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SPECTRAL ANALYSIS OF COLLECTIVELY COMPACT, STRONGLY CONVERGENT OPERATOR SEQUENCES

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A set of linear operators on a normed linear space is collectively compact if and only if the union of the images of the unit ball has compact closure. Bounded linear operators T and T_n, $n = 1, 2, \cdots$, such that $T_n \to T$ strongly and $\{T_n - T\}$ is collectively compact are investigated. The theory somewhat resembles that for $\| T_n - T \| \to 0$. The spectrum of T_n is eventually contained in any neighborhood of the spectrum of T. If $f(T)$ is defined by the operational calculus, then $f(T_n)$ is eventually defined, $f(T_n) \to f(T)$ strongly, and $\{f(T_n) - f(T)\}$ is collectively compact. If $f(T_n)$ and $f(T)$ are spectral projections, the corresponding spectral subspaces eventully have the same dimension. Other results compare eigenvalues and generalized eigenmanifolds of T_n and T.

Let X be a real or complex Banach space, $[X]$ the space of bounded linear operators on X into X, and $\mathscr{B}$ the closed unit ball in X. Then a set $\mathscr{K} \subset [X]$ is collectively compact if and only if the set $\mathscr{K}\mathscr{B} = \{Kx: K \in \mathscr{K},\ x \in \mathscr{B}\}$ has compact closure. Collectively compact sets of operators were introduced in [1] in connection with the approximate solution of integral equations. Various general properties of such sets are developed in [2], which contains further references on the subject.

With slight modification, most of the preliminary results below are valid for operators from one space to another. By rather different methods, Atkinson [3] obtained some of the spectral theory results for the special case with $\{T_n\}$ collectively compact.

For the sake of brevity, strong operator convergence is denoted simply by $T_n \to T$ and norm convergence by $\| T_n - T \| \to 0$.

2. Related hypotheses.

As indicated above, we shall study operators in $[X]$ such that $T_n \to T$ and $\{T_n - T\}$ is collectively compact. But, first, comparisons are made with related conditions. The hypotheses in (a) below were assumed in most of the previous papers on collectively compact sets.

PROPOSITION 2.1. The following are equivalent.
(a) $T_n \to T$ and $\{T_n\}$ collectively compact;
(b) $T_n \to T$, $\{T_n - T\}$ collectively compact, T compact;
(c) $T_n \to T$, $\{T_n - T\}$ collectively compact, some T_n compact.
The proof makes use of the fact that the strong closure of a

collectively compact set is collectively compact [2, Prop. 2.1].

Any totally bounded set of compact operators in $[X]$ is collectively compact, but not conversely [2, Th. 2.5 and Ex. 2.6]. The next proposition indicates the effect of relacing the hypothesis, $\{T_n - T\}$ collectively compact, by the hypothesis, $\{T_n - T\}$ totally bounded (or equivalently $\{T_n\}$ totally bounded).

PROPOSITION 2.2. Let $T,\ T_n \in [X]$, $n \geqq 1$. Then:

(a) $\| T_n - T \| \to 0$ if and only if $T_n \to T$ and $\{T_n - T\}$ totally bounded;

(b) $\| T_n - T \| \to 0$ and each $T_n - T$ compact $\Rightarrow \{T_n - T\}$ collectively compact.

Proof. (a) Clearly, $\| T_n - T \| \to 0$ implies $T_n \to T$ and $\{T_n - T\}$ totally bounded. If $\| T_n - T \| \not\to 0$, then $\| T_{n_i} - T \| > \delta$ for some $\{n_i\}$ and $\delta > 0$. But $T_n \to T$ and $\{T_n - T\}$ totally bounded imply $\| T_{n_{i_j}} - T \| \to 0$ for some $\{n_{i_j}\}$. This is a contradiction. (b) This follows from the fact that a totally bounded set of compact operators is collectively compact.

If $\| T_n - T \| \to 0$, then there is a simpler theory than that presented below. However, $\| T_n - T \| \not\to 0$ in a number of applications to integral equations.

Whenever $T_n \to T$, the principle of uniform boundedness implies that $\{T_n\}$ is bounded or, equivalently, equicontinuous. If follows that strong convergence is uniform on each compact subset of X. In contrast, $\| T_n - T \| \to 0$ if and only if $T_n \to T$ uniformly on each bounded subset of X, or simply on $\mathscr{B}$.

