PACIFIC JOURNAL OF MATHEMATICS
Vol. 25, No. 3, 1968

THE STRUCTURE SPACE OF A COMMUTATIVE LOCALLY M-CONVEX ALGEBRA

R. M. BROOKS

If A is a commutative Banach algebra with identity, then the sets $\mathscr{M}$ (all maximal ideals), $\mathscr{M}_c$ (all closed maximal ideals), $\mathscr{M}_1$ (kernels of nonzero C-valued homomorphisms of A), and $\mathscr{M}_0$ (kernels of nonzero continuous C-valued hommorphisms of A) coincide. If A is a commutative complete locally m-convex algebra, one has only $\mathscr{M}_c = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \mathscr{M}$, and the containments can be proper. Our goal is to investigate $\mathscr{M}$ and its relationship to $\mathscr{M}_0$; specifically (1) to give a description of $\mathscr{M}(A)$ in terms of A and $\mathscr{M}_0(A)$ which is valid for at least the class of F-algebras, (2) to determine when $\mathscr{M}(A)$ is one of the standard compactifications (Wallman, Stone-Čech) of $\mathscr{M}_0(A)$.

For many locally m-convex algebras, especially algebras of functions, one can determine $\mathscr{M}_0$. However, descriptions of $\mathscr{M}$ and its relationship to $\mathscr{M}_0$ seem to be limited to special cases; for example, Hewitt's description of $\mathscr{M}(C(X))$ [5] and Kakutani's description of $\mathscr{M}$ for the algebra of analytic functions in the unit disc [6]. We show that a commutative complete locally m-convex algebra A generates a lattice $\mathscr{L}$ on $\mathscr{M}_0$, and that if we impose a rather natural restriction on A, then $\mathscr{M}$ is the space of ultrafilters of $\mathscr{L}$. We give necessary and sufficient conditions on A in order that (1) $\mathscr{M}$ is the Wallman compactification of ($\mathscr{M}_0$, hull-kernel), (2) $\mathscr{M}$ is the Wallman compactification of ($\mathscr{M}_0$, Gelfand). In the second case, we show that $\mathscr{M} = \beta\mathscr{M}_0$ and obtain a correspondence between $\mathscr{M}_1$ and the A-realcompactification of $\mathscr{M}_0$.

We then specialize to F-algebras and show (1) F-algebras always satisfy the condition imposed in the general situation, (2) $\mathscr{M}$ is the Wallman compactification of ($\mathscr{M}_0$, hull-kernel), and (3) $\mathscr{M} = \beta\mathscr{M}_0$, whenever the algebra is regular.

1. The general case.

A locally m-convex algebra (hereafter LMC algebra) is a locally convex Hausdorff topological algebra A whose topology is given by a family of pseudonorms (submultiplicative, convex, symmetric functionals). For the basic properties of these algebras the reader is referred to [1] or [9]. In this paper we shall restrict our attention to complete algebras with identity element 1. If λ is a complex number we shall write "λ" for "$\lambda \cdot 1$".

The *structure space* of A is the set $\mathscr{M}$ of all maximal ideals of

A, endowed with the hull-kernel ($hk-$) topology. This space is always compact and satisfies the T_1 separation axiom. The *spectrum* of A is the set $\mathscr{M}_0$ of all closed maximal ideals of A.

DEFINITION 1.1. If $S \subseteqq A$, $F \subseteqq \mathscr{M}_0$, $G \subseteqq \mathscr{M}$, $x \in A$, then

(i) $H(S) = \{M \in \mathscr{M} : S \subseteqq M\}$.

(ii) $h(S) = \{M \in \mathscr{M}_0 : S \subseteqq M\}$.

(iii) $kF(=k(F)) = \bigcap \{M \in \mathscr{M}_0 : M \in F\} = \{x \in A : x \in M$ for each $M \in F\}$.

(iv) $K(G) = \bigcap \{M \in \mathscr{M} : M \in G\} = \{x \in A : x \in M$ for each $M \in F\}$.

(v) $H(x) = H(\{x\})$, $h(x) = h(\{x\})$.

The hull-kernel topology is defined in terms of the closure operator:

$$\mathrm{Cl}_{\mathscr{M}}(F) = HK(F) \, , \tag{1.1}$$

or

$$\mathrm{Cl}_{\mathscr{M}}(F) = \bigcap \{H(x) : F \subseteqq H(x)\}, \text{ for each } F \subseteqq \mathscr{M}. \tag{1.2}$$

A simple computation yields

THEOREM 1.1. *The closure operator on $\mathscr{M}_0$ which defines the relative hull-kernel topology on $\mathscr{M}_0$ is given by*

$$\mathrm{Cl}_{\mathscr{M}_0}(F) = hk(F) \tag{1.3}$$

or

$$\mathrm{Cl}_{\mathscr{M}_0}(F) = \bigcap \{h(x) : F \subseteqq h(x)\}, \text{ for each } F \subseteqq \mathscr{M}_0 \, . \tag{1.4}$$

The spectrum can also be endowed with a second natural topology. If $M \in \mathscr{M}_0$, then M is the kernel of a unique continuous homomorphism of A onto C [9, p. 11]. We identify M and the corresponding homomorphism, denoting the value of the homomorphism at an element x of A by $M(x)$, and endow $\mathscr{M}_0$ with the relative weak $-(w^*-)$ topology from A^*, the conjugate space of A. This topology is the weakest such that all of the functions $\hat{x} : \mathscr{M}_0 \to C$ defined by $\hat{x}(M) = M(x)$ for each $x \in A$ are continuous. We state without proof the basic properties of the mapping $x \to \hat{x}$ of A into $C(\mathscr{M}_0)$ (cf [9, Props. 7.3, 8.1, and 9.2] and [4, Ex. 7M]).

