sequence $M_1, M_2, M_3, \cdots$ and* *there exists a sequence* $f_1, f_2 f_3, \cdots$ *of homeomorphisms from* $R^3$ *onto* $R^3$ *such that* (1) *for each* $i$, $f_{i+1} | (R^3 - \operatorname{Int} M_i) = f_i | (R^3 - \operatorname{Int} M_i)$, *and* (2) $f_1[M_1], f_2[M_2], f_3[M_3], \cdots$ *is a defining sequence for* $G$. *Then* $F$ *and* $G$ *are equivalent.*

*Proof.* We shall define a homeomorphism $h$ from $R^3/F$ onto $R^3/G$ such that $h[\operatorname{Cl} P_F[H_F]] = \operatorname{Cl} P_G[H_G]$.

Suppose $x$ is a point of $R^3/F$. Consider first the case where $x \notin \operatorname{Cl} P_F[H_F]$. Then $P_F^{-1}(x)$ is a one-point set and so there is a point $y$ of $R^3$ such that $P_F(y) = x$. Further, $y \notin \operatorname{Cl} H_F$. Hence for some $n_y$, if $i > n_y$, $f_i(y) = f_{n_y}(y)$. Then define $h(x)$ to be the point $P_G f_{n_y}(y)$ of $R^3/G$.

Suppose $x \in \operatorname{Cl} P_F[H_F]$. Then there is a sequence $M_{1j_1}, M_{2j_2}, M_{3j_3}, \cdots$ such that for each $k$, $M_{kj_k}$ is the component of $M_k$ containing $P_F^{-1}(x)$. It is true, further, that $P_F^{-1}(x) = \bigcup_{k=1}^{\infty} M_{kj_k}$. Since $f_1[M_1], f_2[M_2], f_3[M_3], \cdots$ is a defining sequence for $H_G$, then $\bigcap_{k=1}^{\infty} f_k[M_{kj_k}]$ is an element $g_x$ of $G$. Define $h(x)$ to be the point $z$ of $R^3/G$ such that $P_G[g_x] = \{z\}$.

It is not hard to show, using the hypothesis, that $h$ is a homeomorphism from $R^3/F$ onto $R^3/G$ such that $h[\operatorname{Cl} P_F[H_F]] = \operatorname{Cl} P_G[H_G]$.

THEOREM 3. *Suppose* $F$ *and* $G$ *are monotone decompositions of* $R^3$ *such that* $\operatorname{Cl} P_F[H_F]$ *and* $\operatorname{Cl} P_G[H_G]$ *are compact* 0*-dimensional sets. Suppose* $F$ *is definable by* 3*-cells-with-handles and each element of* $G$ *is semi-cellular. Then* $F$ *and* $G$ *are equivalent if and only if there exists a defining sequence* $M_1, M_2, \cdots$ *for* $F$ *and a sequence* $f_1, f_2, f_3, \cdots$ *of homeomorphisms from* $R^3$ *onto* $R^3$ *such that* (1) *for each* $i$,

$$f_{i+1} | (R^3 - \operatorname{Int} M_i) = f_i | (R^3 - \operatorname{Int} M_i) \text{ and } (2)\ f_1[M_1], f_2[M_2], f_3[M_3], \cdots$$

*is a defining sequence for* $G$.

Theorem 3 is a corollary of Theorems 1 and 2.

We shall indicate now some conditions under which a monotone decomposition $F$ of $R^3$ satisfies the hypothesis of Theorem 3 for $F$.

LEMMA 2. *Suppose that* $F$ *is a monotone decomposition of* $R^3$ *such that* $\operatorname{Cl} P_F[H_F]$ *is a compact* 0*-dimensional set. Then* $F$ *is definable by* 3*-cells-with-handles provided it is true that if* $g$ *is any element of* $F$, $g_0$ *is any subcontinuum of* $g$ *embeddable in* $R^2$, *and* $h$ *is any embedding of* $g_0$ *in* $R^2$, *then* $h[g_0]$ *does not separate* $R^2$. *In particular, the condition stated holds provided* $g$ *satisfies any one of the following*:

(1) $g$ *is tree-chainable* (see [3] for definition).

(2) *g is snake-like* (see [3] for definition).
(3) *g is a dendron.*
(4) *g is an arc.*

Lemma 2 may be established by the methods of [2].

4. **The dogbone space.** In this section it is proved that there is a decomposition $F$ which is equivalent to the dogbone decomposition and such that $F$ has only countably many nondegenerate elements. The notation and terminology of [5] will be used in this section.

LEMMA 3. *Suppose $f$ is a homeomorphism of $A$ into $R^3$, $B = f[A]$, $B_i = f[A_i]$, $P_1, P_2, P_3, \cdots$, and $P_m$ are disjoint horizontal planes in $R^3$, there exist positive integers $j$ and $k$ such that $1 \leqq j \leqq k \leqq m$ and $B$ intersects only $P_j, P_{j+1}, \cdots$, and $P_k$, and for each positive integer $i$, $1 \leqq i \leqq m$, each component of $B \cap P_i$ is a tame disc, $B \cap (\bigcup_{s<i} P_s)$ is contained in some component of $B - P_i$, $B \cap (\bigcup_{s>i} P_s)$ is contained in some component of $B - P_i$, and $B \cap P_i$ is contained in some component of $B - \bigcup_{s \neq i} P_s$. Then there exists a homeomorphims $h$ of $R^3$ onto itself such that* (1) *$h$ is point-wise fixed outside of $B$,* (2) *$h[B_1]$ intersects $P_j, P_{j+1}, \cdots$, and $P_k$,* (3) *each of $h[B_2], h[B_3]$, and $h[B_4]$ intersects at most $k - j$ of the $P_i's$, and* (4) *for $i = 1, 2, 3$, or $4$, $h[B_i] \cap (\bigcup_{t=1}^m P_t)$ has the same properties as $B \cap (\bigcup_{t=1}^m P_t)$.*

*Proof.* Adjust $\bigcup_{i=1}^4 f^{-1}[B_i] = \bigcup_{i=1}^4 A_i$ by a homeomorphism $g$ of $A$ onto itself such that $g$ is fixed on the boundary and $g$ carries $\bigcup_{i=1}^4 A_i$ to the positions indicated in Figure 2. Let $h$ be $fgf^{-1}$. It can be assumed that $h[B_i]$ has small cross sectional diameter, $\bigcup_{i=1}^4 h[B_i]$

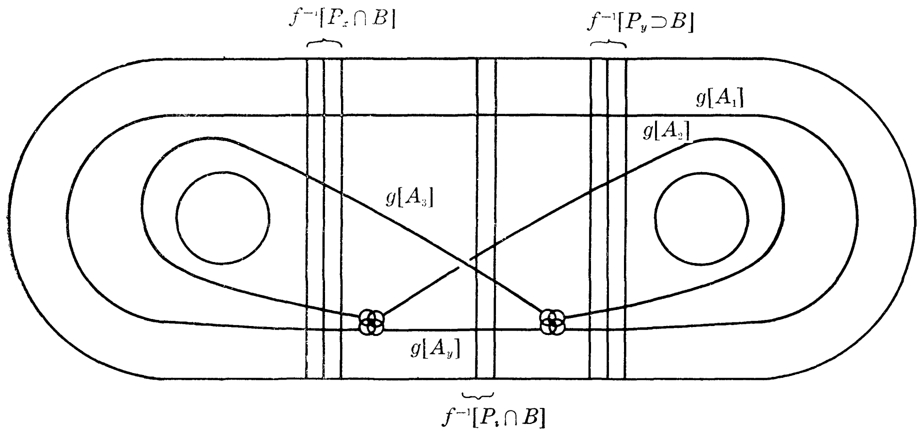


FIGURE 2.

and $\bigcup_{i=1}^{m} P_i$ are in relative general position, and each component of $(\bigcup_{i=1}^{4} h[B_i]) \cap (\bigcup_{i=1}^{m} P_i)$ is a disc.

