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ON SYMMETRY IN CERTAIN GROUP ALGEBRAS

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A complex Banach algebra A with involution $x \to x^*$ is *symmetric* if Sp $(x^*x) \subset [0, \infty)$ for each $x \in A$. It is shown that (i) if A is symmetric, the algebra of all $n \times n$ matrices with elements from A is symmetric, and (ii) the group algebra of any semi-direct product of a finite group with a locally compact group having a symmetric group algebra is again symmetric.

An involution $x \rightarrow x^*$ in A is said to be *hermitian* if Sp $(x) \subset (-\infty, \infty)$ for every self-adjoint $x \in A$. In [1] R. Bonic studied the natural involution in the group algebra of certain discrete groups and raised the question: *Is the group algebra of a semi-direct product of a finite group with a discrete Abelian group necessarily symmetric?* The present work is devoted to proving the more general result that the group algebra of any semi-direct product of a finite group with a locally compact group whose group algebra is symmetric, is again symmetric. The proof in part depends upon showing that the algebra of $n \times n$ matrices with elements from a symmetric Banach algebra has a naturally defined symmetric involution. (We restrict our attention to continuous involutions.)

I am indebted to the referee for pointing out that if G is discrete, our Theorem 2 follows from a result of A. Hulanicki (Corollary 2, page 286 of [4]). Also, while it is easy to show that every symmetric involution is necessarily hermitian and that the notions are equivalent for commutative algebras, the equivalence for noncommutative algebras was an open question until quite recently. Mr. S. Shirali has announced a positive solution to this question which will be contained in his Doctoral Dissertation at Harvard University.

1. Algebras of matrices.

Let A be a Banach algebra with a continuous involution $x \rightarrow x^*$. A linear functional f on A is *positive* if $f(x^*x) \geqq 0$ for all $x \in A$. If A contains an identity e, such a functional satisfies $f(y^*x) = f(x^*y)$ for all $x, y \in A$, and if A is symmetric, then

$$\text{Sp}(x) \subset \{f(x) \mid f \text{ a positive functional, } f(e) = 1\} \tag{1.1}$$

whenever $x \in A$ and $x^*x = xx^*$. (For a proof of these and other facts about symmetric Banach algebras, see [5].) In the following, $\nu(x)$ denotes the spectral radius of x.

LEMMA 1. *Let A be a Banach algebra with identity and continuous involution, and let f be a positive linear functional on A. Then*

(i) $|f(x^*hx)| \leqq f(x^*x)\nu(h)$ *whenever* $x, h \in A$ *and* $h^* = h$.

(ii) $\left|f\left(\sum_{i=1}^{n} y_i^* x_i\right)\right|^2 \leqq f\left(\sum_{i=1}^{n} y_i^* y_i\right) f\left(\sum_{i=1}^{n} x_i^* x_i\right)$ *whenever* $x_i, y_i \in A$.

(iii) $f\left(\left(\sum_{i=1}^{n} y_i^* x_i\right)^* \left(\sum_{i=1}^{n} y_i^* x_i\right)\right) \leqq f\left(\sum_{i=1}^{n} x_i^* x_i\right) \nu\left(\sum_{i=1}^{n} y_i^* y_i\right)$ *whenever* $x_i, y_i \in A$.

Proof. For (*i*), see [5, Th. 4.5.2]. Part (ii) is a generalized Cauchy inequality and is easy to prove using the properties of f mentioned above. If the left side of (iii) is 0, there is nothing to prove. Otherwise, we use (i) and (ii) to write

$$\begin{aligned}
&\left(f\left(\left(\sum_{i=1}^{n} y_i^* x_i\right)^* \left(\sum_{j=1}^{n} y_j^* x_j\right)\right)\right)^2 \\
&= \left(f\left(\sum_{i=1}^{n} x_i^* \left(y_i \sum_{j=1}^{n} y_j^* x_j\right)\right)\right)^2 \\
&\leqq f\left(\sum_{i=1}^{n} x_i^* x_i\right) f\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} y_j^* x_j\right) y_i^* y_i \left(\sum_{j=1}^{n} y_j^* y_j\right)\right) \\
&\leqq f\left(\sum_{i=1}^{n} x_i^* x_i\right) f\left(\left(\sum_{j=1}^{n} y_j^* x_j\right)^* \left(\sum_{j=1}^{n} y_j^* x_j\right)\right) \nu\left(\sum_{i=1}^{n} y_i^* y_i\right).
\end{aligned}$$

We obtain (iii) by cancelling a common factor from both sides.