THEOREM 1.2. *The mapping $x \to \hat{x}$ is a homomorphism of A onto a subalgebra $\hat{A}$ of $C(\mathscr{M}_0)$ which contains the constant functions and separates the points of $\mathscr{M}_0$. The kernel of this homomorphism is the radical $\mathscr{R}(A)$ of A, and $\mathscr{R}(A) = \bigcap \{M : M \in \mathscr{M}_0\} = \bigcap \{M : M \in \mathscr{M}\} = \{x \in A : (1 + ax)$ is regular (invertible) for each $x \in A\}$ is a closed ideal in A. Hence, $\mathscr{M}_0$ is dense in $\mathscr{M}$.*

DEFINITION 1.2. A commutative LMC algebra A with identity is called *regular* provided that for each w^*-closed subset F on $\mathscr{M}_0$ and each point $M \in \mathscr{M}_0 - F$ there exists an element x of A such that $\hat{x}(M) = 1$ and $\hat{x} = 0$ on F (equivalently, $x \in kF - M$).

THEOREM 1.3. (Proposition II, p. 223 of Naimark [7]). *The hull-kernel topology on $\mathscr{M}_0$ is weaker than the w^*-topology. They agree if, and only if, A is regular.*

DEFINITION 1.3. A commutative LMC algebra A with identity is called *w^*-normal* (respectively, *hk-normal*) provided that for each pair F_1, F_2 of disjoint, w^*-closed (respectively, hk-closed) subsets of $\mathscr{M}_0$ there exists $x \in A$ such $\hat{x} = 0$ on F_1 and $\hat{x} = 1$ on F_2.

We note that if A is w^*-normal, then A is regular, the two topologies on $\mathscr{M}_0$ agree and $\mathscr{M}_0$ is a normal space. If A is hk-normal, we cannot conclude that $\mathscr{M}_0$ with the hk-topology is normal; since, in general, the elements of $\hat{A}$ are not hk-continuous.

If $\{x_1, \cdots, x_n\} \subseteqq A$, we write $h(x_1, \cdots, x_n)$ instead of $h(\{x_1, \cdots, x_n\})$, and denote the ideal in A generated by this family by $(x_1, \cdots, x_n)$. We note that $h(x_1, \cdots, x_n) = h((x_1, \cdots, x_n))$ and that $h(x_1, \cdots, x_n) = \bigcap \{h(x_i): i = 1, \cdots, n\}$ and $H(x_1, \cdots, x_n) = \bigcap \{H(x_i): i = 1, \cdots, n\}$.

THEOREM 1.4. *The first three statements about the finite family $\{x_1, \cdots, x_n\} \subseteqq A$ are equivalent. Each of these implies the fourth.*

(i) $h(x_1, \cdots, x_n) = \phi$ *implies* $(x_1, \cdots, x_n) = A$.

(hH)(ii) $h(x_1, \cdots, x_n) = \phi$ *implies* $H(x_1, \cdots, x_n) = \phi$.

(iii) $H(x_1, \cdots, x_n) = \mathrm{Cl}_{\mathscr{M}}\, h(x_1, \cdots, x_n)$.

(iv) $\mathrm{Cl}_{\mathscr{M}}\, h(x_1, \cdots, x_n) = \bigcap \{\mathrm{Cl}_{\mathscr{M}}\, h(x_i): i = 1, \cdots, n\}$.

Proof. (i) if, and only if, (ii): $H(x_1, \cdots, x_n) = \phi$ if, and only if, $(x_1, \cdots, x_n)$ is not contained in any maximal ideal if, and only if, $(x_1, \cdots, x_n) = A$.

(i) implies (iii):

$$\begin{aligned}\mathrm{Cl}_{\mathscr{M}}\, h(x_1, \cdots, x_n) &= Hk(h(x_1, \cdots, x_n)) = HK(h(x_1, \cdots, x_n)) \\ &\subseteqq HK(H(x_1, \cdots, x_n)) = H(x_1, \cdots, x_n)\,.\end{aligned}$$

Suppose $M \notin Hk(h(x_1, \cdots, x_n))$. Then $kh(x_1, \cdots, x_n) + M = A$ and there exist $z \in kh(x_1, \cdots, x_n)$, $w \in M$ such that $z + w = 1$. Then $h(z, w) = \phi$ and $h(x_1, \cdots, x_n, w) = \phi$ (since $h(x_1, \cdots, x_n) \subseteqq h(z)$). By (i) we have $(x_1, \cdots, x_n, w) = A$ and $(w \in M)$ at least one $x_i \notin M$. Thus,

$$M \notin H(x_1, \cdots, x_n)\,.$$

(iii) implies (ii): obvious.

(iii) implies (iv): clear, since $H(x_1, \cdots, x_n) = \bigcap_{i=1}^n H(x_i)$.

We consider throughout the remainder of this section only algebras which satisfy condition (hH) ((ii) of Theorem 1.4). We note the following formulation of (hH). If $\{a_1, \cdots, a_n\} \subseteqq A$ we consider the equation $\sum_{i=1}^n a_i x_i = 1$ and ask for condition on A which insure solvability in A. (hH) is the assumption that the vacuousness of $h(a_1, \cdots, a_n)$ is sufficient. Arens [2] gave sufficient conditions in terms of the solvability of certain related equations in Banach algebras. We shall show below that in F-algebras the vacuousness of $h(a_1, \cdots, a_n)$ is sufficient for the solvability of the equation in A.

