PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 3, 1968

BISECTION INTO SMALL ANNULI

MOSES GLASNER, RICHARD KATZ, AND MITSURU NAKAI

In a Riemannian manifold the modulus of a relatively compact set with border consisting of two sets of components is introduced to measure its magnitude from the viewpoint of harmonic functions. The existence of a subdivision into two sets each having modulus arbitrarily close to one is established.

1. Let M be a Riemannian manifold, i.e. a connected orientable C^∞ n-manifold that carries a metric tensor g_{ij}. Consider a bordered compact region $E \subset M$ whose border is the union of two nonempty disjoint sets α and β of components. We shall call the configuration (E, α, β) an *annulus*.

Let h be the harmonic function on E with continuous boundary values 0 on α and $\log \mu > 0$ on β such that

$$\int_\alpha * dh = 2\pi \ . \tag{1}$$

The number $\mu > 1$ is called the *modulus* of the annulus (E, α, β) and we set

$$\mu = \mathrm{mod}(E, \alpha, \beta) \ .$$

Let w be the *harmonic measure* of β with respect to E, i.e. the harmonic function on E with continuous boundary values 0 on α and 1 on β. By using Green's formula we obtain

$$\log \mu = \frac{2\pi}{D_E(w)} \ , \tag{2}$$

where $D_E(w)$ denotes the Dirichlet integral $\int_E dw \wedge * dw$ of w over E.

An illustration of these concepts is obtained by taking the annulus $E = \{x \mid r \leqq |x| \leqq R\}$ in n-dimensional $(n \geqq 3)$ Euclidean space. The harmonic measure of $|x| = R$ with respect to E is

$$w = \frac{|x|^{2-n} - r^{2-n}}{R^{2-n} - r^{2-n}}$$

and the modulus of $(E, |x| = r, |x| = R)$ is given by

$$\log \mu = \pi^{1-(n/2)}(2-n)\Gamma\left(\frac{n}{2}\right)(R^{2-n} - r^{2-n}) \ .$$

Note that $\mu > 1$, in a sense, measures the relative thickness of E and that $\mu \to 1$ as $R - r \to 0$.

Our result gains interest if we generalize the notion of annulus slightly. Let (E_j, α_j, β_j) $(j = 1, \cdots, m)$ be annuli such that $E_i \cap E_j = \varnothing$ for $i \neq j$. Set $E = \bigcup_{j=1}^m E_j$, $\alpha = \bigcup_{j=1}^m \alpha_j$, $\beta = \bigcup_{j=1}^m \beta_j$. Then we shall also call the configuration (E, α, β) an annulus. The modulus $\mu = \operatorname{mod}(E, \alpha, \beta)$ and the harmonic measure of E with respect to β are defined exactly as for a connected annulus. Moreover, formula (2) is valid and consequently we have

$$(3) \qquad \frac{1}{\log \mu} = \sum_{j=1}^{m} \frac{1}{\log \mu_j},$$

where $\mu_j = \operatorname{mod}(E_j, \alpha_j, \beta_j)$.

2. Let M be a noncompact Riemannian manifold throughout this number. A function which is positive and harmonic on M except for a fundamental singularity is called a *Green's function* if it majorizes no nonconstant positive harmonic functions on M. If a Green's functions exists, then M is called *hyperbolic*; otherwise it is called *parabolic*.

An increasing sequence (Ω_n) of bordered compact regions is called an *exhaustion* of M if $\cup \Omega_n = M$. Note that the configuration $(\Omega_{n+1} - \bar{\Omega}_n, \partial\Omega_n, \partial\Omega_{n+1})$ is an annulus and denote its modulus by μ_n.

The parabolicity of a noncompact Riemannian manifold M is characterized by the following

MODULAR CRITERION. *There exists an exhaustion (Ω_n) of M with $\prod \mu_n = \infty$ if and only if M is parabolic.*

In the 2-dimensional case this criterion has been established by Sario [5] and Noshiro [4] and their work can easily be generalized to arbitrary Riemannian manifolds (cf. Smith [7], Glasner [2]).

One naturally asks whether a convergent modular product has any bearing on the hyperbolicity of a manifold. The main result of this paper is that any annulus can be separated into two annuli each having modulus less than $1 + \varepsilon$. This clearly answers the question in the negative and also settles Problem 3 in Sario [6].

3. Suppose the annulus (E, α, β) has components (E_j, α_j, β_j) $(j = 1, \cdots, m)$. Let γ_j be a hypersurface in E_j such that $E_j - \gamma_j = E'_j \cup E''_j$, $E'_j \cap E''_j = \varnothing$, and $(E'_j, \alpha_j, \gamma_j)$ and $(E''_j, \gamma_j, \beta_j)$ are annuli. Set $\gamma = \bigcup_{j=1}^m \gamma_j$. We shall call γ a *bisecting surface* of (E, α, β). Also set $E' = \bigcup_{j=1}^m E'_j$ and $E'' = \bigcup_{j=1}^m E''_j$. We are now able to state the

THEOREM. *Given an annulus (E, α, β) and $\varepsilon > 0$ there exists a bisecting surface γ of (E, α, β) such that*

(4) $\quad \mathrm{mod}\,(E', \alpha, \gamma) < 1 + \varepsilon,\ \mathrm{mod}\,(E'', \gamma, \beta) < 1 + \varepsilon$.