The set A_n of all $n \times n$ matrices with elements from A can be made into an algebra by defining the operations exactly as for matrices of scalars. Furthermore, if $X \in A_n$, $X = [x_{ij}]$, the mapping $X^* = [y_{ij}]$, where $y_{ij} = x_{ji}^*$, is easily seen to be an involution in A_n. (We use the same symbol for the involution in the two algebras since confusion seems unlikely.) Finally,

$$\|X\| = \max_{i=1,\cdots,n} \sum_{j=1}^{n} \|x_{ij}\|, \qquad X \in A_n,$$

is a Banach algebra norm for A_n.

THEOREM 1. *If A is symmetric then A_n is symmetric for any positive integer n.*

We note that it is sufficient to prove the theorem for the case in which A has an identity e. For otherwise, let A_e denote the algebra obtained by adjoining an identity to A. It is known [2 or 5] that A_e is symmetric if and only if A is symmetric. So, to show that A_n is symmetric we simply observe that $(A_n)_e$ is *-isomorphic to a closed *-subalgebra of $(A_e)_n$. The isomorphism here is

$$[x_{ij}] + \lambda E \leftrightarrow [x_{ij} + \lambda\delta_{ij}e] .$$

Any closed *-subalgebra of a symmetric Banach algebra is again symmetric, so it is enough to know that $(A_e)_n$ is symmetric.

LEMMA 2. *The theorem is true for* $n = 2$.

Proof. Let $X \in A_2$, $X = [x_{ij}]$. Then $X^*X = [y_{ij}]$ where

$$y_{ij} = x_{1i}^* x_{1j} + x_{2i}^* x_{2j} , \quad \text{and} \quad y_{ij} = y_{ji}^* , \qquad i,j = 1,2 .$$

To prove that A_2 is symmetric, it is enough to show that $-1 \notin \mathrm{Sp}(X^*X)$. That is, if E is the identity matrix in A_n, $E = [\delta_{ij}\, e]$, then $E + X^*X$ possesses an inverse. We will exhibit this inverse.

It is first necessary to establish the invertibility of two elements of A. As in [5], if $x \in A$ satisfies $\mathrm{Sp}(x) \subset [0, \infty)$ we write $x \geqq 0$. The symmetry of A implies [5, Lemma 4.7.10]

$$y_{11} = x_{11}^* x_{11} + x_{21}^* x_{21} \geqq 0 .$$

Thus $e + y_{11}$ has an inverse, say d_1. Next we consider $y_{22} - y_{21}d_1y_{12}$. If f is a positive linear functional on A, $f(e) = 1$, then

$$\begin{aligned} f(y_{21}d_1y_{12}) &\leqq f(y_{21}y_{12})\nu(d_1) \\ &\leqq f(y_{22})\nu(y_{11})\nu(d_1) \\ &\leqq f(y_{22}) \end{aligned}$$

from Lemma 1 (iii) and known properties of ν. It then follows that $f(y_{22} - y_{21}d_1y_{12}) \leqq 0$ and, as a consequence of (1.1),

$$y_{22} - y_{21}d_1y_{12} \geqq 0 .$$

We now know that $e + y_{22} - y_{21}d_1y_{12}$ has an inverse, say d_2. It is then an easy matter to verify that the matrix

$$\begin{bmatrix} d_1 + d_1y_{12}d_2y_{21}d_1 & -d_1y_{12}d_2 \\ -d_2y_{21}d_1 & d_2 \end{bmatrix}$$

is an inverse for $E + X^*X$. Hence A_2 is symmetric.

LEMMA 3. *The theorem holds for* $n = 2^k$, *where* k *is any positive integer.*

Proof. The proof is by induction, the case $k = 1$ being covered by Lemma 2. If we assume the result for $k = m$, then it follows for $k = m + 1$ from the fact that A_{2m+1} is *-isomorphic to $(A_{2m})_2$ by partitioning. In fact, every matrix in A_{2m+1} corresponds to a 2×2 matrix of matrices from A_{2m}, and this correspondence is easily proved to be a *-isomorphism.

Proof of Theorem 1. If n is a positive integer, choose k a positive integer so large that $m = 2^k > n$. Then A_m is symmetric, by Lemma 3, and the closed *-subalgebra of A_m consisting of all matrices with 0 in the last $(m - n)$ rows and columns is obviously *-isomorphic to A_n. It follows that A_n is itself symmetric, and the proof is complete.

2. Group algebras and semi-direct products. If F is a locally compact group, let I_F denote a left invariant Haar integral on F and let Δ_F be the corresponding modular function. Thus $J_F(x) = I_F(x \cdot 1/\Delta_F)$ is a right invariant Haar integral on F. The *group algebra* of F is the Banach space $L^1(F)$ of all complex-valued functions on F which are absolutely integrable with respect to the corresponding left Haar measure, μ_F. This algebra has an involution defined by $x^*(f) = x(f^{-1})\Delta_F(f^{-1}), f \in F$. (Here again we use *, in different positions, to denote both convolution and the involution.)

