NONLINEAR ELLIPTIC CONVOLUTION EQUATIONS OF WIENER-HOPF TYPE IN A BOUNDED REGION

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The existence of a solution of a nonlinear perturbation of an elliptic convolution equation of Wiener-Hopf type in a bounded region \( G \) of \( \mathbb{R}^n \) is proved. More explicitly, let \( A \) be an elliptic convolution operator on \( G \) of order \( \alpha, \alpha > 0 \); \( A_j \) the principal part of \( A \) in a local coordinate system and \( A_j(x^i, \xi) \) be the symbol of \( A_j \) with a factorization with respect to \( \xi \) of the form: \( A_j(x^i, \xi) = A_j^0(x^i, \xi)A_j^1(x^i, \xi) \) for \( x_0^i = 0 \). \( A_j^0, A_j^1 \) are homogeneous of orders 0, \( \alpha \) in \( \xi \) respectively; the first admitting an analytic continuation in \( \text{Im} \xi_n > 0 \), the second in \( \text{Im} \xi_n \leq 0 \). Let \( T_k, k = 0, \cdots, [\alpha] - 1 \) be bounded linear operators from \( H^{s}(\mathbb{R}^n) \) into \( L^2(G) \) where \( H^{s}(\mathbb{R}^n), k \geq 0 \) are the Sobolev-Slobodetskii spaces of generalized functions.

The purpose of the paper is to prove the solvability of:

\[
Au_+ + \lambda^\alpha u_+ = f(x, T_0 u_+, \cdots, T_{[\alpha] - 1} u_+) \quad \text{on } G; \quad u_+ \in H^0_s(G) \quad \text{for large } |\lambda| \quad \text{and on a ray } \arg \lambda = \theta \quad \text{such that } \quad A_j + \lambda^\alpha \neq 0 \quad \text{for} \quad |\xi| + |\lambda| \neq 0 \quad \text{and for all } j. \quad f(x, \xi_0, \cdots, \xi_{n-1}) \text{ has at most a linear growth in } (\xi_0, \cdots, \xi_{n-1}) \text{ and is continuous in all the variables.}
\]

Linear elliptic convolution equations in a bounded region for arbitrary \( \alpha \) and with symbols having the above type of factorization (\( \lambda = 0 \)) have been considered recently by Visik-Eskin [3]. Those equations are similar to integral equations since no boundary conditions are required.

The notation and terminology are those of Visik-Eskin and are given in §1. The theorems are proved in §2.

1. Let \( s \) be an arbitrary real number and \( H^s(\mathbb{R}^n) \) be the Sobolev-Slobodetskii space of (generalized) functions \( f \) such that:

\[
\|f\|_s^2 = \int_{\mathbb{R}^n} (1 + |\xi|^s)^s |\hat{f}(\xi)|^2 d\xi < +\infty
\]

where \( \hat{f}(\xi) \) is the Fourier transform of \( f \).

We denote by \( H^s(\mathbb{R}^n_+) \), the space consisting of functions defined on \( \mathbb{R}^n_+ = \{x: x_n > 0\} \) and which are the restrictions to \( \mathbb{R}^n_+ \) of functions in \( H^s(\mathbb{R}^n) \). Let \( l\hat{f} \) be an extension of \( \hat{f} \) to \( \mathbb{R}^n \), then:

\[
\|\hat{f}\|_s^2 = \|f\|_{H^s(\mathbb{R}^n_+)} = \inf \|l\hat{f}\|_s.
\]

The infimum is taken over all extensions \( l\hat{f} \) of \( \hat{f} \).

The \( \hat{H}^s_\alpha = \{f_\alpha; f_\alpha(x) = f(x) \text{ if } x_n > 0, f \in L^2(\mathbb{R}^n), f_\alpha(x) = 0 \text{ if } x_n \leq 0\} \)
and similarly for $\tilde{H}_s$.

We denote by $H^+_s$, the space of functions $f_+$ with $f_+ \in \tilde{H}_s$ and $f_+ \in H^+(R^n)$ on $R^n$.

$\tilde{H}_s^+$ is the subspace of $H^+(\mathbb{R}^n)$ consisting of functions with supports in $\text{cl}(R^n)$. $\tilde{H}^+_s$, $\tilde{H}^-_s$ denote respectively the spaces which are the Fourier images of $H^+_s$, $H^-(\mathbb{R}^n)$, $\tilde{H}^+_s$.

