PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 3, 1968

POINT-LIKE 0-DIMENSIONAL DECOMPOSITIONS OF S^3

H. W. LAMBERT AND R. B. SHER

This paper is concerned with upper semicontinuous decompositions of the 3-sphere which have the property that the closure of the sum of the nondegenerate elements projects onto a set which is 0-dimensional in the decomposition space. It is shown that such a decomposition is definable by cubes with handles if it is point-like. This fact is then used to obtain some properties of point-like decompositions of the 3-sphere which imply that the decomposition space is a topological 3-sphere. It is also shown that decompositions of the 3-sphere which are definable by cubes with one hole must be point-like if the decomposition space is a 3-sphere.

In this paper we consider upper semicontinuous decompositions of S^3, the Euclidean 3-sphere. In particular, we shall restrict ourselves to those decompositions G of S^3 which have the property that the union of the nondegenerate elements of G projects onto a set whose closure is 0-dimensional in the decomposition space of G. We shall refer to such decompositions as 0-dimensional decompositions of S^3. Numerous examples of such decompositions appear in the literature. (One should note that some of the examples and results to which we refer are in E^3, Euclidean 3-space, but the corresponding examples and results for S^3 will be obvious in each case.)

In § 3, a technique of McMillan [10] is used to show that point-like 0-dimensional decompositions of S^3 are definable by cubes with handles. Armentrout [2] has shown this in the case where the decomposition space is homeomorphic with S^3. The proof of this theorem shows that compact proper subsets of S^3 with point-like components are definable by cubes with handles.

In § 4 we give some properties of point-like 0-dimensional decompositions of S^3 which imply that the decomposition space is homeomorphic with S^3. These properties were suggested by Bing in § 7 of [6].

It is not known whether monotone 0-dimensional decompositions of S^3 which yield S^3 must have point-like elements. Partial results in this direction have been obtained by Armentrout [2], Bean [5], and Martin [9]. Bing, in § 4 of [6], has presented an example of a decomposition of S^3 which yields S^3 even though it is not a point-like decomposition, but this example is not 0-dimensional. In § 5 we show that a 0-dimensional decomposition of S^3 that yields S^3 must have point-like elements if it is definable by cubes with one hole.

2. Definitions and notation. Let G be an upper semicontinuous decomposition of S^3, the 3-sphere. We denote the decomposition space of G by S^3/G, the union of the nondegenerate elements of G by H_G, and the projection map from S^3 onto S^3/G by P.

The decomposition G is said to be *monotone* if each element of G is a continuum. If cl $P(H_G)$ is 0-dimensional in S^3/G, then G is a *0-dimensional decomposition* of S^3. If each element of G has a complement in S^3 which is homeomorphic with E^3, Euclidean 3-space, then G is a *point-like decomposition* of S^3.

The sequence $M_1, M_2, M_3, \cdots$ is a *defining sequence* for G if and only if $M_1, M_2, M_3, \cdots$ is a sequence of compact 3-manifolds with boundary in S^3 such that (1) for each positive integer i, $M_{i+1} \subset \operatorname{Int} M_i$, and (2) g is a nondegenerate element of G if and only if g is a nondegenerate component of $\bigcap_{i=1}^{\infty} M_i$. Here, as in the remainder of the paper, subsets of S^3 which are manifolds will be assumed to be polyhedral subsets of S^3. It is well known that if G is a 0-dimensional decomposition of S^3, a defining sequence exists for G. If a defining sequence $M_1, M_2, M_3, \cdots$ exists for G such that for each positive integer i, each component of M_i is a cube with handles, G is said to be *definable by cubes with handles*. If a defining sequence $M_1, M_2, M_3, \cdots$ exists for G such that for each positive integer i, each component of M_i is a cube with one hole, G is said to be *definable by cubes with one hole*.

3. Some consequences of a result of McMillan. The following lemma is a special case of Lemma 1 of [11]. Its proof follows from the very useful technique used by McMillan to prove Theorem 1 of [10].