THEOREM 1.5. *Suppose F_1 and F_2 are disjoint subsets of $\mathscr{M}_0$. The following statements are equivalent.*

(i) $\mathrm{Cl}_{\mathscr{M}} F_1 \cap \mathrm{Cl}_{\mathscr{M}} F_2 = \phi$.

(ii) *There exists $x \in A$ such that $\hat{x} = 0$ on $F_1, \hat{x} = 1$ on F_2.*

(iii) $kF_1 + kF_2 = A$.

Proof. (i) if, and only if, (iii): $\mathrm{Cl}_{\mathscr{M}} F_i = HkF_i, i = 1, 2,$ and $HkF_1 \cap HkF_2 = H(kF_1 + kF_2)$. The equivalence follows (kF is always a closed ideal in A for $F \subseteqq \mathscr{M}_0$).

(ii) if, and only if, (iii): If (iii) $kF_1 + kF_2 = A$ we choose $x \in kF_1$, $y \in kF_2$ such that $x + y = 1$. Then $\hat{x} = 0$ on F_1 and $\hat{x} = 1$ on F_2. The converse is immediate.

COROLLARY 1.5. *Disjoint w^*-closed (respectively, hk-closed) subsets of $\mathscr{M}_0$ have disjoint closures in $\mathscr{M}$ if, and only if, A is w^*-normal (respectively, hk-normal).*

We now give our description of $\mathscr{M}$, assuming A satisfies (hH). The result is stated in terms of a lattice compactification of $\mathscr{M}_0$. The basic facts about these compactifications may be found in [12] and [13], and in the form used here in [3].

DEFINITION 1.4. A lattice $\mathscr{L}$ (with respect to $\cup$ and $\cap$) of subsets of $\mathscr{M}_0$ is called an *α-lattice* provided that for each $B \in \mathscr{L}$ and $M \in \mathscr{M}_0 - B$ there exists $D \in \mathscr{L}$ such that $M \in D, B \cap D = \phi$. $\mathscr{L}$ is called a *β-lattice* provided that for each pair M_1, M_2 of distinct points of $\mathscr{M}_0$ there exists $B \in \mathscr{L}$ such that $M_1 \in B, M_2 \in \mathscr{M}_0 - B$. $\mathscr{L}$ is said to be *normal* provided that for each pair B, D of disjoint members of $\mathscr{L}$ there exists a pair B_1, D_1 of elements of $\mathscr{L}$ such that $B \subseteqq B_1, D \subseteqq D_1, B \cap D_1 = \phi = B_1 \cap D$, and $B_1 \cup D_1$ belongs to every ultrafilter in $\mathscr{L}$ (in the presence of (α), this is equivalent to the statement that $B_1 \cup D_1 = \mathscr{M}_0$).

$w\mathscr{L} (= w(\mathscr{M}_0, \mathscr{L}))$ is the set of all ultrafilters in $\mathscr{L}$. For each

$E \in \mathscr{L}$ we define $C(E) = \{\mathscr{U} \in w\mathscr{L}: E \in \mathscr{U}\}$ and define a topology on $w\mathscr{L}$ by taking the family $\{C(E): E \in \mathscr{L}\}$ as a base for the closed sets ($E \rightarrow C(E)$ is a lattice homomorphism of $\mathscr{L}$ into the power set of $w\mathscr{L}$). The space $w\mathscr{L}$ is always compact and satisfies the T_1 separation axiom. The assumption that $\mathscr{L}$ is an α-lattice is equivalent to assuming that for each $M \in \mathscr{M}_0$ the family $\mathscr{U}_M = \{E \in \mathscr{L}: M \in E\}$ is an ultrafilter in $\mathscr{L}$. If $\mathscr{L}$ is an α-lattice then the function $\varphi: \mathscr{M}_0 \rightarrow w\mathscr{L}$ defined by $\varphi(M) = \mathscr{L}_M$ maps $\mathscr{M}_0$ onto a dense subspace of $w\mathscr{L}$. If $\mathscr{L}$ is an α-lattice, then (β) is equivalent to the statement that φ is one-to-one. Normality of $\mathscr{L}$ is equivalent to the statement that $w\mathscr{L}$ is Hausdorff. If we fix a topology $\mathscr{T}$ on $\mathscr{M}_0$ then φ is continuous (assuming (α)) if, and only if, each element of $\mathscr{L}$ is $\mathscr{T}$-closed, and φ is a homeomorphism if, and only if, such element of $\mathscr{L}$ is $\mathscr{T}$-closed, $\mathscr{L}$ is a β-lattice, and $\mathscr{L}$ forms a base for the $\mathscr{T}$-closed subsets of $\mathscr{M}_0$. (For proofs, see Theorems 2.5 and 2.7 of [3]). Finally, if $E \in \mathscr{L}$, then $(\varphi E)^-$ (the closure in $w\mathscr{L}$ will be denoted by "$-$") $= C(E)$, and for any subset F of $\mathscr{M}_0$, $(\varphi F)^- = \bigcap \{C(A): F \subseteqq A\}$ (Theorem 2.6 of [3]).

LEMMA 1.6. *The family* $\mathscr{L} = \{h(x_1, \cdots, x_u): \{x_1, \cdots, x_u\} \subseteqq A\}$ *is an* $\alpha-$, β*-lattice of hk-closed subsets of* $\mathscr{M}_0$ *which forms a base for the hk-closed sets. Thus, the mapping* $\varphi(M \rightarrow \mathscr{U}_M)$ *is a homeomorphism of* $(\mathscr{M}_0, hk)$ *onto a dense subspace of* $w\mathscr{L}$.