We will now construct the decomposition $F$. Let $P_1, P_2, \cdots$, and $P_r$ be horizontal planes which intersect $A$ as shown in Figure 3. Apply Lemma 3 to $A$ and $P_1, P_2, \cdots$, and $P_r$ to obtain a homeomorphism $h_1$, and let $B_i = h_1[A_i]$. See Figure 3. Apply Lemma 3 to $B_i$ and $P_1, \cdots$, and $P_r$ to obtain a homeomorphism $h_2^i$. Let $h_2 = h_2^1 h_2^2 h_2^3 h_2^4 h_1$ and let $B_{ij} = h_2[A_{ij}]$. This process is continued until $B_{ij\cdots m}$ intersects at most one of the $P_i's$. When $B_{ij\cdots m}$ intersects only $P_s$, then discs are added to the collection of discs $B_{ij\cdots m} \cap P_s$ so that the total collection cuts $B_{ij\cdots m}$ in the same manner as $P_1, \cdots$, and $P_r$ cut up $A$, and so that each component of $B_{ij\cdots m}$ in the complement of the collection of discs has diameter less than one half the diameter of $B$. A modified version of Lemma 3 is now applied to $B_{ij\cdots m}$ and the collection of disjoint discs.

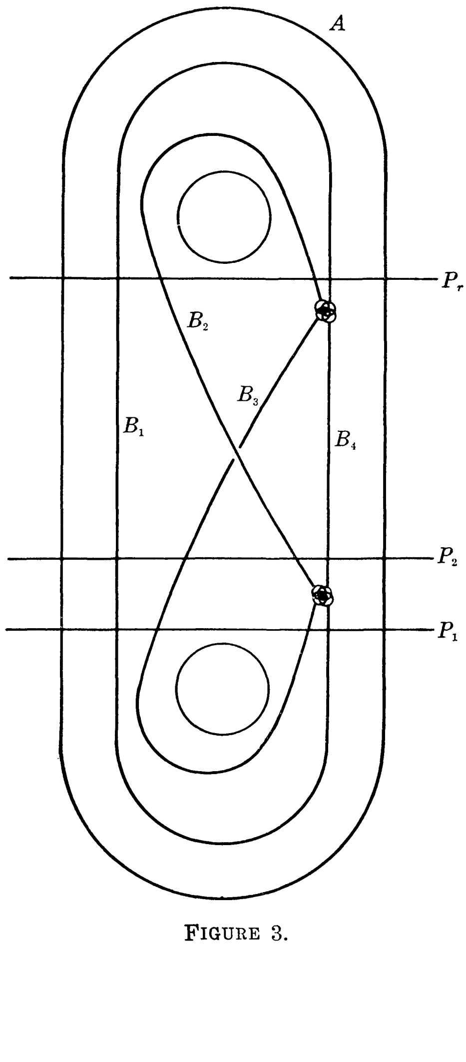


FIGURE 3.

$F$ is the decomposition whose nondegenerate elements are the nondegenerate components of $A \cap (\cap B_i) \cap (\cup B_{ij}) \cap (\cup B_{ijk}) \cup \cdots$. It is clear, using Theorem 2, that $F$ is equivalent to the dogbone decomposition.

THEOREM 4. *$F$ has only countably many nondegenerate elements.*

*Proof.* There is a one to one correspondence between the components of $A \cap (\cup B_i) \cap (\cap B_{ij}) \cap \cdots$ and the set of all sequences into $\{1, 2, 3, 4\}$, where the sequence $t$ corresponds to $A \cap B_{t(1)} \cap B_{t(1)t(2)} \cap \cdots$. It will next be shown that $f$ is a nondegenerate element of $F$ if and only if the sequence corresponding to $f$ converges to 1.

Suppose $t$ is a sequence into $\{1, 2, 3, 4\}$, $t$ converges to 1, and $f$ corresponds to $t$. Then there exist disjoint discs $E_1$ and $E_2$ and an

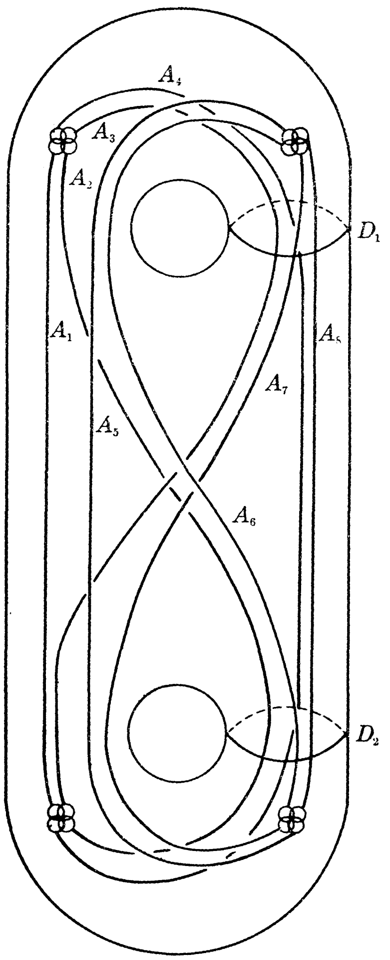


FIGURE 5.

integer $m$ such that if $n \geqq m$, then $B_{t(1)t(2)\cdots t(n)}$ intersects $E_1$ and $E_2$. Hence $f$ is a nondegenerate element.

Suppose $t$ is a sequence into $\{1, 2, 3, 4\}$ which does not converge to 1. Let $\{q_i\}$ be an increasing sequence such that for each $i$, $t(q_i) \neq 1$. Then $B_{t(1)t(2)\cdots t(q_r)}$ intersects at most one of $P_1, P_2, \cdots$, and $P_r$. For some $n$, $B_{t(1)\cdots t(q_n)}$ intersects at most one of the discs used to define the homeomorphism $h_{t(q_r)+1}$. Hence $\lim_{n\to\infty}(\text{diam } B_{t(1)\cdots t(n)})$ is zero.