Let $\tilde{f}(\xi)$ be a smooth decreasing (i.e., $|\tilde{f}(\xi)| \leq M|\xi_n|^{-1-\varepsilon}$ for large $|\xi_n|$ and for some $\varepsilon > 0$) function. The operator $\Pi^+$ is defined as:

$$\Pi^+ \tilde{f}(\xi) = \frac{1}{2} \tilde{f}(\xi) + i(2\pi)^{-1} \text{v.p.} \int_{-\infty}^{\infty} \tilde{f}(\xi', \eta_n)(\xi_n - \eta_n)^{-1} d\eta_n$$

where $\xi' = (\xi_1, \cdots, \xi_n)$.

For any $\tilde{f}$, then the above relation is understood as the result of the closure of the operator $\Pi^+$ defined on the set of smooth and decreasing functions.

$\Pi^+$ is a bounded mapping from $\tilde{H}_s$ into $\tilde{H}_s^+$ if $0 \leq s < 1/2$ and is a bounded mapping from $\tilde{H}_s$ into $\tilde{H}_s^+$ if $s \geq 1/2$.

Set: $\xi^- = \xi_n - i|\xi'|$; $(\xi^- - i)^s$ is analytic for any $s$ if $\text{Im} \xi_n \leq 0$ and:

$$||f||^s = ||\Pi^+(\xi^- - i)^s l\tilde{f}(\xi)||_s$$

where $l\tilde{f}$ is any extension of $f$ to $\mathbb{R}^n$ (Cf. [3], p. 93 relation (8.1)).

Let $G$ be a bounded open set of $\mathbb{R}^n$ with a smooth boundary. $H^s(G)$ denotes the restriction to $G$ of functions in $H^s(\mathbb{R}^n)$ with the norm:

$$||u||_s = \inf ||v||_{H^s(\mathbb{R}^n)}; \quad v = u \text{ on } G.$$

By $H^s_s(G)$, we denote the space of functions $f$ defined on all of $\mathbb{R}^n$, equal to 0 on $\mathbb{R}^n/\text{cl}(G)$ and coinciding in $\text{cl}G$ with functions in $H^s(\mathbb{R}^n)$.

**DEFINITION 1.** $\tilde{A}(\xi)$ is in $0_\alpha$ if and only if:

(i) $\tilde{A}(\xi)$ is a homogeneous function of order $\alpha$ in $\xi$.
(ii) $\tilde{A}$ is continuous for $\xi \neq 0$.

**DEFINITION 2.** $\tilde{A}_s(\xi)$ is in $0_\alpha$ if and only if:

(i) $\tilde{A}_s(\xi)$ is in $0_\alpha$.
(ii) $\tilde{A}_s(\xi', \xi_n)$ has an analytic continuation with respect to $\xi_n$ in the half-plane $\text{Im} \xi_n > 0$ for each $\xi'$.

Similar definition for $0_\alpha$:

**DEFINITION 3.** $\tilde{A}$ is in $E_\alpha$ if and only if:

(i) $\tilde{A}$ is in $0_\alpha$. 

(ii) \( \tilde{A}(\xi) \neq 0 \) for \( \xi \neq 0 \).
(iii) \( \tilde{A}(\xi) \) has, for \( \xi' \neq 0 \), continuous first order derivatives, bounded if \( |\xi'| = 1, \xi' \neq 0 \).

**Definition 4.** \( \tilde{A}(x, \xi', \xi_n) \) is in \( D_0^s \) if and only if:

(i) \( \tilde{A}(x, \xi) \) is infinitely differentiable with respect to \( x \) and \( \xi; \xi \neq 0 \).
(ii) \( \tilde{A}(x, \xi) \) is in \( 0_a \) for \( x \) in \( \mathbb{R}^n \).
(iii) \( a_{k}(x) = \left( \frac{\partial^k}{(\partial \xi')^k} \tilde{A}(x, 0, -1) = (-1)^k \exp (-i\alpha \pi) \frac{\partial^k}{(\partial \xi')^k} \tilde{A}(x, 0, 1) \right) \) in \( \mathbb{R}^n; 0 \leq |k| < \infty; k = (k_1, \ldots, k_n) \).