Proof. The family $\mathscr{L}$ is closed under finite intersections, since $h(x_1, \cdots, x_n) = \bigcap_{i=1}^n h(x_1, \cdots, x_n)$ for each finite family $\{x_1, \cdots, x_n\}$ in A. Moreover, $h(x_1, \cdots, x_n) \cup h(y_1, \cdots, y_n) = \bigcap \{h(x_i y_i): i = 1, \cdots, n; j = 1, \cdots, m\}$, the latter being an element of $\mathscr{L}$. Thus, $\mathscr{L}$ is a lattice on $\mathscr{M}_0$ consisting of hk-closed sets which forms a base for the hk-closed sets of $\mathscr{M}_0$.

If $M \in \mathscr{M}_0 - h(x_1, \cdots, x_n)$, then $(x_1, \cdots, x_n) + M = A$ and there exists $z \in M$ such that $\hat{z} = 1$ on $h(x_1, \cdots, x_n)$. But this implies $M \in h(z)$ and $h(z) \cap h(x_1, \cdots, x_n) = \phi$. Thus, $\mathscr{L}$ is an α-lattice. That $\mathscr{L}$ is an β-lattice is immediate.

We note that in general $\mathscr{L}$ is not a normal lattice. For example, if A is the algebra of all functions on the open unit disc D to the complex plane which are analytic on D, then $\mathscr{M}_0$ and D are in a natural one-to-one correspondence. In this case, $\mathscr{L}$ is the lattice of all discrete subsets of D plus the set D itself. It is clear that $\mathscr{L}$ is not normal.

THEOREM 1.6. $\mathscr{M}$ *is* $w\mathscr{L}$ (*i.e., there exists a homeomorphism* σ *of* $\mathscr{M}$ *onto* $w\mathscr{L}$ *such that* $\sigma(M) = \varphi(M)$ *for each* $M \in \mathscr{M}_0$).

Proof. For each $M \in \mathscr{M}$ we let $\sigma(M)$ be the subfamily of $\mathscr{L}$ consisting of all $E \in \mathscr{L}$ such that $M \in \mathrm{Cl}_{\mathscr{M}} E$. It is clear that $\sigma(M)$ is a filter in $\mathscr{L}$. We use the criterion "A filter $\mathscr{F}$ in $\mathscr{L}$ is an ultrafilter if, and only if, for each $E \in \mathscr{L} - \mathscr{F}$ there exists $F \in \mathscr{F}$ such that $E \cap F = \phi$" ([12, p. 105]) to establish that $\sigma(M)$ is an ultrafilter. If $E \in \mathscr{L} - \sigma(M)$, then $E = h(x_1, \cdots, x_n)$ for some family $\{x_1, \cdots, x_n\}$ in A and $M \notin \mathrm{Cl}_{\mathscr{M}} h(x_1, \cdots, x_n) = H(x_1, \cdots, x_n)$. We have $\{M\} = \bigcap \{H(y_1, \cdots, y_n): M \in H(y_1, \cdots, y_m)\}$ and this family is a descending family of compact sets of $\mathscr{M}$ whose intersection is contained in the open set $\mathscr{M} - H(x_1, \cdots, x_n)$. Thus, there exists a family $\{y_1, \cdots, y_m\}$ in A such that $M \in H(y_1, \cdots, y_m) \subseteqq \mathscr{M} - H(x_1, \cdots, x_n)$. But then $h(y_1, \cdots, y_m) \in \sigma(M)$ and is disjoint from $h(x_1, \cdots, x_n)$.

If $\mathscr{U}$ is an ultrafilter in $\mathscr{L}$, we let $\mathscr{U}^* = \{\mathrm{Cl}_{\mathscr{M}} E: E \in \mathscr{U}\}$. Then, $\mathscr{U}^*$ is a descending family of compact subsets of $\mathscr{M}$ and has a nonempty intersection. It is easily verified that there is a unique element M of in $\cap \mathscr{U}^*$ and that $\sigma(M) = \mathscr{U}$. It follows that the mapping σ is one-to-one, onto, and that for each $M \in \mathscr{M}_0$ $\sigma(M) = \mathscr{U}_M = \varphi(M)$. The equality $\sigma[H(x_1, \cdots, x_n)] = C[h(x_1, \cdots, x_n)]$ for each finite family $\{x_1, \cdots, x_n\}$ in A yields the fact that σ is a homeomorphism.

We state without proof the following theorem on lattice compactifications (cf. [3, Th. 3.1]).

THEOREM 1.7. *If $\mathscr{L}'$ is a second α-lattice on $\mathscr{M}_0$, $\mathscr{L} \subseteqq \mathscr{L}'$, and ψ is the mapping of $\mathscr{M}_0$ into $w\mathscr{L}'$, then the following statements are equivalent.*

(i) *If $F_1, F_2 \in \mathscr{L}'$, then $F_1 \cap F_2 = \phi$ if, and only if, $(\varphi F_1)^- \cap (\varphi F_2)^- = \phi$.*

(ii) *If $F_1, F_2 \in \mathscr{L}'$, then $\varphi(F_1 \cap F_2)^- = (\varphi F_1)^- \cap (\varphi F_2)^-$.*

(iii) *$w\mathscr{L}' = w\mathscr{L}$ (i.e., there exists a homeomorphism τ of $w\mathscr{L}'$ onto $w\mathscr{L}$ such that $\tau\varphi(M) = \psi(M)$ for each $M \in \mathscr{M}_0$).*

We apply this theorem to our situation. We identity $\mathscr{M}$ and w$\mathscr{L}$ here and let $\mathscr{C}(hk)$ and $\mathscr{C}(w^*)$ denote the lattices of all hk-closed subsets of $\mathscr{M}_0$ and all w^*-closed subsets of $\mathscr{M}_0$, respectively. $W(\mathscr{M}_0, \mathscr{T})$ denotes the Wallman compactification of the topological space $(\mathscr{M}_0, \mathscr{T})$.