EXAMPLE 6. There exists a point-like decomposition $F$ such that if $K$ is any point-like decomposition equivalent to $F$, then $K$ has uncountably many nondegenerate elements. Let $A$ be a solid double torus and let $A_1, A_2, \cdots, A_7$, and $A_8$ be solid double tori embedded in $A$ as shown in Figure 4. Inside each of the $A_i$'s eight double tori are embedded like the $A_i$'s are in $A$, etc. Suppose $K$ is equivalent to $F$ and let $A'_{ij\cdots m}$ correspond to $A_{ij\cdots m}$. Let $D'_1$ and $D'_2$ be disjoint discs in $A'$ which are embedded in $A'$ in the same manner as $D_1$ and $D_2$ are embedded in $A$. See Figure 4. It follows from the arguments in [5] that two of the $A'_i$'s intersect both $D'_i$ and $D'_2$, and inside each of those two of the $A_{ij}$'s interset both $D'_1$ and $D'_2$, etc. It follows that $K$ has uncountably many nondegenerate elements.

**5. A decomposition not equivalent to the dogbone.** In this section $G$ will denote the point-like decomposition of $R^3$ described by Bing in [7], and the notation and terminology of that paper will be used. It will be proved that any point-like decomposition equivalent to $G$ has at least one nondegenerate element which is not locally connected. Let $T_0$ denote a round solid torus in $R^3$. Let $T_{00}$ and $T_{01}$ be disjoint solid tori embedded in the interior of $T_0$ as shown in Figure 5. Inside each $T_{0i}$ two tori are embedded, etc. $G$ is the decomposition of $R^3$ whose nondegenerate elements are the nondegenerate components of $T_0 \cap (\cup T_{0i}) \cap (\cup T_{0ij}) \cap \cdots$. $G$ has countably many nondegenerate elements, each of which is indecomposable.

*Property P.* Suppose $T$ is a solid torus. A disc $D$ has Property $P$ with respect to $T$ if and only if $D$ is a polyhedral disc in general position with respect to $T$ and Bd $D$ is a simple closed curve on Bd $T$ which circles Bd $T$ meridianally.

*Property A.* A collection of sets $\{T, D_1, \cdots, D_n\}$ has Property $A$ if and only if (1) $T$ is a solid torus, (2) for $1 \leqq i \leqq n$, $D_i$ is a disc which has Property $P$ with respect to $T$ and no proper subdisc of $D_i$ has Property $P$ with respect to $T$, (3) if $i \neq j$ then $D_i$ and $D_j$ are disjoint, and (4) if $C$ is a longitudinal curve on Bd $T$ which intersects

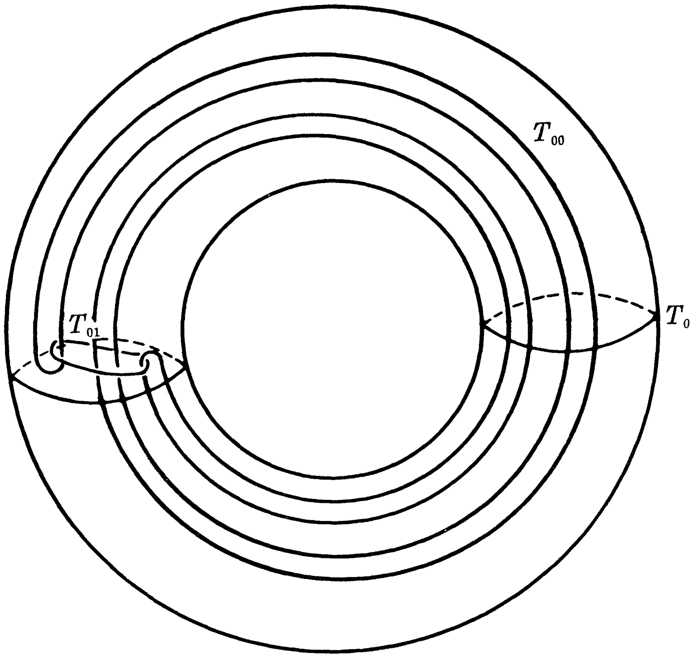


FIGURE 5.

each $\mathrm{Bd}\, D_i$ in a single point $q_i$, then the ordering of the $q_i$'s on $C$ is $q_1 q_2 \cdots q_n q_1$.

Suppose $\{T, D_1, \cdots, D_n\}$ has Property $A$. A collection $(Q_1, \cdots, Q_n\}$ is a division of $T$ determined by $\{T, D_1, \cdots, D_n\}$ if and only if for each $i$, $1 \leqq i < n$, if $D_i'$ denotes the component of $D_i - (R^3 - T)$ which contains $\mathrm{Bd}\, D_i$, for $1 \leqq i < n$, $Q_i'$ denotes the component of $T - (\cup D_i')$ whose closure intersects both $D_i'$ and $D'_{(i+1)}$, $Q_i = \bar{Q}_i'$, and $Q_n$ is the closure of the component of $T - (D_1' \cup D_n')$ which is disjoint from $Q_1'$.

LEMMA 4. *If* $\{T_0, D_1, \cdots, D_n\}$ *has Property* $A$ *and* $\{Q_1, \cdots, Q_n\}$ *is a division of* $T_0$ *determined by* $\{T_0, D_1, \cdots, D_n\}$, *then there exist an integer* $i$ *and discs* $E_1, E_2, \cdots,$ *and* $E_m$ *such that* (1) $i = 0$ *or* 1, (2) $\{T_{0i}, E_1, \cdots, E_m\}$ *has Property* $A$, (3) *if* $\{R_1, \cdots, R_m\}$ *is the division of* $T_{0i}$ *determined by* $\{T_{0i}, E_1, \cdots, E_m\}$, *then there exists integers* $i_1, i_2, \cdots,$ *and* $i_{2n}$, *such that* $1 \leqq i_1 < i_2 < \cdots < i_{2n} \leqq m$ *and for some*

$$t,\ 1 \leqq t \leqq n,\ R_{i_1} \subset Q_t,\ R_{i_2} \subset Q_{t-1},\ \cdots,\ R_{i_t} \subset Q_1,\ R_{i_{t+1}} \subset Q_n,\ \cdots,\ R_{i_n} \subset Q_{t+1},\ R_{i_{n+1}} \subset Q_{t+1},\ \cdots,\ R_{i_{2n}} \subset Q_t\ ,$$

*and* (4) *if* $R_{i_k} \subset Q_j$, *then* $E_{i_k}$ *is contained in one of* $D_j$ *and* $D_{j+1}$ *and* $E_{i_k+1}$ *is contained in the other.*

*Proof.* Consider the universal covering space for $T_0$. It is represented by Figure 6 where it appears that $T_0$ has been rolled out onto a cylinder. It follows from the proof of [7, Th. 5] that for some $k$, either each center for $T_{00}$ intersects two adjacent copies of $D_k$ in the universal covering space, or each center for $T_{01}$ intersects two adjacent copies of $D_k$ in the universal covering space. Assume each center for $T_{00}$ intersects two adjacent copies of $D_k$ in the universal covering space and let $i = 0$. Let $C$ be a center for $T_{00}$ such that $C \cap (\cup D_j)$ is a finite set, and if $C'$ is a center for $T_{00}$, then $C' \cap (\cup D_j)$ contains at least as many elements as $C \cap (\cup D_j)$. This last condition implies that if $r \in C \cap D_j$, then there is a subdisk $E$ of $D_j$ which has Property $P$ with respect to $T_{00}$ and $E \cap C = r$. It can be assumed without loss of generality that $T_{00}$ is polyhedral and Bd $T_{00}$ and $\cup D_j$ are in relative general position.