**Definition 5.** Let \( A \) be a bounded linear operator from \( H^s \) into \( H^{s-a}(\mathbb{R}^n) \). Then any bounded linear operator \( T \) from \( H^s_{a-1} \) into \( H^{s-a}(\mathbb{R}^n) \), (or from \( H^s \) into \( H^{s-a+1}(\mathbb{R}^n) \)) is called a right (left) smoothing operator with respect to \( A \).

\( T \) is a smoothing operator with respect to \( A \) if it is both a left and right smoothing operator.

Let \( \tilde{A}(\xi) \) be in \( 0_a \) for \( \alpha > 0 \). For \( u_+ \) in \( H^s_+ \), \( s \geq 0 \), with support in \( \text{cl} (\mathbb{R}^n) \), set: \( Au_+ = F^{-1}(\tilde{A}(\xi)\bar{u}_+(\xi)) \) where \( F^{-1} \) is the inverse Fourier transform. It is well defined in the sense of generalized functions.

Let \( \tilde{A}(x, \xi) \) be an element of \( E_a \) for each \( x \) in \( \text{cl} G \) and \( \tilde{A}(x, \xi) \) be infinitely differentiable with respect to \( x \) and \( \xi \). Since \( G \) is a bounded set of \( \mathbb{R}^n \), we may assume that \( G \) is contained in a cube of side \( 2p \) centered at \( 0 \). We extend \( \tilde{A}(x, \xi) \) with respect to \( x \) to all of \( \mathbb{R}^n \) by setting \( \tilde{A}(x, \xi) = 0 \) if \( |x| \geq p - \varepsilon \) for \( \varepsilon > 0 \). We get a finite function, homogeneous of order \( \alpha \) with respect to \( \xi \).

We take the expansion into Fourier series of \( \tilde{A}(x, \xi) \):

\[ \tilde{A}(x, \xi) = \sum_{k=-\infty}^{\infty} \psi_k(x) \exp \left[ (i\pi k x)/p \right] \tilde{L}_k(\xi); \quad k = (k_1, \ldots, k_n) \]

where:

\[ \tilde{L}_k(\xi) = (2p)^{-n} \int_{-p}^{p} \exp \left[ (i\pi k x)/p \right] \tilde{A}(x, \xi) dx \]

\( \psi_k(x) = 1 \) for \( |x| \leq p - \varepsilon \); \( \psi_k(x) = 0 \) for \( |x| \geq p \); \( \psi_k(x) \in C^\infty(\mathbb{R}^n) \). We have: \( |\tilde{L}_k(\xi)| \leq C \left| \xi \right|^a (1 + |k|)^{-M} \) for arbitrary positive \( M \). Let \( u_+ \) be in \( H^s_+(G) \), we define:

\[ (1.1) \quad Au_+ = \sum_{k=-\infty}^{\infty} \psi_k(x) \exp \left[ (i\pi k x)/p \right] L_k u_+ \]

where \( L_k u_+ = L_k u_+ \) is defined as before since \( \tilde{L}_k(\xi) \) is independent of \( x \).
Denote by $P^+$, the restriction operator of functions defined on $\mathbb{R}^n$ to $G$. We consider an elliptic convolution equation of order $\alpha$, on $G$ of the form:

$$P^+Au_+ = \sum_j P^+\varphi_j A\psi_j u_+ + Tu_+$$

$T$ is a smoothing operator. The $\varphi_j$ is a finite partition of unity corresponding to a covering $N_j$ of $\text{cl}G$ with diam $(N_j)$ sufficiently small. The $\psi_j$ are in $C^\infty_c(\mathbb{R}^n)$ with $\varphi_j \psi_j = \varphi_j$ and $\text{supp} (\psi_j) \subseteq N_j$.

Suppose $\tilde{A} \in D_0^s$, then the operator $\varphi_j A\psi_j$ taken in local coordinates may be written as:

$$\varphi_j A\psi_j = \varphi_j A_j \psi_j + T_j$$

where $A_j$ is a convolution operator of the form (1.1) and $T_j$ is a smoothing operator (Cf. [3] Appendix 2).

2. The main result of the paper is the following theorem:

**Theorem 1.** Let $A$ be an elliptic convolution operator on $G$, of order $\alpha > 0$, and of the form (1.2). Suppose that:

(i) $\tilde{A}_j(x^i, \xi) \in E_{\alpha} \cap D_0^s$.