Let F and G be locally compact groups, and let $f \to \phi_f$ be a homomorphism of F into the group of automorphisms of G such that $(f, g) \to \phi_f(g)$ is a continuous mapping of $F \times G$ into G. In particular, each ϕ_f is continuous (and hence a homeomorphism). Let $S = F \times G$ and define a multiplication in S by

$$(f_1, g_1)(f_2, g_2) = (f_1 f_2, g_1 \phi_{f_1}(g_2)), \qquad (f_i, g_i) \in S, i = 1,2 .$$

Then S becomes a locally compact group which we denote by $F \times_\phi G$. We note in passing that the inverse of (f, g) is $(f^{-1}, \phi_{f^{-1}}(g^{-1}))$.

We now observe that the automorphisms ϕ_f induce a group of bounded linear transformations Φ_f of $L^1(G)$ defined by

$$\Phi_f(x) = x \circ \phi_{f^{-1}} \quad \text{for} \quad f \in F, x \in L^1(G) ,$$

and the mapping $f \to \Phi_f$ is a homomorphism of F onto this group. To see that the range of Φ_f is contained in $L^1(G)$, it is sufficient to note that each ϕ_f maps the measurable subsets of G onto measurable subsets, and that for some $\delta(f) > 0$

$$\mu_G(\phi_f(E)) = \delta(f)\mu_G(E) \tag{2.1}$$

is satisfied by every measurable set $E \subset G$. Because ϕ_f is a homeomorphism, it maps Borel sets of G onto Borel sets, and because it is also an automorphism, the measure

$$\mu_G^f(B) = \mu_G(\phi_f(B)) , \qquad B \text{ a Borel set} ,$$

is left-invariant. This measure clearly satisfies conditions (iv)-(vii) of [3, p. 194] and consequently, by the uniqueness of left Haar measure, (2.1) is satisfied for some $\delta(f) > 0$ and all Borel sets. Furthermore, the outer measure

$$\mu^*(E) = \inf \{\mu_G(A) \mid A \text{ is open, } E \subset A\}$$

induced by μ_G also satisfies

$$\mu^*(\phi_f(E)) = \delta(f)\mu^*(E)$$

for every subset $E \subset G$. It is then easy to verify (using [3, Th. 11.32] for example) that (2.1) holds for every measurable set E. In particular, if G is compact, any topological automorphism of G is measure preserving.

Clearly the mapping δ is a homomorphism of F into the multiplicative group of positive real numbers and

$$I_G(\Phi_f(x)) = I_G(x \circ \phi_{f-1}) = \delta(f)I_G(x) , \qquad x \in L^1(G) .$$

In these terms, the modular function for S can be expressed as

$$\Delta_S(f, g) = \delta(f^{-1})\Delta_F(f)\Delta_G(g) .$$

The principal concern of this paper is the case in which F is finite. In this case the functions Δ_F and δ are obviously identically 1.

THEOREM 2. *Let F be a finite group, and let G be a locally compact group whose group algebra is symmetric. Then any semidirect product $S = F \times_\phi G$ has a symmetric group algebra.*

Proof. Let $x \in L^1(S)$, $x = x(f, g)$. For each $f \in F$ the function $x_f(g) = x(f, g)$ is, by Fubini's theorem, in $L^1(G)$. Conversely, if $y_f \in L^1(G)$ for each $f \in F$ and y is defined by $y(f, g) = y_f(g)$, then $y \in L^1(S)$. In this manner $L^1(S)$ is identified with the space of all $L^1(G)$-valued functions defined on F. Now,

$$x^*(f, g) = x(f^{-1}, \phi_{f-1}(g^{-1}))\Delta_G(\phi_{f-1}(g^{-1})) = \Phi_f((x_{f-1})^*)(g)$$

and

$$\begin{aligned} x^*x(f, g) &= I_S(x^*[r, s]x[(r, s)^{-1}(f, g)]) \\ &= I_F(I_G(\Phi_r((x_{r-1})^*)(s)\Phi_r(x_{r-1f})s^{-1} g))) \\ &= \sum_{f \in F} \Phi_r((x_{r-1})^*) * \Phi_r(x_{r-1f})(g) . \end{aligned}$$

To see that $L^1(S)$ is symmetric we must show that $(-x^*x)$ is both right and left quasi-regular. For example, we must exhibit functions $y_f \in L^1(G)$ such that y as defined above satisfies $y + x^*xy - x^*x = 0$. We compute x^*xy.