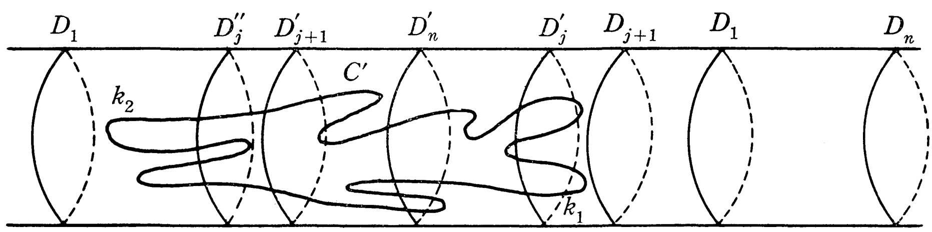


FIGURE 6.

Let $C'$ denote one of the copies of $C$ in the universal covering space as shown in Figure 6. Assume that one of the copies of $D_j$, say $D_j'$, is the rightmost one of the copies of the $D_k$'s that intersect $C'$. Let $D_j''$ be the first copy of $D_j$ to the left of $D_j'$ and let $D_k'$ be the first copy of $D_k$ to the right of $D_j''$. Let $t$ be $j - 1$ if $2k \leqq j \leqq n$ or $t$ be $n$ if $j = 1$. Let $k_1$ be a point in $C'$ to the right of $D_j'$ and let $k_2$ be a point in $C'$ to the left of $D_j''$. Let $A$ be an arc in $C'$ from $k_1$ to $k_2$ and $B$ be the arc in $C'$ from $k_2$ to $k_1$ which intersects $A$ only in the end points. Let $r_2'$ be the first point of $A$ in $D_{j-1}'$ and let $r_1'$ be the last point of $A \cap D_j'$ preceding $r_2'$. Let $r_4'$ be the first point of $A$ in $D_{j-2}'$ and let $r_3'$ be the last point of $A \cap D_{j-1}'$ preceding $r_4'$. Continue this procedure to obtain points $r_5', r_6', \cdots, r_{2n-1}'$, and $r_{2n}'$. Let $r_{2n+2}'$ be the first point of $B$ in $D_{j+1}'$ and let $r_{2n+1}'$ be the last point of $B \cap D_j''$ preceding $r_{2n+2}'$. Continue this to get $r_1', r_2', \cdots, r_{4n-1}'$ and $r_{4n}'$. Let $r_i$ be the point in $C$ corresponding to $r_i'$.

The $r_i$'s have the ordering $r_1 r_2 \cdots r_{4n} r_1$ on $C$, and determine disks $E_1, \cdots,$ and $E_{4n}$ on $T_{00}$. It can be assumed that each of the $E_i$'s is a subdisk of $\cup D_k$, each has property $P$ with respect to $T_{00}$, no proper subdisk of $E_i$ has property $P$ with respect to $T_{00}$, if $r_i = r_{i+1}$ then $E_i = E_{i+1}$, otherwise the $E_i$'s form a disjoint collection, and finally

$E_j \cap C = r_j$. If the collection $\{E_1, \cdots, E_{4n}\}$ is reindexed to give a disjoint collection $\{E_1, \cdots, E_m\}$ then clearly $m \geqq 2n$ and there exist integers $i_1, \cdots$, and $i_{2n}$ which satisfy the conclusion of the lemma.

THEOREM 5. *If $F$ is a point-like decomposition equivalent to $G$, then some nondegenerate element of $F$ is not locally connected.*

*Proof.* By Theorem 1 and Corollary 1, $F$ is a toroidal decomposition of $R^3$ and there exists a sequence of homeomorphisms $\{h_i\}_{i=0}^{\infty}$ of homeomorphisms such that $h_i$ is from $R^3$ onto $R^3$, if $j > k$, then $h_j \mid R^3 - \cup T_{0i_1\cdots i_k} = h_k$, and the nondegenerate elements of $F$ are the nondegenerate components of $h_0[T_0] \cap h_1[\cup T_{0i}] \cap \cdots$.

Let $D_1$ and $D_2$ be disjoint discs, each of which has Property $P$ with respect to $h_0[T_0]$, and such that no proper subdisc of either $D_1$ or $D_2$ has property $P$ with respect to $h_0[T_0]$. Then $h_1^{-1}[D_1]$ and $h_1^{-1}[D_2]$ are discs, each of which has Property $P$ with respect to $T_0$, and no proper subdisc of either has Property $P$ with respect to $T_0$. Let $R_1$ and $R_2$ be the division of $T_0$ determined by $\{T_0, h_1^{-1}[D_1], h_1^{-1}[D_2]\}$.

By Lemma 4, there exist an integer $t_1$ in $\{0,1\}$ and disks $E_{11}, E_{12}, \cdots$, and $E_{1m(1)}$ such that, $\{T_{0t_1}, E_{11}, \cdots, E_{1m(1)}\}$ has Property $A$, and if $\{R_{11}, \cdots, R_{1m(1)}\}$ is a division of $T_{0t_1}$ determined by $\{T_{0t_1}, E_{11}, \cdots, E_{1m(1)}\}$, then there exist integers $j_{11}$ and $j_{12}$, $j_{11} < j_{12}$, such that, $R_{1j_{11}}$ and $R_{1j_{12}}$ are contained in $R_1$, $E_{1j_1}$ and $E_{1j_2}$ are contained in one of $h^{-1}[D_1]$ and $h_1^{-1}[D_2]$, and $E_{1(j_1+1)}$ and $E_{1(j_2+1) \bmod m(1)}$ are contained in the other. Then $\{T_{0t_1}, h_2^{-1}h_1[E_{11}], \cdots, h_2^{-1}h_1[E_{1m(1)}]\}$ has Property $A$, and by applying Lemma 4 again, there exist an integer $t_2$ in $\{0, 1\}$ and discs $E_{21}, \cdots$, and $E_{2m(2)}$ such that $\{T_{0t_1t_2}, E_{21}, \cdots, E_{2m(2)}\}$ has Property $A$, and if $\{R_{21}, \cdots, R_{2m(2)}\}$ is a division determined by $\{T_{0t_1t_2}, E_{21}, \cdots, E_{2m(2)}\}$, then there exist integers $j_{21} < j_{22} < j_{23} < j_{24}$ such that

$$R_{2j_{21}} \subset R_{1j_{11}}, R_{2j_{22}} \subset R_{2j_{12}}, R_{2j_{23}} \subset R_{1j_{12}}, \quad \text{and} \quad R_{2j_{24}} \subset R_{1j_{11}} .$$

Continuing this process by induction it follows that

$$(h_0[T_0] \cap h_1[T_{0t_1}] \cap h_2[T_{0t_1t_2}] \cap \cdots) - (D_1 \cup D_2)$$

has an infinite number of components, each of which intersects both $D_1$ and $D_2$, and hence is not locally connected.