(ii) $\tilde{A}_j(x^i, \xi)$ has for $x_+^i = 0$ a factorization of the form:

$$\tilde{A}_j(x^i, \xi) = \tilde{A}_j^i(x^i, \xi) \tilde{A}_j^j(x^j, \xi)$$

where $\tilde{A}_j^i \in D_{\alpha}^s; \tilde{A}_j^j \in D_0^s$ for all $x^i \in N_j \cap G$.

(iii) There exists a ray $\arg \lambda = \theta$ such that $\tilde{A}_j(x^i, \xi) + \lambda^\alpha \neq 0$ for $|\xi| + |\lambda| = 0, x^i \in N_j \cap G$.

Let $f(x, \zeta_0, \cdots, \zeta_{[\alpha]-1})$ be a function measurable in $x$ on $G$, continuous in all the other variables. Suppose there exists a positive constant $M$ such that:

$$|f(x, \zeta_0, \cdots, \zeta_{[\alpha]-1})| \leq M \left\{ 1 + \sum_{j=0}^{[\alpha]-1} |\zeta_j| \right\}.$$  

Let $T_k; k = 0, \cdots, [\alpha] - 1$ be bounded, linear operators from $H^s(G)$ into $L^2(G)$. Then for $|\lambda| \geq \lambda_0 > 0; \arg \lambda = \theta$; there exists a solution $u$ in $H^s(G)$ of:

$$P^+(A + \lambda^\alpha)u_+ = f(x, T_0 u_+, \cdots, T_{[\alpha]-1} u_+) \quad \text{on} \ G.$$  

The solution is unique if $f$ satisfies a Lipschitz condition in $(\zeta_0, \cdots, \zeta_{[\alpha]-1})$.

To prove the theorem, we shall do as in [2]. First, following Visik-Agranovich [4], we establish an a priori estimate and show the existence and the uniqueness of a solution of a linear elliptic convolution
equation depending on a large parameter in a bounded region. Then we use the Leray-Schauder fixed point theorem to prove Theorem 1.

We have:

**Theorem 2.** Let $A$ be an elliptic convolution operator, of order $\alpha > 0$, of the form (1.2). Suppose that all the hypotheses of Theorem 1 are satisfied. Let $f \in L^2(G)$; then there exists a unique solution $u_+$ in $H^\alpha(G)$ of:

$$P^+(A + \lambda^\alpha)u_+ = f \text{ on } G; \quad |\lambda| \geq \lambda_0 > 0 \quad \arg \lambda = \theta.$$

Moreover:

$$||u_+||_\alpha + ||\lambda||^\alpha ||u_+||_0 \leq M ||f||_0$$

where $M$ is independent of $\lambda, u_+$.

**Proof of Theorem 1.** Let $v$ be an element of $H^\alpha(G)$ and $0 \leq t \leq 1$. Consider the linear elliptic convolution equation:

$$P^+(Au_+ + \lambda^\alpha u_+) = f(x, tT_\alpha v, \cdots, tT_{(\alpha-1)}v).$$

With the hypotheses of the theorem, $f(x, tT_\alpha v, \cdots, tT_{(\alpha-1)}v)$ is in $L^2(G)$. It follows from Theorem 2 that there exists a unique solution $u_+$ in $H^\alpha(G)$ of the problem.

Let $\mathcal{S}(t)$ be the nonlinear mapping from $[0,1] \times H_\alpha(G)$ into $H_\alpha(G)$ defined by $\mathcal{S}(t)v = u_+$ where $u_+$ is the unique solution of the above problem.

The theorem is proved if we can show that $\mathcal{S}(1)$ has a fixed point.

**Proposition 1.** $\mathcal{S}(t)$ is a completely continuous mapping from $[0,1] \times H_\alpha(G)$ into $H_\alpha(G)$.

**Proof.** (i) $\mathcal{S}(t)$ is continuous. Suppose that $t_n \to t; \ t_n, t \in [0,1] \ v_n \to v$ in $H_\alpha(G)$. Set: $u_n = \mathcal{S}(t_n)v_n$. Then from Theorem 2, we get:

$$||u_n - u||_\alpha \leq M ||f(\cdot, t_n T_\alpha v_n, \cdots, t_n T_{(\alpha-1)}v_n) - f(\cdot, tT_\alpha v, \cdots, tT_{(\alpha-1)}v)||_0.$$

It follows from Lemmas 3.1 and 3.2 of [1] that $u_n \to u$ in $H_\alpha(G)$.