COROLLARY 1.7. *$\mathscr{M} = W(\mathscr{M}_0, hk)$ if, and only if, A is hk-normal. If A is regular, then $(\mathscr{M}_0, w^*)$ is embedded homeomorphically in $\mathscr{M}$ and $\mathscr{M} = W(\mathscr{M}_0, w^*)$ if, and only if, A is normal. In this case, $\mathscr{M}$ is Hausdorff and $\mathscr{M} = \beta\mathscr{M}_0$.*

Proof. The first statement is clear in view of Theorem 1.7 and Corollary 1.5, where we let $\mathscr{L}' = \mathscr{C}(hk)$. The second statement follows from the same two theorems, where $\mathscr{L}' = \mathscr{C}(w^*)$. Finally, if A is normal, then $\mathscr{M}_0$ is a normal space and $W(\mathscr{M}_0)$ (we suppress $\mathscr{T}$ since the topologies agree) is Hausdorff [13, p. 119], hence $\mathscr{M} = W(\mathscr{M}_0) = \beta\mathscr{M}_0$ (cf. [7, Exercises 5P and 5R] or [3, Th. 3.2]).

EXAMPLE 1.1. We give an example to show first that in general a commutative LMC algebra can be completely regular, but not normal (the concepts are equivalent for F-algebras, see § 2), and secondly that $\mathscr{M}$ may be $\beta\mathscr{M}_0$ while A is not normal. We let Ω be the first uncountable ordinal and w the first ordinal with countably many prodecessors, Ω' is the set of all ordinals up to and including Ω, w' the set of all ordinals up to and including w, $T' = \Omega' x w'$ with the product topology (each of Ω', w' being endowed with the order topology), and $T = T' - \{(\Omega, w)\}$. T is a locally compact Hausdorff space which is not normal and $\beta T = T'$ (cf. [4, pp. 123-124]). We let $A = C(T)$ with the compact-open topology. Then $(\mathscr{M}_0, w^*) = T$, $w^* = hk$ on T and $\mathscr{M} = \beta T = T'$. But A is not normal.

We next consider for a normal algebra A satisfying the condition (hH) the problem of identifying the subspace of $\mathscr{M}$ which consists of the maximal ideals of A which are kernels of (possibly discontinuous) homomorphisms of A onto C. We denote this subspace by $\mathscr{M}_1$.

Since A is normal, $\mathscr{M} = \beta\mathscr{M}_0$ and for each $x \in A$ the function $\hat{x}$ on $\mathscr{M}_0$ is a continuous mapping of $\mathscr{M}_0$ into the one-point compactification $\boldsymbol{C}^* = \boldsymbol{C} \cup \{\infty\}$ of $\boldsymbol{C}$. Thus $\hat{x}$ has an extension x^*, a $\boldsymbol{C}^*$-valued continuous function on $w\mathscr{L}(=\beta\mathscr{M}_0)$. Discussions of this extension and of the realcompactification of a space are found in Chapters 7 and 8 of [4]. The realcompactification of $\mathscr{M}_0$, $\upsilon\mathscr{M}_0$, is the subspace of $\beta\mathscr{M}_0$ consisting of all $\mathscr{V} \in \beta\mathscr{M}_0$ such that for each $z \in C(\mathscr{M}_0)$ $\mathscr{V} \in z^{*-1}(\boldsymbol{C})$, i.e. z^* does not take on the value ∞ at $\mathscr{V}$, where z^* is the extension of the mapping $z\colon \mathscr{M}_0 \to \boldsymbol{C}^*$ to $\beta\mathscr{M}_0$.

DEFINITION 1.5. $\upsilon_A\mathscr{M}_0$ *(the A-realcompactification of* $\mathscr{M}_0$) $= \{\mathscr{U} \in w\mathscr{L}\colon x^*(\mathscr{U}) \in \boldsymbol{C}$ *for each* $x \in A\}$.

THEOREM 1.8. *If* $\mathscr{U} \in \upsilon_A\mathscr{M}_0$ *and* $\mathscr{U} = \sigma(M)$, *then* $M = \{x \in A\colon x^*(\mathscr{U}) = 0\}$.

Proo. If $M \in \mathscr{M}$ and $\sigma(M) = \mathscr{U} \in \upsilon_A\mathscr{M}_0$, then the set $I = \{x \in A\colon x^*(\mathscr{U}) = 0\}$ is an ideal in A. Moreover, if $x \in M$, then $\mathscr{U} \in C[h(x)] = h(x)^-$ and since x^* is continuous on $w\mathscr{L}$ and agrees with $\hat{x}$ on $\mathscr{M}_0$, $\hat{x}^*(\mathscr{U}) = 0$. Therefore, $M \subseteq I$ and $I \neq A (1 \notin I)$. Hence, $M = I$.

THEOREM 1.9. *The restriction of the mapping* $\sigma\colon \mathscr{M} \to w\mathscr{L}$ *to*

$\mathscr{M}_1$, *is a homeomorphism of* $\mathscr{M}_1$ *onto* $\upsilon_A\mathscr{M}_0$.

Proof. If $\mathscr{U} \in \upsilon_A \mathscr{M}_0$ and $\mathscr{U} = \sigma(M)$, then the mapping $x \to x^*(\mathscr{U})$ is a homomorphism of A onto $\boldsymbol{C}$ with kernel M and $M \in \mathscr{M}_1$.