3. General convergence results. The following proposition will play an important role in our analysis.

PROPOSITION 3.1. Assume $T_n \to T$ and $\mathscr{K} \subset [X]$ collectively compact. Then

$$\| (T_n - T)K \| \to 0 \quad \text{uniformly for} \quad K \in \mathscr{K} .$$

Proof. Since $\overline{\mathscr{K}\mathscr{B}}$ is compact, $(T_n - T)Kx \to 0$ uniformly for $K \in \mathscr{K}$ and $X \in \mathscr{B}$.

The following convention will be used in the next proposition as well as later in the paper. Suppose $T_{\alpha n},\ T_\alpha \in [X]$ for $n \geqq 1$ and α in an index set A. Then the phrase, $T_{\alpha n} \to T_\alpha$ uniformly for $\alpha \in A$, will mean that, for each $x \in X$, $T_{\alpha n}x \to T_\alpha x$ uniformly for $\alpha \in A$. In the following proposition $\mathscr{K}$ need not consist of compact operators.

PROPOSITION 3.2. Assume $T_n \to T$ and $\mathscr{K} \subset [X]$ totally bounded.

Then

$$T_n K \to TK \quad \text{uniformly for} \quad K \in \mathscr{K} .$$

Proof. Fix $x \in X$ and define the map f: $\bar{\mathscr{K}} \to X$ by $f(K) = Kx$. Since $\bar{\mathscr{K}}$ is compact and f is continuous, the set $f(\bar{\mathscr{K}}) = \bar{\mathscr{K}}x$ is compact. The assertion follows.

The next result is essentially a corollary of Proposition 3.1.

PROPOSITION 3.3. Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. Then $\|(T_n - T)^2\| \to 0$.

Although by itself the condition $\|(T_n - T)^2\| \to 0$ does not have very many useful consequences, it becomes more important when the operators have appropriate additional properties.

THEOREM 3.4. *Let X be a Hilbert space. Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. Then $\| T_n - T\| \to 0$:*

(a) *if each T_n is self adjoint;*

(b) *if each $T_n - T$ is normal;*

(c) *if and only if the set $\{T_n^* - T^*\}$ is collectively compact.*

Proof. If (a) or (b) holds, then $\|(T_n - T)^2\| = \| T_n - T\|^2$ and $\| T_n - T\| \to 0$ by Proposition 3.3. Alternatively, all three cases follow from Proposition 2.2 and [2, Th. 3.4, 3.5].

4. Convergence of projections. For the main theorem in this section we shall need the following known result, which is usually proved by contradiction. We give a short direct proof. (Note: dimensions may be finite or ∞.)

LEMMA 4.1. *Let $T_n \to T$. Then dim $T_nX \geqq$ dim TX eventually.*

Proof. Let $\{Tx_j\colon j = 1, \cdots, m\}$ be linearly independent and define $\mathscr{C} = \{\sum_{j=1}^m c_j x_j\colon \max |c_j| = 1\}$. Then $\mathscr{C}$ and $T\mathscr{C}$ are compact, so $T_n \to T$ uniformly on $\mathscr{C}$. Since $\min_{x \in \mathscr{C}} \| Tx\| > 0$, eventually $\min_{x \in \mathscr{C}} \| T_n x\| > 0$ and $\{T_n x_j\colon j = 1, \cdots, m\}$ is linearly independent. The desired result follows.

THEOREM 4.2. *Let E and E_n, $n \geqq 1$, be projections in $[X]$ such that $E_n \to E$ and $\{E_n - E\}$ is collectively compact. Then, eventually,* $\dim E_nX = \dim EX$.

Proof. In view of Lemma 4.1, we may assume $\dim EX < \infty$. Then E is compact and, by Proposition 2.1, $\{E_n\}$ is collectively compact.

Suppose that $\dim E_n X \geqq m$ for $n \geqq 1$. By the Riesz lemma, there exist linearly independent sets $\{x_{nk}: k = 1, \cdots, m\} \subset E_n X$, $n \geqq 1$, such that

$$\| x_{nk} \| = 1 , \qquad \| x_{nk} - \sum_{j=1}^{k-1} c_j x_{nj} \| \geqq 1$$

for all n, k and $\{c_j\}$. Since $x_{nk} = E_n x_{nk} \in \overline{\{E_n\}\mathscr{B}}$, which is compact, there exists a subsequence $\{n_i\}$ and elements $x_k \in X$ such that

$$x_{n_i k} = E_{n_i} x_{n_i k} \longrightarrow x_k \quad \text{for} \quad k = 1, \cdots, m .$$

Then

$$x_k \| = 1 , \qquad \| x_k - \sum_{n=1}^{k-1} c_j x_j \| \geqq 1$$

for all k and $\{c_j\}$, so $\{x_k: k = 1, \cdots, m\}$ is linearly independent. Now $E_n \longrightarrow E$ implies $E_{n_i} x_{n_i k} \longrightarrow E x_k$, so that $x_k = E x_k \in EX$ for all k. Thus

$$\dim E_n X \geqq m \quad \text{for all} \quad n \Rightarrow \dim EX \geqq m .$$

Apply this result to an arbitrary subsequence of $\{E_n\}$ to conclude that $\dim E_n X \leqq \dim EX$ eventually. Lemma 4.1 gives the reverse inequality.