(ii) $\mathcal{S}(t)$ is compact. Suppose that $||v_n||_\alpha \leq M$. Then from the weak compactness of the unit ball in a Hilbert space and from the generalized Sobolev imbedding theorem, we get:

$v_{n_j} \to v$ weakly in $H_\alpha(G)$ and strongly in $H^{\alpha-\varepsilon}(G); \ 0 < \varepsilon, \alpha - \varepsilon \geq 0$. 

Applying the argument of the first part, we get the compactness of $\mathcal{A}(t)$.

**Proposition 2.** $I - \mathcal{A}(0)$ is a homeomorphism of $H^2(G)$ into itself. If $v = \mathcal{A}(t)v$, for $0 < t \leq 1$; then: $\|v\|_\alpha \leq M$ where $M$ is independent of $t$.

**Proof.** The first assertion is trivial.
Suppose that $v = \mathcal{A}(t)v$. It follows from Theorem 2 that:

$$\|v\|_\alpha + |\lambda|^\alpha \|v\|_0 \leq M \|f(\cdot, tT_0v, \cdots, tT_{(\alpha-1)}v)\|_0 \leq M(1 + \|v\|_{(\alpha-1)}).$$

It is well-known that:

$$\|v\|_{(\alpha-1)} \leq 1/2M \|v\|_\alpha + C \|v\|_0.$$  

Taking $|\lambda|$ sufficiently large, we have: $\|v\|_\alpha \leq M$. $\mathcal{A}(t)$ satisfies the hypotheses of the Leray-Schauder fixed point theorem (the uniform continuity condition as in [2] is not necessary). So $\mathcal{A}(1)$ has a fixed point, i.e. $\mathcal{A}(1)u_+ = u_+.$

The uniqueness of the solution in the case $f(x, \zeta_0, \cdots, \zeta_{(\alpha-1)})$ satisfies a Lipschitz condition in $(\zeta_0, \cdots, \zeta_{(\alpha-1)})$ follows trivially from the estimate of Theorem 2. We shall not reproduce it.

**Proof of Theorem 2.** As usual, we consider first the case of the positive half-space $R^n_+$ with the convolution operator $A$ having a constant symbol.

**Lemma 1.** Let $\hat{A}(\xi)$ be an element of $E_\alpha (\alpha > 0)$. Suppose that: $\hat{A}(\xi) = \hat{A}_+(\xi)\hat{A}_-(\xi)$ with $\hat{A}_+(\xi)$ in $0^+_\alpha$, $\hat{A}_-(\xi)$ in $0^-_\alpha$. Let $P^+$ be the restriction operator of functions in $R^n$ to $R^n_+$ and $A$ be the convolution operator with symbol $\hat{A}(\xi)$. Suppose there exists a ray $\text{arg} \lambda = \theta$ such that: $\hat{A}(\xi) + \lambda^\alpha \neq 0$ for $|\xi| + |\lambda| \neq 0$. If $f$ is in $H^\alpha(R^n_+)$, then there exists a unique solution $u$ in $H^\alpha_+$ of:

$$P^+(A + \lambda^\alpha)u_+ = f \text{ on } R^n_+; |\lambda| \geq \lambda_0 > 0.$$

Moreover:

$$\|u_+\|_\alpha^+ + |\lambda|^\alpha \|u_+\|_0^+ \leq M \|f\|_\alpha^+$$

where $M$ is independent of $\lambda, u_+, f$.

**Proof.** Set $\hat{A}(\xi, \lambda) = \hat{A}(\xi) + \lambda^\alpha$. It is homogeneous of order $\alpha$ in $(\xi, \lambda)$. Since $\hat{A}(\xi)$ is in $E_\alpha$, we have the following factorization with respect to $\xi$, which is unique up to a constant multiplier:
\[
\tilde{A}(\xi) = \tilde{A}_+(\xi)\tilde{A}_-(\xi)
\]
(Cf. Theorem 1.2 of [3], p. 95). The same proof with \(\xi_+ = \xi + i|\xi'|\) replaced by \(\xi_+ = \xi + i(\xi + |\xi'|)\) and \(\xi_-\) replaced by:
\[
\xi_- = \xi - i(\xi + |\xi'|)
\]
gives:
\[
\tilde{A}(\xi, \lambda) = \tilde{A}_+(\xi, \lambda)\tilde{A}_-(\xi, \lambda).
\]

Moreover:
If \(\tilde{A}_+(\xi)\) is in \(O_0\), then \(\tilde{A}_+(\xi, \lambda)\) is also in \(O_0\) (is homogeneous of order 0 in \((\xi, \lambda)\)). Similarly for \(\tilde{A}_-(\xi, \lambda)\).