$$\begin{aligned} x^*xy(f, g) &= I_S(x^*x[p, q]y[(p, q)^{-1}(f, g)]) \\ &= I_F\Big(I_G\Big(\sum_{f\in F} \Phi_r((x_{r^{-1}})^*) * \Phi_r(x_{r^{-1}p})(q)\Phi_p(y_{p^{-1}f}(q^{-1}g))\Big)\Big) \\ &= \sum_{f\in F} \sum_{p\in F} \Phi_r((x_{r^{-1}})^*) * \Phi_r(x_{r^{-1}p}) * \Phi_p(y_{p^{-1}f})(g) \ . \end{aligned}$$

Let the group F be written $F = \{f_1 = e, f_2, \cdots, f_n\}$. Then the equations which must be satisfied are

$$\begin{aligned} y_{f_i} &+ \sum_{j=1}^n \sum_{k=1}^n \Phi_{r_j}((x_{r_j^{-1}})^*) * \Phi_{r_j}(x_{r_j^{-1}p_k}) * \Phi_{p_k}(y_{p_k^{-1}f_i}) \\ &- \sum_{j=1}^n \Phi_{r_j}((x_{r_j^{-1}})^*) * \Phi_{r_j}(x_{r_j^{-1}f_i}) = 0 \ . \qquad i = 1, 2, \cdots, n \ . \end{aligned}$$

These are equivalent to

$$\begin{aligned} y_{f_i} &+ \sum_{j=1}^n \sum_{m=1}^n \Phi_{r_j}((x_{r_j^{-1}})^*) * \Phi_{r_j}(x_{r_j^{-1}f_iq_m^{-1}}) * \Phi_{f_iq_m^{-1}}(y_{q_m}) \\ &- \sum_{j=1}^n \Phi_{r_j}((x_{r_j^{-1}})^*) * \Phi_{r_j}(x_{r_j^{-1}f_i}) = 0 \ . \qquad i = 1, 2, \cdots, n \ . \end{aligned}$$

Transforming both sides by $\Phi_{f_i^{-1}}$ we obtain the equations

$$\begin{aligned} \Phi_{f_i^{-1}}(y_{f_i}) &+ \sum_{j=1}^n \sum_{m=1}^n \Phi_{f_i^{-1}r_j}((x_{r_j^{-1}})^*) * \Phi_{f_i^{-1}r_j}(x_{r_j^{-1}f_iq_m^{-1}}) * \Phi_{q_m^{-1}}(y_{q_m}) \\ &- \sum_{j=1}^n \Phi_{f_i^{-1}r_j}((x_{r_j^{-1}})^*) * \Phi_{f_i^{-1}r_j}(x_{r_j^{-1}f_i}) = 0 \ , \qquad i = 1, 2, \cdots, n \ . \end{aligned}$$

Finally,

$$\begin{aligned} \Phi_{f_i^{-1}}(y_{f_i}) &+ \sum_{k=1}^n \sum_{m=1}^n \Phi_{s_k}((x_{s_k^{-1}f_i^{-1}})^*) * \Phi_{s_k}(x_{s_k^{-1}q_m}) * \Phi_{q_m}(y_{q_m^{-1}}) \\ &- \sum_{k=1}^n \Phi_{s_k}((x_{s_k^{-1}f_i^{-1}})^*) * \Phi_{s_k}(x_{s_k^{-1}}) = 0 \ , \qquad i = 1, 2, \cdots, n \ . \end{aligned} \tag{2.2}$$

It is evidently enough to determine the functions $\Phi_{f_i^{-1}}(y_{f_i})$, for from them the y_{f_i} can be obtained on transforming by Φ_{f_i}. Consider the matrix $A = [a_{ij}]$ of elements from $L^1(G)$ defined by

$$a_{ij} = \Phi_{s_i}(x_{s_i^{-1}f_j^{-1}}) \ , \qquad i, j = 1, 2, \cdots, n \ .$$

Since $L^1(G)$ is symmetric we know, by Theorem 1, that $-A^*A$ has a quasi-inverse, say $C = [c_{ij}]$ with $c_{ij} \in L^1(G)$. It follows from $C + A^*AC - A^*A = 0$ that

$$c_{i1} + \sum_{k=1}^n \sum_{m=1}^n a_{ki}^* a_{ki} c_{m1} - \sum_{k=1}^n a_{ki}^* a_{k1} = 0 \ , \qquad i = 1, 2, \cdots, n \ .$$

Thus $\Phi_{q_m}(q_m^{-1}) = c_{m1}$, $m = 1, 2, \cdots, n$ is a solution of the equations (2.2). A left quasi-inverse for $(-x^*x)$ can be computed in a similar manner. Hence $L^1(S)$ is symmetric.

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