A slight modification of the proof of Theorem 4.2 yields the following result, which will be needed later.

PROPOSITION 4.3. Let E and E_n, $n \geqq 1$, be projections in $[X]$ for which $E_n \longrightarrow E$ and $\{E_n\}$ is collectively compact. Let $Z \subset EX$ and $Z_n \subset E_n X$ be subspaces such that: $x_n \longrightarrow x$, $x_n \in Z_n \Rightarrow x \in Z$. Then $\dim Z_n \leqq \dim Z$ eventually.

5. **Resolvent sets, spectra, and resolvent operators.** It is convenient to compactify the scalar field with the point ∞ if X is complex and the points $\pm\infty$ if X is real. For each $T \in [X]$, let $(\infty - T)^{-1} = 0$ in the complex case and $(\pm \infty - T)^{-1} = 0$ in the real case. The spectrum, resolvent set, and extended resolvent set of T are denoted by $\sigma(T)$, $\rho(T)$ and $\tilde{\rho}(T)$, respectively.

LEMMA 5.1. *For each $T \in [X]$ and each closed set $\Lambda \supset \tilde{\rho}(T)$, the set $\{(\lambda - T)^{-1}: \lambda \in \Lambda\}$ is compact.*

Proof. The set Λ is compact and the map $\lambda \longrightarrow (\lambda - T)^{-1}$ is continuous.

LEMMA 5.2. *Assume $T_n \longrightarrow T$ and $\{T_n - T\}$ collectively compact. Then*

$$\| [(T_n - T)(\lambda - T)^{-1}]^2 \| \longrightarrow 0 \quad \textit{for} \quad \lambda \in \tilde{\rho}(T) .$$

The convergence is uniform on each closed set $\Lambda \subset \tilde{\rho}(T)$.

Proof. If $\mathscr{K}, \mathscr{M}, \mathscr{N} \subset [X]$, with $\mathscr{K}$ collectively compact, $\mathscr{M}$ bounded, and $\mathscr{N}$ compact, then the sets $\mathscr{K}\mathscr{M}$ and $\mathscr{N}\mathscr{K}$ are collectively compact [2, Prop. 2.3]. Hence, by Lemma 5.1, the set

$$\{(\lambda - T)^{-1}(T_n - T)(\lambda - T)^{-1} \colon \lambda \in \Lambda,\ n \geqq 1\}$$

is collectively compact. An application of Proposition 3.1 completes the proof.

THEOREM 5.3. *Assume* $T_n \to T$ *and* $\{T_n - T\}$ *collectively compact. Let* Ω *be any neighborhood of* $\sigma(T)$ *and* $\Lambda = \tilde{\rho}(T) - \Omega$. *Then:*

(a) *there exists* N *such that* $\sigma(T_n) \subset \Omega$ *and* $\Lambda \subset \tilde{\rho}(T_n)$ *for all* $n \geqq N$;
(b) *the set* $\{(\lambda - T_n)^{-1} \colon \lambda \in \Lambda,\ n \geqq N\}$ *is bounded;*
(c) *the functions* $\lambda \to (\lambda - T_n)^{-1}$, $n \geqq N$, *are equicontinuous on* Λ;
(d) $(\lambda - T_n)^{-1} \to (\lambda - T)^{-1}$ *uniformly for* $\lambda \in \Lambda$.

(Estimates corresponding to (b), (c), (d) are given in the proof.)