Let \(\tilde{f}(x)\) be an extension of \(f\) to \(\mathbb{R}^n\). Consider:
\[
\tilde{u}_+(\xi) = (\tilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}.
\]

For \(|\lambda| \neq 0\), \(\tilde{u}_+(\xi)\) has an analytic continuation in \(\text{Im} \xi_n > 0\) and:
\[
\int |\tilde{u}_+(\xi', \xi_n + i\tau)|^2 d\xi' d\xi_n \leq C,
\]
\(C\) is independent of \(\tau > 0\). So: \(\tilde{u}_+(\xi) \in H^\circ_0\). (Cf. [3], p. 91).

We get:
\[
||u_+||^2 = ||\Pi^+ (\xi_+ - i)^n \tilde{u}_+(\xi)||^2_0 \\
\leq ||(\xi_+ - i)^n (\tilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}||_0.
\]

Since \(\tilde{A}_+(\xi, \lambda)\) is homogeneous of order 0 in \((\xi, \lambda)\), we have:
\[
\tilde{A}_+(\xi, \lambda) = \tilde{A}_+(\xi/(|\xi| + |\lambda|), \lambda/(|\xi| + |\lambda|)).
\]

Let \(c = \text{Min} |\tilde{A}_+(\xi, \lambda)|\) for \(|\xi| + |\lambda| = 1, \arg \lambda = \theta\). Then \(c > 0\) and is independent of \(\lambda\).

So:
\[
||u_+||^2 \leq c^{-1} ||(\xi_+ - i)^n \Pi^+ \tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}||_0 \\
\leq C ||\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}||_0.
\]

We may write:
\[
\tilde{A}_-(\xi, \lambda) = (|\xi| + |\lambda|)^n \tilde{A}_-(\xi/(|\xi| + |\lambda|), \lambda/(|\xi| + |\lambda|)).
\]

Let \(C = \text{Min} |\tilde{A}_-(\xi, \lambda)|\) for \(|\xi| + |\lambda| = 1, \arg \lambda = \theta\). Then \(C > 0\) and is independent of \(\lambda\).

We obtain:
\[
||u_+||^2 \leq C ||\tilde{f}(\xi)||_0 \leq C_\varepsilon ||f||^\circ_0.
\]

A similar argument gives:
\[ \| u_+ \|_{s}^+ \leq C |\lambda|^{-\sigma} \| f \|_{s}^+ . \]

So:
\[ \| u_+ \|_{s}^+ + |\lambda|^{-\sigma} \| u_+ \|_{s}^+ \leq C \| f \|_{s}^+ . \]

\( C \) is independent of \( \lambda, f, u_+ \).

A direct verification shows that \( u_+ \) is a solution of the equation. It remains to show that the solution is unique. Let \( v_+ \) be an element of \( H_0^+ \). Suppose that \( v_+ \) is also a solution of the equation. Then as in [3], \( \tilde{v}_+(\xi) \), its Fourier transform is given by an expression of the same form as \( \tilde{u}_+(\xi) \) with \( \tilde{l}_f(\xi) \) replaced by \( \tilde{l}_f(\xi) \). \( l_f \) being an extension of \( f \) to \( \mathbb{R}^n \).

Set \( l_x f = l f - l_x f \). Then \( l_x f \in H_0^- \), so \( \tilde{l}_x f \in \mathcal{H}_\mathbb{C} \). \( \tilde{l}_x f(\tilde{A}_-(\xi, \lambda))^{-1} \) is analytic in \( \text{Im} \xi_n \leq 0 \) for \( |\lambda| \neq 0 \) and moreover:
\[ \int |\tilde{l}_x f(\xi', \xi_n + i\tau)|^2 |\tilde{A}_-(\xi', \xi_n + i\tau)|^{-2} d\xi d\xi_n \leq C \]
where \( C \) is independent of \( \tau \leq 0 \).

Hence \( \tilde{l}_x f(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \) is in \( \mathcal{H}_\mathbb{C} \) (Cf. [3], p. 91), so:
\[ \Pi^+ \tilde{l}_x f(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}) = 0 . \]

Therefore: \( \tilde{A}_+(\xi, \lambda)(\tilde{u}_+(\xi) - \tilde{v}_+(\xi)) = 0 \).