In fact, countably many of the nondegenerate elements fail to be locally connected. To see this let $v_i$ denote $(t_i + 1) \bmod 2$. Let $D_{21}$ and $D_{22}$ be disjoint discs which have Property $P$ with respect to $h_1[T_{0v_1}]$ and repeat the above argument. Similarly for each of $h_2[T_{0t_1}]$, $h_3[T_{0t_1t_2v_3}]$, $h_4[T_{0t_1t_2t_3v_4}]$, etc.

COROLLARY 2. *There does not exist a point-like decomposition*

*$F$ equivalent to $G$ such that each nondegenerate element of $F$ is an arc.*

COROLLARY 3. *The decomposition $G$ is not equivalent to the dogbone decomposition.*

EXAMPLE 7. There does exist a decomposition $F$ equivalent to $G$ such that some nondegenerate element of $F$ is an arc.

Construction of $F$. Let $T_{00}$ and $T_{01}$ be embedded in $T_0$ as shown in Figure 7. If this pattern is used at each stage, then $T_{01} \cap T_{011} \cap T_{0111} \cap \cdots$ is an arc.

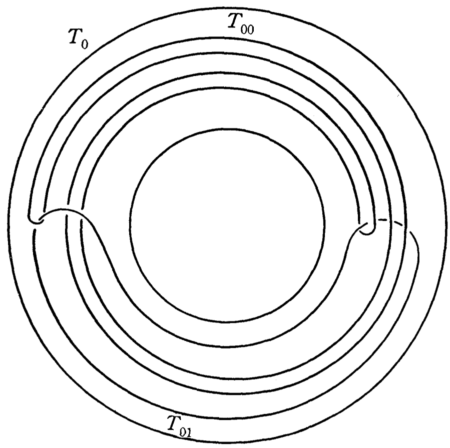


FIGURE 7.

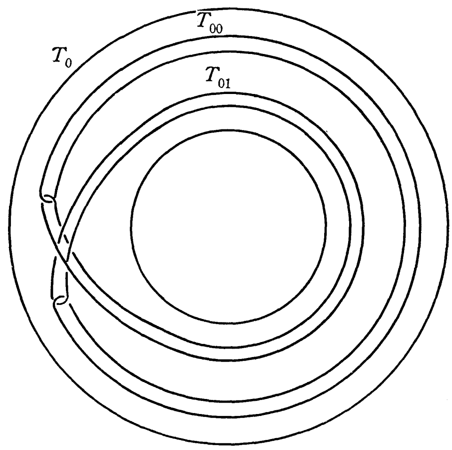


FIGURE 8.

EXAMPLE 8. There exists a decomposition $F$ equivalent to $G$ such that $F$ has uncountably many nondegenerate elements and each is an indecomposable continuum.

Construction of $F$. Let $T_{00}$ and $T_{01}$ be embedded in $T_0$ as shown in Figure 8. This pattern is used at each stage.

**6. Tamely finnable 3-cells.** In this section we show that if a 3-cell $C$ in $R^3$ is tamely finnable, then there is a 3-cell $C'$ in $R^3$ with a flat spot on its boundary such that the decomposition of $R^3$ whose only nondegenerate element is $C$ is equivalent to the decomposition of $R^3$ whose only nondegenerate element is $C'$. A 3-cell $C$ in $R^3$ is *tamely finnable* if and only if there exists a tame disc $D$ in $R^3$ such that $D \cap C$ is an arc $\alpha$ and $\alpha \subset (\text{Bd } D) \cap (\text{Bd } C)$. The statement that Bd $C$ has a *flat spot* means that Bd $C$ contains a polyhedral disc. We begin by describing several sets and functions which will be used in the proof.

Let $R$ be the 3-cell $\{(x, y, z) : |x| \leqq 1, |y| \leqq 2, |z| \leqq 1\}$, $R^+$ be $(R \cap \{(x, y, z) : y \geqq 0\})$, and $R^-$ be $(R \cap \{(x, y, z) : y \leqq 0\})$. For each subset $X$ of $R$, let $X^+$ denote $X \cap R^+$ and $X^-$ denote $X \cap R^-$.

If $P$ and $Q$ are points in $R^3$, let $[P, Q]$ denote the straight line interval from $P$ to $Q$ Let $D_1$ be $\{(x, y, z) : x^2 + y^2 \leqq 1\}$, $D_2$ be

$$\bigcup \{[(x, 0, 0), (x, y, 1)] : x^2 + y^2 = 1\} ,$$

$D_3$ be

$$\bigcup \{[(x, y, 0), (x, y, 1)] : x^2 + y^2 = 1\} ,$$

$D_4$ be

$$\bigcup \{[(x, y, 0), (x, 0, -1)] : x^2 + y^2 = 1\} ,$$

and $D^5$ be $\{(x, y, 0) : x^2 + y^2 \leqq 1\}$.

Let $K$ be the 3-cell bounded by $D_1 \cup D_2$, $L$ be $\mathrm{Cl}\,(R - K)$, $M$ be the 3-cell bounded by $D_1 \cup D_3 \cup D_4$, and $N$ be $\mathrm{Cl}\,(R - M)$; see Figure 9.

Let $g_1$ be a homeomorphism of $L^+$ onto $N^+$ such that $g_1$ is fixed on $\mathrm{Bd}\, L^+ \cap \mathrm{Bd}\, R$, $g_1[\{(x, 0, 0) : -1 \leqq x \leqq 1\}]$ is $\{(x, y, 0) : x^2 + y^2 = 1, y \geqq 0\}$ and $g_1$ moves points only along lines parallel to the $y$-axis. Let $g_2$ be a homeomorphism of $L^-$ onto $N^-$ such that $g_2$ is fixed on

$$\mathrm{Bd}\, L^- \cap \mathrm{Bd}\, R, \qquad g_2[\{(x, 0, 0) : -1 \leqq x \leqq 1\}]$$

is $\{(x, y, 0) : x^2 + y^2 = 1, y \leqq 0\}$ and $g_2$ moves points only along lines parallel to the $y$-axis.