If $M \in \mathscr{M}_1$ and $\mathscr{U} = \sigma(M)$, then for each $x \in A$ there exists $\lambda \in C(\lambda = M(x))$ such that $x - \lambda \in M$. We fix $x \in A$ and the corresponding (unique) $\lambda \in C$. If $x - \lambda \in M$, then $M \in H(x - \lambda)$ and $\mathscr{U} \in C[h(x - \lambda)]$. This implies $(x - \lambda)*(\mathscr{U}) = 0$. Since $\lambda*(\mathscr{U}) = \lambda \in C$, we have $x*(\mathscr{U}) = [(x - \lambda) + \lambda]*(\mathscr{U}) = (x - \lambda)*(\mathscr{U}) + \lambda*(\mathscr{U}) = \lambda \in C$ and $\mathscr{U} \in \upsilon_A \mathscr{M}_0$.

We wish to acknowledge here our indebtedness to Donald L. Plank of the Case Western Reserve University who communciated to the author theorems analogous to 1.8 and 1.9 for a real algebra A of functions on a completely regular space X satisfying: $BC(X) \subseteqq A \subseteqq C(X)$, where $BC(X)$ is the algebra of all bounded real-valued functions on X to $\boldsymbol{R}$.

2. A special case. We consider in this section the special case: A is a commutative F-algebra with identity 1 — a complete LMC algebra whose topology is given by a countably family of pseudonorms. In this case we can assume that the family $\{p_n\}_{n=0}^{\infty}$ satisfies: $p_n(x) \leqq p_{n+1}(x)$ for each $n \geqq 0$ and each $x \in A$. The fact the F-algebras are inverse limits of Banach algebras is important for our purposes. We let $N_k = \{x \in A: p_k(x) = 0\}$, Π_k the natural map of A onto A/N_k and A_k the completion of A/N_k with respect to the norm defined by $\| \Pi_k x \| = p_k(x)$. Each A_n is a commutative Banach algebra with identity. For each $n \geqq 0$ there is a norm-decreasing homomorphism Π_n^{n+1} of A_{n+1} onto a dense subalgebra of A_n which is defined on A/N_{n+1} by $\Pi_n^{n+1}(\Pi_{n+1}x) = \Pi_n x$ and extended to A_{n+1}. For $n \leqq m$, $\Pi_n^m: A_m \to A_n$ is defined by the obvious composition. The resulting family of algebras and homomorphisms is an inverse limit system and A is isomorphic and pseudo-isometric to the inverse limit of this system. An important consequence of this is the following fact. If $\{\xi_n\}_{n=0}^{\infty}$ is a sequence where $\xi_n \in A_n$ and $\Pi_n^m \xi_m = \xi_n$ whenever $n \leqq m$, then there exists $x \in A$ such that $\Pi_n x = \xi_n$ for each $n \geqq 0$. For details of this construction and the basic facts about such systems, the reader is referred to [9].

We state without proof two theorems, the first is just Theorem 4.2 of [2] in our terminology, the second is immediate.

THEOREM 2.1. *Suppose* $\{a_1, \cdots, a_m\}$ *is a family of elements of* A *such that* $(\Pi_n a_1, \cdots, \Pi_n a_m) = A_n$ *for each* $n \geqq 0$. *Then* $(a_1, \cdots, a_m) = A$.

THEOREM 2.2. *If* $\{\xi_1, \cdots, \xi_m\}$ *is a finite family in* A_n *and* $\hat{\xi}_1, \cdots, \hat{\xi}_m$ *have no common zeros on* $\mathscr{M}(A_n)$ *(the structure space of* A_n*), then* $(\xi_1, \cdots, \xi_m) = A_n$.

The spectrum $\mathscr{M}_0$ of A has the following structure. $\mathscr{M}_0 = \bigcup \{\mathscr{M}^k : k = 0, 1, 2, \cdots\}$, where each $\mathscr{M}^k$ is homeomorphic to $\mathscr{M}(A_k)$, the structure space of A_k. The homeomorphism σ_k of $\mathscr{M}(A_k)$ into $\mathscr{M}_0$ is defined by $[\sigma_k(M^k)](x) = M^k(\Pi_k x)$ for each $M^k \in \mathscr{M}(A_k)$ and $x \in A$.

THEOREM 2.3. *If A is a commutative F-algebra with identity, then A satisfies the condition (hH).*

Proof. We fix a family $\{x_1, \cdots, x_n\}$ in A satisfying $h(x_1, \cdots, x_n) = \phi$ and show $H(x_1, \cdots, x_n) = \phi$. We need only show that for each $k \geqq 0$ the family $\{\Pi_k x_1, \cdots, \Pi_k x_n\}$ generates the improper ideal A_k. We fix $k \geqq 0$. For each i we have $(\Pi_k x_i)(M^k) = x_i(\sigma_k M^k)$. Therefore, the family $\{(\Pi_k x_1)\hat{}, \cdots, (\Pi_k x_n)\hat{}\}$ has a common zero on $\mathscr{M}(A_k)$ if, and only if, the intersection of $\mathscr{M}^k$ and $h(x_1, \cdots, x_n)$ is nonempty. We have assumed that $h(x_1, \cdots, x_n)$ is empty. Thus, Theorem 2.2 implies $(\Pi_k x_1, \cdots, \Pi_k x_n) = A_k$ for each $k \geqq 0$, and we obtain $(x_1, \cdots, x_n) = A$.

Thus, F-algebras always satisfy the conditions of Theorem 1.4 and $\mathscr{M} = w\mathscr{L}$. We next extend Theorem 2.1 to pin down further the space $\mathscr{M}$. We note that Theorem 2.4 is immediate for Banach algebras (since $\mathscr{M}_0 = \mathscr{M}$) and false for commutative LMC algebras in general (cf. Example 1.1 above).