LEMMA 1. (*McMillan*). *In S^3, let M' be a compact polyhedral 3-manifold with boundary such that BdM' is connected, and let M be a compact polyhedral 3-manifold with boundary such that $M \subset Int\, M'$, and each loop in M can be shrunk to a point in $Int\, M'$. Then there is a cube with handles C such that $M \subset IntC \subset C \subset Int\, M'$.*

LEMMA 2. *If G is a point-like 0-dimensional decomposition of S^3, then there is a defining sequence $M_1, M_2, M_3, \cdots$ for G such that for each positive integer i, each component of M_i has a connected boundary.*

Proof. Let $M_1', M_2', M_3', \cdots$ be a defining sequence for G, let n be a positive integer, and let K be a component of M_n'. Let g be a component of $\bigcap_{i=1}^{\infty} M_i'$ which lies in K and let U be an open subset of K containing g such that cl $U \cap BdK = \varnothing$. Since g is point-like, there is a 3-cell C such that $g \subset \operatorname{Int} C \subset C \subset U$. There is an integer j such that L, the component of M_j' containing g, lies in $\operatorname{Int} C$. Since

C separates no points of BdK in K, L separates no points of BdK in K.

Using compactness of $\bigcap_{i=1}^{\infty} M_i'$, one obtains a finite collection $L_1, \cdots, L_k$ of mutually exclusive defining elements whose interiors cover $(\bigcap_{i=1}^{\infty} M_i') \cap K$ and so that no L_i separates points of BdK in K. It follows easily that $\bigcup_{i=1}^{k} L_i$ separates no points of BdK in K. By suitable relabeling, we suppose then, that if i is a positive integer and K is a component of M_i', $K \cap M_{i+1}'$ does not separate points of BdK in K. We construct disjoint arcs in K-M_{i+1}' connecting the boundary components of K and "drill-out" these arcs to replace K by a compact 3-manifold with connected boundary. Doing this for each component of each M_i', we obtain a defining sequence $M_1, M_2, M_3, \cdots$ as required by the conclusion of the lemma.

THEOREM 1. *If G is a point-like 0-dimensional decomposition of S^3, then G is definable by cubes with handles.*

Proof. Using Lemma 2, there is a defining sequence $M_1', M_2', M_3', \cdots$ for G such that each component of each M_i' has a connected boundary. Let n be a positive integer and N a component of M_n'. Since G is point-like, there is no loss of generality in supposing that each loop in $M_{n+1}' \cap N$ can be shrunk to a point in $\operatorname{Int} N$. From Lemma 1, there is a cube with handles, C, such that $(M_{n+1}' \cap N) \subset \operatorname{Int} C \subset C \subset \operatorname{Int} N$. Hence, there is a sequence $M_1, M_2, M_3, \cdots$ of compact 3-manifolds with boundary such that (1) for each positive integer i, $M_{i+1}' \subset \operatorname{Int} M_i \subset M_i \subset \operatorname{Int} M_i'$, and (2) each component of M_i is a cube with handles. The sequence $M_1, M_2, M_3, \cdots$ is a defining sequence for G and so G is definable by cubes with handles.

The proof of the next theorem follows from the proof of Theorem 1.

THEOREM 2. *If M is a closed subset of S^3 such that each component of M is point-like, then there exists a sequence $M_1, M_2, M_3, \cdots$ of compact 3-manifolds with boundary such that* (1) *for each positive integer i, $M_{i+1} \subset \operatorname{Int} M_i$,* (2) *each component of M_i is a cube with handles, and* (3) $M = \bigcap_{i=1}^{\infty} M_i$.

The concept of equivalent decompositions of S^3 was introduced in [4] and the following theorem follows immediately from Theorem 1 of this paper and Theorem 8 of [4].

THEOREM 3. *If G is a point-like 0-dimensional decomposition of S^3, then G is equivalent to a point-like 0-dimensional decomposition of S^3 each of whose nondegenerate elements is a 1-dimensional continuum.*

In the remaining two sections, we shall utilize some of the above results to investigate certain properties of 0-dimensional decompositions of S^3.

4. Properties of point-like 0-dimensional decompositions of S^3. In this section we give two properties, each of which is both necessary and sufficient to imply S^3/G is homeomorphic to S^3.

A space X will be said to have the *Dehn's Lemma property* if and only if the following condition holds: If D is a disk and f is a mapping of D into X such that on some neighborhood of $f(\mathrm{Bd}D)$, f^{-1} is a function, and U is neighborhood of the set of singular points of $f(D)$, then there is a disk D' in $f(D) \cup U$ such that $\mathrm{Bd}D' = f(\mathrm{Bd}D)$.

A space X will be said to have the *map separation property* if and only if the following condition holds: If D is a disk and $f_1, \cdots, f_n$ are maps of D into X such that (1) for each i, on some neighborhood of $f_i(BdD), f_i^{-1}$ is a function, (2) if $i \neq j$, $f_i(\mathrm{Bd}D) \cap f_j(D) = \emptyset$, and (3) U is a neighborhood of $f_1(D) \cup \cdots \cup f_n(D)$, then there exist maps $f_1', \cdots, f_n'$ of D into X such that (1) for each $i, f_i' \mid \mathrm{Bd}D = f_i \mid \mathrm{Bd}D$, (2) $f_1'(D) \cup \cdots \cup f_n'(D) \subset U$, and (3) if $i \neq j, f_i'(D) \cap f_j'(D) = \emptyset$.