This was established by Sario [5] for doubly connected plane regions using Koebe's distortion theorem. All proofs for the 2-dimensional case known to the authors use either a distortion theorem, in essence, or an estimate (cf. Akaza-Kuroda [1]) obtained by means of Möbius transformations (Nakai-Sario [3]) which cannot be generalized to higher dimensions. Therefore, one is led to estimate the Dirichlet integral of the harmonic measure directly and the proof presented here seems to even give a more elementary proof for the 2-dimensional case.

4. Denote by $C(a, b) = C_{x_0}(a, b)$ the Euclidean cylinder

$$\sum_{j=1}^{n-1} (x^i - x_0^i)^2 < a^2,\ x_0^n < x^n < x_0^n + b\ , \tag{5}$$

where $a, b > 0$ and $x_0 = (x_0^1, \cdots, x_0^n)$ is a fixed point. Let $\mathfrak{F}(a, b)$ be the class of C^1 functions f on $C(a, b)$ with continuous boundary values 0 on $\overline{C(a, b)} \cap \{x^n = x_0^n\}$ and 1 on $\overline{C(a, b)} \cap \{x^n = x_0^n + b\}$. Also denote by D^e the Dirichlet integral with respect to the Euclidean metric. We set s equal to the surface area of $\sum_{i=1}^{n-1}(x^i)^2 = 1$, $x^n = 0$ and state the

LEMMA. *For every* $f \in \mathfrak{F}(a, b)$,

$$D^e_{C(a,b)}(f) \geqq \frac{sa^{n-1}}{b} \tag{6}$$

and equality holds for $f_0(x) = b^{-1}(x^n - x_0^n)$.

Clearly (6) is valid with equality for f_0. To prove (6) for an arbitrary f we may assume $f \in C^1$ in a neighborhood of $\overline{C(a, b)}$. By Green's formula we have

$$D^e_{C(a,b)}(f - f_0, f_0) = \int_{\partial C(a,b)} (f - f_0)\frac{\partial f_0}{\partial n} ds = 0\ ,$$

since $f - f_0 = 0$ on the upper and lower boundary of the cylinder and $(\partial f_0/\partial n) = 0$ on the side of the cylinder. Consequently Schwarz's inequality yields

$$D^e_{C(a,b)}(f) \cdot D^e_{C(a,b)}(f_0) \geqq (D^e_{C(a,b)}(f, f_0))^2 = (D^e_{C(a,b)}(f_0))^2\ ,$$

which completes the proof.

5. We are ready to prove the main result. Take a point $x_0 \in \alpha$ and a point $y_0 \in \beta$. Let $x^1, \cdots, x^n$ be a local coordinate system at

$x_0 = (x_0^1, \cdots, x_0^n)$ valid in a neighborhood U of x_0 such that $U \cap \alpha$ is given by $x^n = x_0^n$ and x^n increases as x moves from α to E. Similarly, let $y^1, \cdots, y^n$ be a local coordinate system at $y_0 = (y_0^1, \cdots, y_0^n)$ valid in a neighborhood V of y_0 such that $V \cap \beta$ is given by $y^n = y_0^n$ and y^n increases as y moves from β to E. Choose a constant $c > 0$ so small that

$$\sqrt{g} \mid U \cup V > \sqrt{c} \tag{7}$$

and also

$$(g^{ij} \mid U \cup V)\xi_i\xi_j \geqq \sqrt{c} \sum_{i=1}^{n} (\xi_i)^2 \tag{8}$$

for every vector $(\xi_1, \cdots, \xi_n)$. Now choose $a > 0$ sufficiently small to insure that $\sum_{i=1}^{n-1}(x^i - x_0^i) < a^2$ with $x^n = x_0^n$ and $\sum_{i=1}^{n-1}(y^i - y_0^i)^2 < a^2$ with $y^n = y_0^n$ are contained in $U \cap \alpha$ and $V \cap \beta$, respectively. Finally choose $b > 0$ so that

$$0 < b < \frac{csa^{n-1}\log(1+\varepsilon)}{2\pi}\,, \tag{9}$$

$$\overline{C_{x_0}(a, b)} - \{x^n = x_0^n\} \subset E, \quad \overline{C_{y_0}(a, b)} - \{y^n = y_0^n\} \subset E$$

and

$$\overline{C_{x_0}(a, b)} \cap \overline{C_{y_0}(a, b)} = \varnothing\ .$$

Now take a bisecting surface γ of (E, α, β) subject to the requirements

$$\gamma \cap (C_{x_0}(a, b) \cup C_{y_0}(a, b)) = \varnothing$$

and

$$\gamma \supset [\overline{C_{x_0}(a, b)} \cap \{x^n = x_0^n + b\}] \cup \overline{C_{y_0}(a, b)} \cap \{y^n = y_0^n + b\}]\ .$$

Let w' (resp. w'') be the harmonic measure of γ (resp. β) with respect to E' (resp. E''). Since $E' \supset C_{x_0}(a, b)$, by using (7) and (8) we obtain

$$D_{E'}(w') > D_{C_{x_0}(a,b)}(w') \geqq cD^{e}_{C_{x_0}(a,b)}(w')\ . \tag{10}$$

Hence by using (6) and (9) we have

$$\frac{2\pi}{D_{E'}(w')} < \log(1+\varepsilon)$$

and in view of (2) we conclude that

$$\operatorname{mod}(E', \alpha, \gamma) < 1 + \varepsilon\ .$$

A similar consideration for E'' establishes (4).

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Received March 17, 1967. Moses Glasner's work was done partly while he was an NSF fellow. Mitsuru Nakai's work was sponsored in part by the U.S. Army Research Office-Durham, Grant DA-AROD-31-124-G855.

UNIVERSITY OF CALIFORNIA, LOS ANGELES AND
NAGOYA UNIVERSITY