Let $M'$ be $M \cap \{(x, y, z) : z \geqq 0\}$ and let $g_3$ be a continuous function from $M'$ onto $K$ such that $g_3[D_5] = \{(x, 0, 0) : -1 \leqq x \leqq 1\}$, $g_3$ is the identity on $D_1$, $g_3 \mid D_3^+ = g_1^{-1}$, $g_3 \mid D_3^- = g_2^{-1}$, and $g_3$ is a homeomorphism on $(M' - D_5)$

THEOREM 6. *Let $C$ be a 3-cell in $R^3$ such that $C$ is tamely finnable. Then there exists a 3-cell $C'$ in $R^3$ such that $C'$ has a flat spot and the decomposition of $R^3$ whose only nondegenerate element is $C$ is equivalent to the decomposition of $R^3$ whose only nondegenerate element is $C'$.*

*Proof.* Let $C$ be the 3-cell and $D$ be a tame disc such that $D \cap C$ is an arc $\alpha$ lying on $\mathrm{Bd}\, D \cap \mathrm{Bd}\, C$. There exists a homeomorphism $h$ of $R^3$ onto itself such that (1) $h[\alpha] = \{(x, 0, 0) : -1 \leqq x \leqq 1\}$, (2) $h[D] = \{(x, 0, z) : |x| \leqq 1, 0 \leqq z \leqq 1\}$, and (3) $h[\mathrm{Bd}\, C - \alpha]$ and $K \cup (R^+ \cap R^-)$ are disjoint.

Let $F$ be a homeomorphism from $R - (K \cup (R^+ \cap R^-))$ onto $R - M$

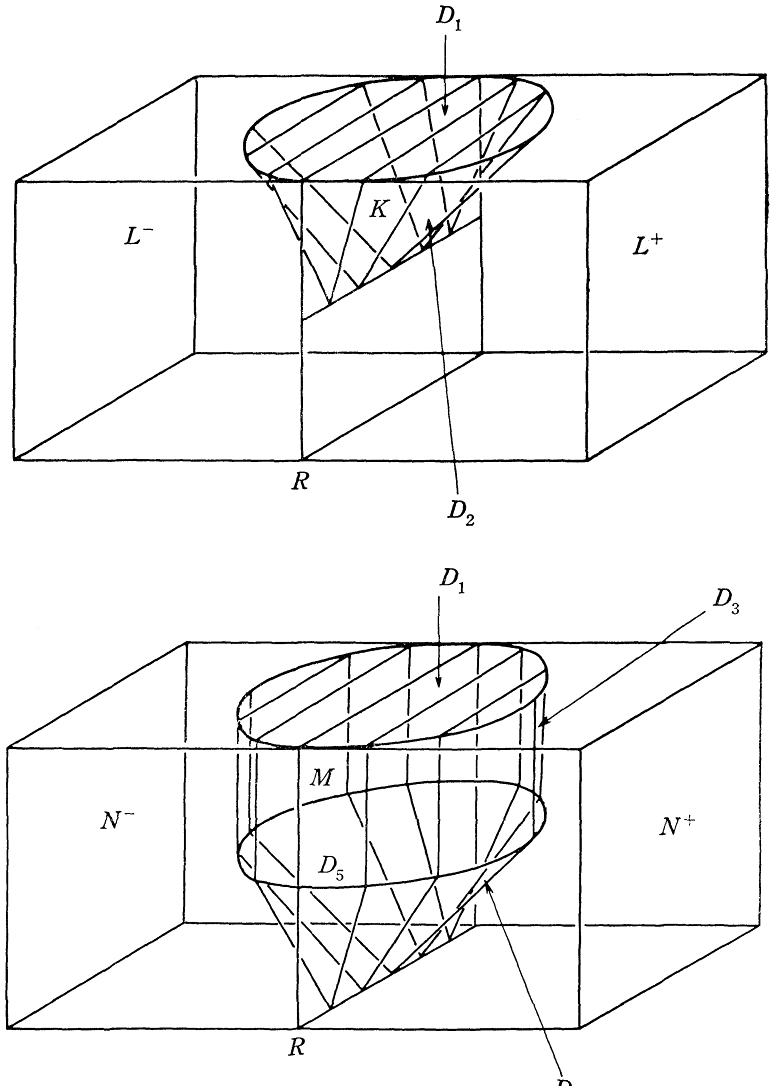


FIGURE 9.

such that if $x \in [R-(K \cup (R^+ \cap R^-))]^+$, $F(x) = g_1(x)$, and if $x \in [R-(KU(R^+ \cap R^-))]^-$, $F(x)=g_2(x)$. Extend $F$ to $R^3-R$ such that if $x \in (R^3-R)$, $F(x)=x$. Let $S$ be $D_5 \cup Fh[\mathrm{Bd}\, C-\alpha]$.

It is easily seen that $S$ is a 2-sphere in $R^3$ which bounds a 3-cell $C'$ and $\mathrm{Bd}\, C'$ has a flat spot, the disk $D_5$. It remains to show that the decomposition of $R^3$ corresponding to $C$ and $C'$ respectively are equivalent.

We will define a function $\Phi$ from $R^3$ onto itself such that $|\Phi[C'] = h[C]$ and $\Phi \mid \mathrm{Ext}\, C'$ is a homeomorphism. If $P \in R^3-M$, $\Phi(P)=F^{-1}(P)$. If $P \in M'$, $\Phi(P)=g_3(P)$. If $P \in M \cap \{(x, y, z)\colon z \leqq 0\}$ and $P=(x, y, z)$, let $\Phi(P)$

be $(x, 0, z)$.

Now the function $h^{-1}\Phi$ is a continuous function from $R^3$ onto itself which maps $C'$ onto $C$ and is a homeomorphism outside $C'$. It follows that the corresponding decompositions are equivalent.

COROLLARY 4. *If $C$ is a 3-cell in $R^3$ and $C$ is tamely finnable, then there exists a disc $D$ in $R^3$ such that the decomposition of $R^3$ whose only nondegenerate element is $C$ is equivalent to the decomposition of $R^3$ whose only nongenerate element is $D$.*

*Proof.* This follows from Theorem 3 of [10].

The statement that $K$ is a *crumpled cube* means that $K$ is homeomorphic to the union of a 2-sphere and its interior in $R^3$.

THEOREM 7. *If $K$ is a crumpled cube in $R^3$, there exists a 3-cell $C$ in $R^3$ such that the decomposition of $R^3$ whose only nondegenerate element is $K$ is equivalent to the decomposition of $R^3$ whose only nondegenerate element is $C$.*

*Proof.* Apply Theorem 2 of [8].

**7. Improving elements of decompositions.** Suppose $K$ is a 3-*cell-with-n-handles* in $R^3$ and $C, C_1, C_2, C_3, \cdots$, and $C_n$ is a standard decomposition of $K$. If $i$ is a positive integer less than or equal to $n$, let $D_{i,1}$ and $D_{i,2}$ be the two components of $C \cap C_i$. Let $p$ be an element of $\operatorname{Int} C$, and if $i$ and $j$ are integers, $1 \leqq i \leqq n$, $1 \leqq j \leqq 2$, let $p_{i,j}$ be an element of $\operatorname{Int} D_{i,j}$. Let $T$ be $\{(x, y, z) \in R^3: x^2 + y^2 \leqq 1, |z| \leqq 1\}$. If $i$ is a positive integer less than or equal to $n$, there is a homeomorphism $f_i$ of $C_i$ onto $T$ such that $f_i[D_{i,1}] = \{(x, y, z): x^2 + y^2 \leqq 1, z = 1\}$, $f_i[D_{i,2}] = \{(x, y, z): x^2 + y^2 \leqq 1, z = -1\}$, $f_i(p_{i,1}) = (0, 0, 1)$ and $f_i(p_{i,2}) = (0, 0, -1)$. Let $\alpha_i$ be $f_i^{-1}[\{(0, 0, z): |z| \leqq 1\}]$.