It is a well known (and useful) fact that S^3 has the Dehn's Lemma property and the map separation property.

THEOREM 4. *If G is a point-like 0-dimensional decomposition of S^3, then S^3/G is homeomorphic with S^3 if and only if S^3/G has the Dehn's Lemma property.*

Proof. The "if" portion of the theorem is the only part that requires proof. Let U be an open set containing $\mathrm{cl}\, H_G$ and $\varepsilon > 0$. We shall construct a homeomorphism $h_\varepsilon : S^3 \rightarrow S^3$ such that if $x \in S^3 - U, h_\varepsilon(x) = x$ and if $g \in G$, diam $h_\varepsilon(g) < \varepsilon$. It will follow from Theorem 3 of [2] that S^3/G is homeomorphic with S^3.

By Theorem 1, G is definable by cubes with handles. Hence, there exist disjoint cubes with handles $C_1, \cdots, C_n$ such that cl $H_G \subset \bigcup_{i=1}^n \mathrm{Int}\, C_i \subset \bigcup_{i=1}^n C_i \subset U$. Let $W_1, \cdots, W_n$ be pairwise disjoint neighborhoods of $C_1, \cdots, C_n$ respectively such that $\bigcup_{i=1}^n W_i \subset U$. Since C_1 is a cube with (possibly 0) handles, there is a homeomorphism h_0 of S^3 onto S^3 such that $h_0(x) = x$ for $x \in S^3 - W_1$ and $h_0(C_1)$ can be written as the union of a finite number of cubes such that (1) each cube has diameter less than $\varepsilon/2$, (2) no three cubes have a point in common, and (3) the intersection of any two cubes is empty or a disk on the boundary of each. The homeomorphism h_0 can be thought of as pulling C_1 towards a 1-dimensional spine of C_1. Let $D_1, D_2, \cdots, D_k$ be the inverse images under h_0 of the disks obtained by intersecting the various cubes making up $h_0(C_1)$. We note that if a continuum in

C_1 intersects at most one D_i, then its image under h_0 has diameter less than ε. For each $i = 1, \cdots, k$, let D_i' be a subdisk of D_i such that $D_i' \subset \text{Int}\, D_i$ and $D_i \cap \text{cl}\, H_G = \text{Int}\, D_i' \cap \text{cl}\, H_G$. Let D be a disk in S^3 such that $\text{Bd}\, D \cap (\bigcup_{i=1}^n C_i) = \varnothing$ and $\bigcup_{i=1}^k D_i = D \cap (\bigcup_{i=1}^n C_i) = D \cap C_1$. Denote the punctured disk $\text{cl}\,(D - \bigcup_{i=1}^k D_i')$ by D'. Now $P_1 = P \mid D$ is a map of D into S^3/G and P_1^{-1} is a homeomorphism on a neighborhood of $P_1(\text{Bd}\, D)$. The singular set of $P_1(D)$ is contained in $P_1(\bigcup_{i=1}^k \text{Int}\, D_i')$. Let V be an open set in S^3/G containing the singular set of $P_1(D)$ and such that $P^{-1}(V) \subset (\text{Int}\, C_1) - D'$. By hypothesis there exists a disk E in $P_1(D) \cup V$ bounded by $P_1(\text{Bd}\, D)$. Let $E_1, \cdots, E_k$ be the subdisks of E bounded by $P_1(\text{Bd}\, D_1'), \cdots, P_1(\text{Bd}\, D_k')$ respectively, and let $U_1, \cdots, U_k$ be open sets whose closures lie in $P(\text{Int}\, C_1)$ such that for each $i = 1, \cdots, k$, $E_i \subset U_i$, and if $i \neq j$, $\text{cl}\, U_i \cap \text{cl}\, U_j = \varnothing$. By the proof of Theorem 2.1 of [12], each $\text{Bd}\, D_i'$ can be shrunk to a point in $P^{-1}(U_i)$. Each map can be "glued" to the annulus $\text{cl}\,(D_i - D_i')$ to obtain a map from D_i into $D_i \cup P^{-1}(U_i)$ with no singularities on $D_i - P^{-1}(\text{cl}\, U_i)$. We now apply Dehn's Lemma in S^3 to these maps to obtain disjoint disks $F_1, \cdots, F_k$ such that (1) for each i, $\text{Bd}\, D_i = \text{Bd}\, F_i$, (2) $\text{Int}\, F_i \subset \text{Int}\, C_1$, and (3) if $g \in G$, g intersects no more than one of the disks $F_1, \cdots, F_k$. Let h_1' be a homeomorphism of S^3 onto itself fixed on S^3-$\text{Int}\, C_1$ such that for each i, $h_1'(F_i) = D_i$. Let $h_1 = h_0 h_1'$. Note that if $g \in G$ and $g \subset C_1$, $\text{diam}\, h_1(g) < \varepsilon$. Let $h_2, \cdots, h_n$ be homeomorphisms such as h_1 for the sets $C_2, \cdots, C_n$. We define $h_\varepsilon : S^3 \to S^3$ by $h_\varepsilon(x) = h_1 h_2 \cdots h_n(x)$.