THEOREM 2.4. *If I_1 and I_2 are closed ideals in A and if $h(I_1) \cap h(I_2) = \phi$, then $I_1 + I_2 = A$.*

Proof. We shall construct two sequences in $\Pi_n A_n$, show that they yield elements of I_1 and I_2 whose sum is 1. We let $F_1 = h(I_1)$ and $F_2 = h(I_2)$. Since the tail of a sequence is the important thing in determining whether it corresponds to an element of A we assume that $F_1 \cap \mathscr{M}^0 \neq \phi$ and $F_2 \cap \mathscr{M}^0 \neq \phi$. If not we begin the construction with the first integer k so that both F_1 and F_2 meet $\mathscr{M}^k$ and define the first k terms by the maps Π_i^k, $i = 0, \cdots, k-1$.

Since $F_1 \cap F_2 = \phi$, $\sigma_n^{-1}(F_1 \cap \mathscr{M}^n) \cap \sigma_n^{-1}(F_2 \cap \mathscr{M}^n) = \phi$ in $\mathscr{M}(A_n)$ for each $n \geqq 0$. We first note that for each $n \geqq 0$ $\Pi_n(I_1)^-$ and $\Pi_n(I_2)^-$ are closed ideals in A_n and $\Pi_n(I_1)^- + \Pi_n(I_2)^- = A_n$. If not, then there exists $M^n \in (A_n)$ such that $\Pi_n(I_1)^-, \Pi_n(I_2)^- \subseteqq M^n$. Then $I_1, I_2 \subseteqq M = \sigma_n(M^n)$, and $M \in F_1 \cap F_2$, a contradiction. By Lemma 7.8 of [9] I_j is the inverse limit of the sequence $\{\Pi_n(I_j)^-\}$ with the restricted homomorphism, for $j = 1, 2$, and for each pair n, m, $n \leqq m$, $\Pi_n^m[\Pi_m(I_j)^-]$ is dense in $\Pi_n(I_j)^-$, since the former contains $\Pi_n^m[\Pi_m(I_j)] = \Pi_n(I_j)$ which is dense in $\Pi_n(I_j)^-$.

We first choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers such that the series $\sum_{n=1}^{\infty} \varepsilon_n$ converges. Since $\Pi_0(I_1)^- + \Pi_0(I_2)^- = A_0$ we choose

$\xi_0^j \in \Pi_0(I_j)^-, j = 1, 2$ such that $\xi_0^1 + \xi_0^2 = 1$.

We next choose ζ_1^j in $\Pi_1(I_j)^-, j = 1, 2$, such that $\zeta_1^1 + \zeta_1^2 = 1$, then choose η_1^j in $\Pi_1(I_j)^-, j = 1, 2$, such that

$$|| \Pi_0^1 \eta_1^j - \xi_0^j || < \min\left(\varepsilon_1/4, \varepsilon_1/4 \max_{j=1,2} || \xi_1^j ||\right).$$

This is possible because $\Pi_0^1[\Pi_1(I_j)^-]$ is dense in $\Pi_0(I_j)^-, j = 1, 2$. We let $\xi_1^j = \eta_1^j + \zeta_1^j(1 - \eta_1^1 - \eta_1^2), i = 1, 2$. Then

$$\xi_1^j \in \Pi_1(I_j)^-, j = 1, 2; \xi_1^1 + \xi_1^2 = 1,$$

and $|| \Pi_0^1 \xi_1^j - \xi_0^j || < \varepsilon_1$, for each j.

Proceeding inductively we choose for each $n \geqq 1, j=1, 2, \xi_n^j \in \Pi_n(I_j)^-$ such that $|| \Pi_{n-1}^n \xi_n^j - \xi_{n-1}^j || < \varepsilon_n$, and $\xi_n^1 + \xi_n^2 = 1$. Then for $k = 0, 1, \cdots, n-1$ we have

$$|| \Pi_k^n \xi_n^j - \Pi_k^{n-1} \xi_{n-1}^j || < \varepsilon_n . \tag{3.1}$$

From this point on the construction is identical to that given in the proof of Theorem 4.2 of [2]. We sketch the important steps.

We first fix $n \geqq 0$ and let $x_j(n)_k = \Pi_n^k \xi_k^j$ for each $k \geqq n, j = 1, 2$. $\{x_j(n)_k\}_{k=n}^\infty$ is a sequence in $\Pi_n(I_j)^-$ and satisfies

(i) $\Pi_n^{n+1}(x_j(n+1)_k) = x_j(x)_k$ for each $k \geqq n+1, j = 1, 2$;

(ii) $x_1(n)_k + x_2(n)_k = 1$,

(iii) $|| x_j(n)_k - x_j(n)_{k+p} || < \varepsilon_{k+1} + \cdots + \varepsilon_{k+p}$.

Thus the sequences are Cauchy for each n, j and converge to elements $x_j(n)$ in $\Pi_n(I_j)^-$ for each $n \geqq 0, j = 1, 2$. There exist $x_1 \in I_1, x_2 \in I_2$ such that $\Pi_n(x_j) = x_j(n)$ for each $n \geqq 0, j = 1, 2$. Thus, $x_1 + x_2 = 1$.

COROLLARY 2.4.1. *If F_1 and F_2 are disjoint hk-closed subsets of $\mathscr{M}_0$, then $\mathrm{Cl}_{\mathscr{M}} F_1 \cap \mathrm{Cl}_{\mathscr{M}} F_2 = \phi$.*

Proof. Letting $I_1 = kF_1$ and $I_2 = kF_2$ yields $I_1 + I_2 = A$. Apply Theorem 1.5.