But \( \tilde{A}_+(\xi, \lambda) \neq 0 \) for \( |\lambda| \neq 0 \), we get \( \tilde{u}_+ = \tilde{v}_+ \).

Q.E.D.

Set:
\[ A_1 u = \sum_{k=\infty}^{\infty} \psi_0(x) \exp [(ik\pi x)/p] L_k u \]
\[ A_0 u = \sum_{k=\infty}^{\infty} \psi_0(x) \exp [(ik\pi)/p] L_k u \]
where \( L_k, \psi_0 \) are as in § 1.

**Lemma 2.** Let \( A_1, A_0 \) be as above and \( \psi(x) \) be in \( C_0^\infty(\mathbb{R}^n) \) with \( \psi(x) = 0 \) for \( |x - x_0| > \delta \); \( |\psi(x)| \leq K \) where \( K \) is independent of \( \delta \). Then:
\[ \| \psi(A_1 - A_0) u \|_{s-a}^+ \leq C \delta \| u \|_{s}^+ + C(\delta) \| u \|_{s-a}^+ \]
for all \( u \) in \( H_0^+ \), \( s \geq 0 \).

**Proof.** Cf. Lemma 4.7 of [3], p. 119.

**Proof of Theorem 2 (continued).** (1) First, we establish an a-priori estimate of the solutions.
Consider:

\[ P^+ \varphi_j A \psi_j u_+ + \lambda^a P^+(\varphi_j u_+) = P^+(\varphi_j f) - Tu_+ \]

where \( T \) is a smoothing operator with respect to \( \varphi_j A \psi_j \).

It has been shown in [3] (Appendix 2) that in a local coordinates system, the operator \( \varphi_j A \psi_j \) becomes: \( \varphi_j A_j \psi_j + T_j \) where \( A_j \) has for symbol \( \tilde{A}_j(x^i, \xi) \) and \( T_j \) is a smoothing operator.

So, we have:

\[ P^+ \varphi_j A_j(\psi_j u_+) + \lambda^a P^+(\varphi_j u_+) = P^+(\varphi_j f) + T_j^2 u_+ \]

where \( T_j^2 \) is again a smoothing operator.

Let \( A_j0 \) be the convolution operator with symbol \( \tilde{A}_j(x_0^i, \xi) \) evaluated at the point \( x_0^i \). We write:

\[ P^+ \varphi_j A_j0(\psi_j u_+) + \lambda^a P^+(\varphi_j u_+) = P^+(\varphi_j f) + T_j^2 u_+ + P^+ \varphi_j(A_j0 - A_j) \psi_j u_+ . \]

Applying Lemma 4.D.1 of [3] (p. 145), we have:

\[ P^+ \varphi_j A_j0(\psi_j u_+) = P^+ A_j0(\varphi_j u_+) + T_j^2 u_+ \]

where \( T_j^2 \) is a smoothing operator.

Therefore:

\[ (A_j0 + \lambda^a) \varphi_j u_+ = \varphi_j f + T_j^2 u_+ + \varphi_j(A_j0 - A_j)(\psi_j u_+) . \]

The symbols \( \tilde{A}_j0 \) satisfy the hypotheses of Lemma 1. Applying Lemma 1; 2, we obtain:

\[ \| \varphi_j u_+ \|_\sigma^+ + |\lambda|^a \| \varphi_j u_+ \|_\sigma^+ \leq M \| \varphi_j f \|_\sigma^+ + \| u_+ \|_\sigma \]

\[ + 1/2 M \| \psi_j u_+ \|_\sigma + |\psi_j u_+ \|_\sigma + \| \varphi_j u_+ \|_\sigma^+ \]

where we have used the well-known inequality:

\[ \| u_+ \|_{\sigma-1} \leq \varepsilon \| u_+ \|_\sigma + C(\varepsilon) \| u_+ \|_0 . \]

On the other hand: \( \| \psi_j u_+ \|_\sigma^+ \leq M \| u_+ \|_\sigma . \) Summing with respect to \( j, \) we get:

\[ \| u_+ \|_\sigma + |\lambda|^a \| u_+ \|_0 \leq M \| f \|_0 + 1/2 M \| u_+ \|_\sigma \]

\[ + \delta \| u_+ \|_\sigma + K \| u_+ \|_0 \]