REMARK. If G is the upper semicontinuous decomposition of S^3 whose only nondegenerate element is a polyhedral 2-sphere, then S^3/G has the Dehn's Lemma property but S^3/G is not homeomorphic with S^3.

The essential ideas of the proof of the following theorem are so like those of the proof of Theorem 4 that we shall not include the proof here.

THEOREM 5. *If G is a point-like 0-dimensional decomposition of S^3, then S^3/G is homeomorphic with S^3 if and only if S^3/G has the map separation property.*

5. Decompositions of S^3 which yield S^3. Let S, T be polyhedral solid tori such that $S \subset \text{Int}\, T$ and let J be a polygonal center curve of S. Following a definition of Schubert [13] which was used in [7], we let $N(S, T)$ be the $\min_D\{N(J \cap D)$: where D is a polyhedral meridional disk of T and $N(J \cap D)$ is the number of points in $J \cap D\}$.

THEOREM 6. *If G is definable by cubes with one hole and S^3/G*

is homeomorphic to S^3, *then* G *is point-like.*

Proof. Let $M_1, M_2, \cdots$, be the defining sequence for G and let T_0 be a component of some M_n. By hypothesis, T_0 is a cube with one hole. Let g be a component of $\bigcap_{i=1}^{\infty} M_i$ contained in T_0. We first show that there is a defining stage M_{n+m} such that each loop in the component of M_{n+m} containing g can be shrunk to a point in T_0.

For $i = 1, 2, 3, \cdots$, let T_i be the component of M_{n+i} that contains g. Then each T_i is a cube with one hole, $T_{i+1} \subset \text{Int } T_i$, and $\bigcap_{i=1}^{\infty} T_i = g$. Suppose that there is a positive integer s such that each $T_j, j \geqq s$, is a solid torus. If the center curve of each T_{j+1} cannot be shrunk to a point in T_j, then g has nontrivial Cech cohomology, and it follows from Corollary 2 of [8] that S^3/G is not homeomorphic to S^3, contradicting our hypothesis. Hence there is an m such that the center curve of T_m can be shrunk to a point in T_0 and hence each loop in T_m can be shrunk to a point in T_0.

Suppose then that infinitely many of the T_i are not solid tori. We may suppose for convenience that each T_i is not a solid torus. By [1], each $T'_i = S^3 - \text{Int } T_i$ is a solid torus. We now have three cases.

Case I. Suppose there is an m such that $N(T'_{m-1}, T'_m) = 0$. This implies that there is a meridional disk D of T'_m such that $D \cap T'_{m-1} = \emptyset$. Then there is a cube K in T'_m such that $T'_{m-1} \subset \text{Int } K$. It then follows that each loop in $T_m (= S^3 - \text{Int } T'_m)$ can be shrunk to a point in T_0.

We now show that the remaining two cases cannot occur.

Case II. Suppose that there is a positive integer s such that $N(T'_j, T'_{j+1}) = 1$ for $j \geqq s$. Since $P(\bigcap_{i=1}^{\infty} M_i)$ is 0-dimensional there is a positive integer t and a cube K such that $P(T_{s+t}) \subset \text{Int } K \subset K \subset P$ (Int T_s). Let D'_{s+t} be a meridional disk of T'_{s+t}. Using Dehn's Lemma we may adjust $P(D'_{s+t})$ in $P(\text{Int } T'_{s+t})$ so that it is polyhedral, and it follows that $P(T'_{s+t})$ is a solid torus with the adjusted $P(D'_{s+t})$ as a meridional disk. Let J be a longitudinal simple closed curve of T'_{s+t} such that $J \subset \text{Bd } T'_{s+t}$ and J intersects Bd D'_{s+t} at just one point. Let A be an annulus with boundary components A_1 and A_2. By [13], $N(T'_s, T'_{s+t}) = 1$. Hence there is a mapping f of A into T'_{s+t} such that $f \mid A_1$ is a homeomorphism, $f(A_1) = J$, and $f(A_2) \subset T'_s$. Now $P(f(A_2))$ can be shrunk to a point missing K since it is contained in $S^3 - K$; hence $P(f(A_2))$ can be shrunk to a point in $P(T'_{s+t})$. But this implies that the longitudinal simple closed curve $P(J)$ of $P(T'_{s+t})$ can be shrunk to a point in $P(T'_{s+t})$. Hence Case II cannot occur.