Let $f$ be a homeomorphism of $C$ onto the unit ball $\{(x, y, z): x^2 + y^2 + z^2 \leqq 1\}$ such that $f(p) = (0, 0, 0)$. If $i$ and $j$ are integers, $1 \leqq i \leqq n$, $1 \leqq j \leqq 2$, let $b_{ij}$ be the straight line interval from $f(p_{i,j})$ to $f(p)$, and let $\beta_{i,j}$ be $f^{-1}[b_{i,j}]$. Let $S$ be $(\bigcup_{i=1}^{n} \alpha_i) \cup (\bigcup_{i=1}^{n} \alpha_{j=1}^{2} \beta_{i,j})$. We will call $S$ a *special spine* of $K$.

A *partition* of $K$ is a finite collection $\mathscr{P}$ of subsets of $K$ such that (1) if $Q \in \mathscr{P}$, $Q$ is a 3-cell, (2) if $Q \in \mathscr{P}$ and $Q \subset C$, then $Q = C$, (3) if $Q \in \mathscr{P}$ and there is a positive integer $i$ less than or equal to $n$ such that $Q \subset C_i$, then there exist real numbers $a$ and $b$ such that $-1 \leqq a \leqq b \leqq 1$ and $f_i[Q] = \{(x, y, z): a \leqq z \leqq b\} \cap T$, (4) if $Q_1 \in \mathscr{P}$, $Q_2 \in \mathscr{P}$, $Q_1 \neq Q_2$, and $Q_1 \cap Q_2 \neq \emptyset$, then $Q_1 \cap Q_2$ is a disc on $\operatorname{Bd} Q_1 \cap \operatorname{Bd} Q_2$, and (5) $\cup \{Q: Q \in \mathscr{P}\} = K$.

If $K$ is a polyhedral cell with handles in $R^3$, $S$ is a special spine of $K$, and $\varepsilon > 0$, there is clearly a homeomorphism $h$ of $R^3$ onto itself and a partition $\mathscr{P}$ of $K$ such that (1) if $x \in (R^3 - V(K, \in))$, $h(x) = x$, (2) $h[K] \subset V(S, \in)$, and (3) if $Q \in \mathscr{P}$, $(\operatorname{diam} h[Q]) < \in$.

THEOREM 8. *Suppose $F$ is an upper semi-continuous decomposition of $R^3$ and $F$ is definable by 3-cells-with-handles. Then there exists an upper semi-continuous decomposition $G$ of $R^3$ such that $F$ is equivalent to $G$ and each nondegenerate element of $G$ is one dimentional.*

*Proof.* Since $F$ is definable by 3-cells-with-handles, there exists a defining sequence $M_1, M_2, M_3, \cdots$ for $F$ such that for each positive integer $k$, each component of $M_k$ is a 3-cell-with-handles. Let $C_{k,1}, C_{k,2}, \cdots, C_{k,n_k}$ be the 3-cells-with-handles which are the components of $M_k$, and if $j$ is a positive integer less than or equal to $n_k$, let $S_{k,j}$ be a special spine of $C_{k,j}$.

Let $\varepsilon_1$ be a positive number such that $\varepsilon_1 < 1$ and $V(C_{1,1}, \varepsilon_1)$, $V(C_{1,2}, \varepsilon_1), \cdots$, and $V(C_{1,n_1}, \varepsilon_1)$ are mutually disjoint sets. For each positive integer $j$ less than or equal to $n_1$, there exist a partition $P_{1,j}$ of $C_{1,j}$ and a homeomorphism $h_{1,j}$ of $V(C_{1,j}, \varepsilon_1)$ onto itself such that (1) if $x \in V(C_{1,j}, \varepsilon_1) - V(C_{1,j}, \varepsilon_1/2)$, $h_{1,j}(x) = x$, (2) $h_{1,j}[C_{1,j}] \subset V(S_{1,j}, \varepsilon_1)$, and (3) if $Q \in \mathscr{P}_{1,j}$, $(\operatorname{diam} h_{1,j}[Q]) < \varepsilon_1$. Let $h_1$ be a homeomorphism of $R^3$ onto itself such that if $x \notin \bigcup_{i=1}^{n_1} V(C_{1,i}, \varepsilon_1)$, $h_1(x) = x$, and if $i$ is a positive integer less than or equal to $n_1$ and $x \in V(C_{1,i}, \varepsilon_1)$, $h_1(x) = h_{1,i}(x)$.

Let $\delta_1$ be $\min\{(\operatorname{diam} h_1[Q]) \colon Q \in (\bigcup_{i=1}^{n_1} P_{1,i})\}$ and let $\varepsilon_2$ be a positive number such that $\varepsilon_2 < \min\{\delta_1/2, 1/2\}$ and $V(h_1[C_{2,1}], \varepsilon_2)$, $V(h_1[C_{2,2}], \varepsilon_2)$, $\cdots$, and $V(h_1[C_{2,n_2}], \varepsilon_2)$ are mutually disjoint sets each one of which is contained in $h_1[V(M_1, 1/2)]$. For each positive integer $j$ less than or equal to $n_2$, there exist a partition $\mathscr{P}_{2,j}$ of $C_{2,j}$ and a homeomorphism $h_{2,j}$ of $V(h_1[C_{2,j}], \varepsilon_2)$ onto itself such that (1) if

$$x \in V(h_1[C_{2,j}], \varepsilon_2) - V(h_1[C_{2,j}], \varepsilon_2/2),\ h_{2,j}(x) = x,$$

(2) $h_{2,j}[h_1[C_{2,j}]] \subset V(h_1[S_{2,j}], \varepsilon_2)$, (3) if $Q_1 \in \mathscr{P}_{2,j}$, there exists an element $Q_2$ of $\bigcup_{i=1}^{n_1} \mathscr{P}_{1,i}$ such that $h_{2,j}h_1[Q_1] \subset h_1[Q_2]$, and (4) if

$$Q \in \mathscr{P}_{2,j},\ (\operatorname{diam} h_{2,j}h_1[Q]) < \varepsilon_2.$$

Let $h_2$ be a homeomorphism of $R^3$ onto itself such that if

$$x \notin \bigcup_{i=1}^{n_2} V(h_1[C_{2,i}], \varepsilon_2),\ h_2(x) = x,$$

and if $i$ is a positive integer less than or equal to $n_2$ and

$$x \in V(h_1[C_{2,i}], \varepsilon_2),\ h_2(x) = h_{2,i}(x).$$

Continue in this manner obtaining a sequence $h_1, h_2, h_3, \cdots$ of homeomorphisms of $E^3$ onto itself. Let $h$ be $\lim_{n\to\infty}(h_n h_{n-1} \cdots h_1)$, and let $G$ be $\{h[f]: f \in F\}$. It is easily seen that $G$ is an upper semi-continuous decomposition of $E^3$ such that $F$ and $G$ are equivalent. The fact that each nondegenerate element $g$ of $G$ is one-dimensional can be seen by noticing that $g$ intersects the boundaries of the images of the elements of the partitions in a 0-dimensional set.