Taking \( \delta \) small and \( |\lambda| \) sufficiently large, we have:

\[ \| u_+ \|_\sigma + |\lambda|^a \| u_+ \|_0 \leq M \| f \|_0 . \]

So, if there exists a solution, then the solution is unique.
(2) It remains to show the existence of a solution. From Lemma 1, we know that \( P^+(A_{j_0} + \lambda^a) \) has an inverse \( R_{j_0} \). Let \( R_{j_0} \) be the operator \( R_{j_0} \) expressed in the global system of coordinates of \( G \). Consider:

\[
Rf = \sum_j \varphi_j \tilde{R}_{j_0}(\psi_j f) .
\]

\( R \) is a bounded linear mapping from \( L^s(G) \) into \( H^s_c(G) \).

We show that: \( \mathcal{A} Rf = P^+(A + \lambda^a)Rf = f + \mathcal{E}f \) with \( \| \mathcal{E} \| \leq \frac{1}{2} \).

We have:

\[
\mathcal{A} Rf = \sum_j P^+(A + \lambda^a)\varphi_j \psi_j \tilde{R}_{j_0}(\psi_j f) .
\]

Applying Lemma 4.D.1. of [3], we may write:

\[
\mathcal{A} Rf = \sum_j P+\varphi_j(A + \lambda^a)\psi_j \tilde{R}_{j_0}(\psi_j f) + TRf
\]

where \( T \) is a smoothing operator.

We express \( \varphi_j(A + \lambda^a)\psi_j \tilde{R}_{j_0}(\psi_j f) \) in local coordinates. We get:

\[
\varphi_j(A_{j_0} + \lambda^a)\psi_j R_{j_0}(\psi_j f) + \varphi_j(A_j - A_{j_0})\psi_j R_{j_0}(\psi_j f) + T_j R_{j_0}(\psi_j f) .
\]

Using Lemma 4.D.1 of [3] again, we obtain:

\[
\varphi_j(A_{j_0} + \lambda^a)R_{j_0}(\psi_j f) + \varphi_j(A_j - A_{j_0})\psi_j R_{j_0}(\psi_j f) = T_j R_{j_0}(\psi_j f) .
\]

The \( T_j \) are all smoothing operators.

Applying Lemma 1, we have:

\[
\| T_j^2 R_{j_0}(\psi_j f) \|_0^\delta \leq C \| R_{j_0}(\psi_j f) \|_0^\alpha - 1 \leq \epsilon \| f \|_0 + C |\lambda|^{-\alpha} \| f \|_0 .
\]

From Lemmas 1 and 2, we get:

\[
\| \varphi_j(A_j - A_{j_0})\psi_j R_{j_0}(\psi_j f) \|_0^\delta \leq \delta \| \psi_j R_{j_0}(\psi_j f) \|_0^{\delta} + C(\delta) \| \psi_j R_{j_0}(\psi_j f) \|_0^{\alpha - 1}
\]
\[
\leq \delta \| f \|_0 + C(\delta) \| \psi_j R_{j_0}(\psi_j f) \|_0
\]
\[
\leq \delta \| f \|_0 + \epsilon C(\delta) \| R_{j_0}(\psi_j f) \|_\alpha
\]
\[
+ C(\delta)M(\epsilon) \| \tilde{R}_{j_0}(\psi_j f) \|_0
\]
\[
\leq \{ \delta + \epsilon C(\delta) \} \| f \|_0
\]
\[
+ |\lambda|^{-\alpha} M(\epsilon) C(\delta) \| f \|_0 .
\]

Taking \( \epsilon, \delta \) small, \( |\lambda| \) large enough, we have:

\[
\| \mathcal{E} f(\psi_j f) \|_0^\delta \leq \frac{1}{4N} \| f \|_0 .
\]

We obtain:
\[ Rf = f + TRf + \sum_j \mathcal{E}_j(\psi_j f) = f + \mathcal{E}f \]

where \( \mathcal{E}_j \) is the operator \( \mathcal{E}_j \) expressed in the global coordinates system of \( G \). We obtain: \[ \| f \|_0 \leq 1/4 \| f \|_0 + 1/4 \| f \|_0 \]

for large \( |\lambda| \).

Hence \( \| \mathcal{E} \| \leq 1/2 \); therefore \( (I + \mathcal{E})^{-1} \) exists. We define:

\[ \mathcal{E}^{-1} = R(I + \mathcal{E})^{-1}. \]

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**BIBLIOGRAPHY**


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