Case III. Now assume there is a positive integer s such that $N(T'_j, T'_{j+1}) > 1$ for $j \geqq s$. Since each T'_j is knotted in S^3, we may use an argument similar to that used in [7] to conclude that Case III cannot occur.

These three cases now imply that there is a defining stage M_{n+m} such that each loop in the component of M_{n+m} containing g can be shrunk to a point in T_0. Since $T_0 \cap (\bigcap_{i=1}^{\infty} M_i)$ is compact, there is a defining stage $M_p(p \geqq n+m)$ such that each loop in $T_0 \cap M_p$ can be shrunk to a point in T_0. By Lemma 1 there is a cube with handles C such that $T_0 \cap M_p \subset \operatorname{Int} C \subset C \subset \operatorname{Int} T_0$. It then follows that G is definable by cubes with handles. By Bean's result [5], G is a point-like decomposition, and the proof of Theorem 6 is complete.

COROLLARY. *Let f be a mapping of S^3 onto S^3 and let $H = \mathrm{cl}(\{x : x \in S^3$ and $f^{-1}(x)$ is nondegenerate$\})$. If H is a 0-dimensional set which is definable by cubes with one hole, then for each $x \in S^3$, $S^3 - f^{-1}(x)$ is homeomorphic to E^3.*

Proof. Let $G = \{f^{-1}(x) : x \in S^3\}$. It is not hard to show that G is an upper semicontinuous decomposition of S^3 and that S^3/G is homeomorphic to S^3. Since H is definable by cubes with one hole, it follows that G is definable by cubes with one hole. By Theorem 6, G is a point-like decomposition of S^3; hence if $x \in S^3$, then $S^3 - f^{-1}(x)$ is homeomorphic to E^3.

REFERENCES

1. J. W. Alexander, *On the subdivision of 3-space by a polyhedron*, Proc. Nat. Acad. Sci. U.S.A. **10** (1924), 6-8.
2. S. Armentrout, *Decompositions of E^3 with a compact 0-dimensional set of non-degenerate elements*, Trans. Amer. Math. Soc. **123** (1966), 165-177.
3. ———, "Monotone decompositions of E^3," *Topology Seminar Wisconsin*, 1965, Princeton University Press, 1966.
4. S. Armentrout, L. L. Lininger and D. V. Meyer, "Equivalent decompositions of E^3", *Topology Seminar Wisconsin*, 1965, Princeton University Press, 1966.
5. R. J. Bean, *Decompositions of E^3 which yield E^3*, Pacific J. Math. **20** (1967), 411-413.
6. R. H. Bing, "Decompositions of E^3," *Topology of 3-manifolds and related topics*, Prentice-Hall, 1962.
7. J. M. Kister and D.R. McMillan, Jr., *Locally Euclidean factors of E^4 which cannot be imbedded in E^3*, Ann. of Math. **76** (1962), 541-546.
8. K. W. Kwun and F. Raymond, *Almost acyclic maps of manifolds*, Amer. J. Math. **86** (1964), 638-650.
9. J. Martin, "Sewings of crumpled cubes which do not yield S^3," *Topology Seminar Wisconsin*, 1965, Princeton University Press, 1966.
10. D. R. McMillan, Jr., *Cartesian products of contractible open manifolds*, Bull. Amer. Math. Soc. **67** (1961), 510-514.

11. ———, *A criterion for cellularity in a manifold,* Ann. of Math. **79** (1964), 327-337.
12. T. M. Price, *A necessary condition that a cellular upper semicontinuous decomposition of E^n yield E^n*, Trans. Amer. Math. Soc. **122** (1966), 427-435.
13. H. Schubert, *Knoten und Vollringe,* Acta Math. **90** (1953), 131-286.

Received July 25, 1966.

THE UNIVERSITY OF IOWA
THE UNIVERSITY OF GEORGIA