THEOREM 9. *Let $G$ be a monotone upper semi-continuous decomposition of $R^3$ such that $G$ has only countably many nondegenerate elements, and each nondegenerate element is tame (relative to the usual triangulation of $R^3$). Then there exists a homeomorphism $h$ of $R^3$ onto itself such that if $g \in G$, $h[g]$ is polyhedral.*

*Proof.* Let $g_1, g_2, g_3, \cdots$ denote the nondegenerate elements of $G$. Let $\varepsilon_1$ be a positive number such that $\varepsilon_1 < 1/2$. Since $g_1$ is tame, it follows from Theorem 9 of [4] that there exists a homeomorphism $h_1$ of $R^3$ onto itself such that if $x \in R^3 - V(g_1, \varepsilon_1/4)$, $h_1(x) = x$, if $x \in R^3$, $d(x, h_1(x)) < \varepsilon_1/4$, and $h_1[g_1]$ is polyhedral.

Let $\varepsilon_2$ be a positive number such that

$$\varepsilon_2 < (\varepsilon_1/2),\ V(h_1[g_2], \varepsilon_2) \subset h_1[V(g_2, 1/2^2)],$$

and $V(h_1[g_2], \varepsilon_2) \cap h_1[g_1] = \emptyset$. There exists a homeomorphism $h_2$ of $R^3$ onto itself such that if $x \in R^3 - V(h_1[g_2], \varepsilon_2/4)$, $h_2(x) = x$, if $x \in R^3$ then $d(h_2(x), x) < \varepsilon_2/4$, and $h_2 h_1[g_2]$ is polyhedral.

If $n$ is a positive integer and $h_1, h_2, \cdots$, and $h_{n-1}$ are chosen, let $\varepsilon_n$ be a positive number such that

$$\varepsilon_n < 1/2^n,\ V(h_{n-1} \cdots h_1[g_n], \varepsilon_n) \subset h_{n-1} \cdots h_1[V(g_n, 1/2^n)],$$

and

$$V(h_{n-1} \cdots h_1[g_n], \varepsilon_n) \cap \left(\bigcup_{i=1}^{n-1} h_{n-1} \cdots h_1[g_i]\right) = \emptyset.$$

There exists a homeomorphism $h_n$ of $R^3$ onto itself such that if

$$x \in R^3 - V(h_{n-1} \cdots h_1[g_n], \varepsilon_n/4),$$

then $h_n(x) = x$, if $x \in R^3$ then $d(h_n(x), x) < \varepsilon_n/4$, and $h_n \cdots h_1[g_n]$ is polyhedral.

Let $h$ be $\lim_{n\to\infty} h_n h_{n-1} \cdots h_1$. $h$ is the uniform limit of continuous functions, thus $h$ is continuous. It follows from Theorem $C_2$ of [9] that $h$ is onto $R^3$.

To show that $h$ is one-to-one, let $x$ and $y$ be distinct points of $R^3$. If $x$ and $y$ belong to the same element of $G$, then clearly $h(x) \neq h(y)$.

Suppose $x \in g_x$ and $y \in g_y$ and $g_x \neq g_y$ where $g_x$ and $g_y$ are elements of $G$.

Since $G$ is upper semi-continuous, there exists a positive integer $N$ such that if $n$ is an integer greater than $N$ and $x \in V(g_n, 1/2^n)$, then $y \notin V(g_n, 1/2^n)$, and if $n$ is an integer greater than $N$ and $y \in V(g_n, 1/2^n)$, then $x \notin V(g_n, 1/2^n)$.

Now for each positive integer $n$, let $U_n$ be

$$(h_{n-1} \cdots h_1)^{-1}[V(h_{n-1} \cdots h_1)[g_n], \varepsilon_n/4].$$

Then $U_n \subset V(g_n, 1/2^n)$. If for each positive integer $n$ neither $x$ nor $y$ belongs to $U_n$, then

$$h(x) = h_N \cdots h_1(x),\ h(y) = h_N \cdots h_1(y)$$

and

$$h(y) \neq h(x).$$

Suppose there exists a positive integer $n$ such that $n > N$ and $x \in U_n$. Then

$$x \in V(g_n, 1/2^n),\ y \in V(g_n, 1/2^n)$$

and

$$h_{n-1} \cdots h_1(x) \in V(h_{n-1} \cdots h_1[g_n], \varepsilon_n/4).$$

Since

$$V(h_{n-1} \cdots h_1[g_n], \varepsilon_n) \subset h_{n-1} \cdots h_1[V(g_n, 1/2^n)],$$
$$h_{n-1} \cdots h_1(y) \notin h_{n-1} \cdots h_1[V(g_n, 1/2^n)]$$

and

$$d(h^n{}_{-1} \cdots h_1(y), h_{n-1} \cdots h_1(x)) \geqq \varepsilon_n.$$

Then

$$d(h(x), h_{n-1} \cdots h_1(x)) \leqq \varepsilon_n/2,\ d(h(y), h_{n-1} \cdots h_1(y)) \leqq \varepsilon_{n+1}/2 < \varepsilon_n/2$$

and

$$d(h(x), h(y)) \neq 0.$$

Thus $h(x) \neq h(y)$. Hence $h$ is one-to-one.

To show that $h^{-1}$ is continuous, suppose there exists a sequence $x_n \to x$ such that $h^{-1}(x) \not\to h^{-1}(x)$. Picking a subsequence if necessary, it can be assumed that there exists a positive number $\varepsilon$ such that for each $i$, $h^{-1}(x_i) \notin V(h^{-1}(x), \varepsilon)$.

Since $h$ is bounded, $\{h^{-1}(x_i): i \in J\}$ is bounded and $\mathrm{Cl}\,\{h^{-1}(x_i): i \in J\}$ and $\{h^{-1}(x)\}$ are disjoint compact sets. Hence for some

$$\delta > 0,\ d(h[\mathrm{Cl}\,\{h^{-1}(x_i): i \in J\}],\ hh^{-1}(x)) > \delta.$$

Then for each positive integer $i$, $d(x_i, x) > \delta$. This is a contradiction. Thus $h^{-1}$ is continuous.

COROLLARY 5. *If $F$ is an upper semi-continuous decomposition of $R^3$ into tame 3-cells and points, then there exists an upper semi-continuous $G$ into polyhedral 3-cells and points such that $F$ is equivalent to $G$.*

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