COROLLARY 2.4.2. *If A is a commutative F-algebra with identity, then $\mathscr{M} = W(\mathscr{M}_0, hk)$. Moreover, if A is regular, then A is normal and $\mathscr{M} = W(\mathscr{M}_0) = \beta\mathscr{M}_0$.*

Proof. The first statement follows from Corollary 2.4.1, Theorem 1.5, and Corollary 1.7. The second follows from Corollaries 1.7 and 2.4.1.

We note that Rosenfeld [11] has indicated a proof of part of Corollary 2.4.2 (A regular implies A normal) using Silov's theorem. This theorem also yields a proof of Corollary 2.4.1, since $F_1 \cup F_2$ is

hk-closed in $\mathscr{M}_0$ and is $\mathscr{M}_0(B)$, where $B = A/(kF_1 \cap kF_2)$. However, since the application of this theorem yields an element a of A such that $\hat{a} \equiv 0$ on F_1 and $\hat{a} \equiv 1$ on F_2, we can conclude only that $kF_1 + kF_2 = A$. Thus, it does not appear that the proof of Theorem 2.4 can be simplified by the use of this tool.

THEOREM 2.5. *Let I be a closed ideal in A and $B = A/I$. Then B is a commutative F-algebra with identity, $\mathscr{M}_0(B)$ is homeomorphic to $h(I)$ with respect to both the w^*- and hk-topologies, and $\mathscr{M}(B) =$ $\mathrm{Cl}_{\mathscr{M}(A)}\, h(I)$.*

Proof. The first conclusion follows from the open mapping theorem for F-spaces (cf. [8, Lemma 11.3]) and the fact that the natural map Π of A onto B is continuous and open. The range of $\Pi^*: \mathscr{M}_0(B) \rightarrow \mathscr{M}_0(A)$ is easily seen to be $h(I)$ and it is also immediate that Π^* is a w^*- homeomorphism. For convenience we let $F = h(I)$ for the remainder of the proof.

We show that for each $E \subseteqq F$, $\Pi^{*-1}[hk(E)] = h'k'[\Pi^{*-1}(E)]$, where h' and k' are the h- and k- operators for B. $M' \in \Pi^{*-1}[hk(E)]$ if, and only if, $M \in hk(E)$ $(M = \Pi^* M')$ if, and only if, $M(x) = 0$ for each $x \in kE$. And $x \in kE$ if, and only if, $M_1(x) = 0$ for each $M_1 \in E$ if, and only if, $M_1'(\Pi x) = 0$ for each $M_1' \in \Pi^{*-1}(E)$. So $x \in kE$ if, and only if, $\Pi x \in k'[\Pi^{*-1}(E)]$. Thus, from above, $M(x) = 0$ for each $x \in kE$ if, and only if, $M'(\Pi x) = 0$ for each $\Pi x \in k'[\Pi^{*-1}(E)]$, if, and only if, $M' \in h'k'[\Pi^{*-1}(E)]$. The equality is established and it is immediate that Π^* is a homeomorphism with respect to the hk-topologies in $\mathscr{M}_0(B)$ and F.

For each $x \in A$ we have $\Pi^*[h'(\Pi x)] = h(x) \cap F$. Thus, there is a lattice isomorphism of $\mathscr{L}' = \{h(\xi_1, \cdots, \xi_n): \{\xi_1, \cdots, \xi_n\} \subseteqq B\}$ onto $\mathscr{L}_F = \{E \subseteqq F: E = B \cap F$ for some $B \in \mathscr{L}\}$, and there is induced a homeomorphism of $w\mathscr{L}'$ onto $w\mathscr{L}_F$. Therefore, $\mathscr{M}(B)$ is homeomorphic to $w\mathscr{L}_F$. For each $M \in \mathrm{Cl}_{\mathscr{M}(A)} F$ we define $\tau(M) = \{E \in \mathscr{L}_F: M \in \mathrm{Cl}_{\mathscr{M}} E\}$. $M \rightarrow \tau(M)$ is a one-to-one mapping of $\mathrm{Cl}_{\mathscr{M}(A)} F$ onto $w\mathscr{L}_F$. From the easily verified equation $H(x) \cap \mathrm{Cl}_{\mathscr{M}(A)} F = C[h(x) \cap F]$ it follows that τ is a homeomorphism.

BIBLIOGRAPHY

1. R. Arens, *A generalization of normed rings*, Pacific J. Math. **2** (1952), 455-471.
2. ———, *Dense inverse limit rings*, Michigan Math. J. **5** (1958), 169-182.
3. R. M. Brooks, *On Wallman compactifications*, Fund. Math. LX (1967), 157-173.
4. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, New York, 1960.
5. E. Hewitt, *Rings of real-valued continuous functions*, I, Trans. Amer. Math. Soc. **64** (1948), 45-99.

6. S. Kakutani, *Rings of analytic functions,* Lectures on functions of a complex variable, Ann Arbor, 1955.
7. J. L. Kelley, *General Topology,* Van Nostrand, New York, 1955.
8. I. Namioka *et al, Linear Topological Spaces,* Van Nostrand, New York, 1963.
9. E. A. Michael, *Locally multiplicatively-convex topological algebras,* Mem. Amer. Math. Soc. **11** (1952).
10. M. A. Naimark, *Normed Rings,* P. Noordhoff, Ltd., Groningen, The Netherlands, 1960.
11 M. Rosenfeld, *Commutative F-algebras,* Pacific J. Math. **16** (1966), 159-166
12. P. Samuel, *Ultrafilters and compactification of uniform spaces,* Trans. Amer. Math. Soc. **64** (1948), 100-132.
13. H. Wallman, *Lattices and topological spaces,* Ann of Math. (2) **39** (1938), 112-126.

Received September 22, 1966. The research for this paper was supported in part by NSF Grant G 24295.