Proof. For $\lambda \in \rho(T)$, $\lambda - T_n = [I - (T_n - T)(\lambda - T)^{-1}](\lambda - T)$. By Lemma 5.2, there is an N such that

$$\| [(T_n - T)(\lambda - T)^{-1}]^2 \| \leqq \tfrac{1}{2}\,, \qquad \lambda \in \Lambda\,, \quad n \geqq N\,.$$

Recall that if $L \in [X]$ and $\| L^2 \| < 1$ then there exists $(I - L)^{-1} \in [X]$, and

$$\| (I - L)^{-1} \| \leqq \frac{\| I + L \|}{1 - \| L^2 \|}\,.$$

Hence, for $\lambda \in \Lambda$ and $n \geqq N$, there exist $[I - (T_n - T)(\lambda - T)^{-1}]^{-1}$ and $(\lambda - T_n)^{-1}$ in $[X]$,

$$(\lambda - T_n)^{-1} = (\lambda - T)^{-1}[I - (T_n - T)(\lambda - T)^{-1})^{-1}]\,,$$

and

$$\| (\lambda - T_n)^{-1} \| \leqq \frac{\| (\lambda - T)^{-1} \| \, \| I + (T_n - T)(\lambda - T)^{-1} \|}{1 - \| [(T_n - T)(\lambda - T)^{-1}]^2 \|}\,.$$

Since $\{T_n - T\}$ and $\{(\lambda - T)^{-1} \colon \lambda \in \Lambda\}$ are necessarily bounded, (a) and (b) follow. Suppose $\| (\lambda - T_n)^{-1} \| \leqq B$ for $\lambda \in \Lambda$, $n \geqq N$. By the resolvent identity,

$$(\lambda - T_n)^{-1} - (\mu - T_n)^{-1} = -(\lambda - \mu)(\lambda - T_n)^{-1}(\mu - T_n)^{-1}\,,$$
$$\| (\lambda - T_n)^{-1} - (\mu - T_n)^{-1} \| \leqq B^2 |\lambda - \mu|$$

for $\lambda, \mu \in \Lambda$, $n \geqq N$. This implies (c). Note that

$$(\lambda - T_n)^{-1} - (\lambda - T)^{-1} = (\lambda - T_n)^{-1}(T_n - T)(\lambda - T)^{-1}\,,$$

$$\|(\lambda - T_n)^{-1}x - (\lambda - T)^{-1}x\| \leqq B\|(T_n - T)(\lambda - T)^{-1}x\|$$

for $\lambda \in \Lambda$, $n \geqq N$ and $x \in X$. Therefore, Lemma 5.1 and Proposition 3.2 yield (d).

6. **Functions of operators.** Henceforth, X is a complex Banach space. The results of this section involve the functional calculus of an operator in $[X]$. To fix notation we review briefly the definitions. A more extended discussion with proofs may be found in [4].

For each $T \in [X]$, let $\mathfrak{F}(T)$ denote the set of locally analytic functions f defined on (not necessarily connected) neighborhoods $\mathscr{D}(f)$ of $\sigma(T)$. For each $f \in \mathfrak{F}(T)$, there is a contour Γ in $\mathscr{D}(f) \cap \rho(T)$ which consists of a finite number of rectifiable Jordan curves. Suppose that $\sigma(T)$ lies entirely inside Γ. Then the operator

$$f(T) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - T)^{-1}d\lambda$$

is the limit in the norm topology of the usual approximating sums, and $f(T)$ is independent of the particular choice of Γ. The map $f(\lambda) \to f(T)$ is an algebraic homomorphism from $\mathfrak{F}(T)$ into $[X]$.

THEOREM 6.1. *Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. Let $f \in \mathfrak{F}(T)$. Then there is an N such that*

(a) *$f \in \mathfrak{F}(T_n)$ for $n \geqq N$;*

(b) *$f(T_n) \to f(T)$;*

(c) *$\{f(T_n) - f(T)\colon n \geqq N\}$ is collectively compact.*

Proof. Choose $\Gamma \subset \mathscr{D}(f) \cap \rho(T)$ with $\sigma(T)$ inside Γ. By Theorem 5.3(a) there is an N such that $\sigma(T_n)$ is inside Γ for $n \geqq N$. Thus, (a) follows. For $n \geqq N$ and $x \in X$,

$$f(T_n)x - f(T)x = \frac{1}{2\pi i}\int_\Gamma f(\lambda)[(\lambda - T_n)^{-1}x - (\lambda - T)^{-1}x]d\lambda\ .$$

By Theorem 5.3(d), $f(\lambda)[(\lambda - T_n)^{-1}x - (\lambda - T)^{-1}x] \to 0$ uniformly for $\lambda \in \Gamma$ as $n \to \infty$. This implies (b). For $n \geqq N$,

$$f(T_n) - f(T) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - T)^{-1}(T_n - T)(\lambda - T_n)^{-1}d\lambda\ ,$$

where $\{f(\lambda)(\lambda - T)^{-1}(T_n - T)(\lambda - T_n)^{-1}\colon \lambda \in \Gamma,\ n \geqq N\}$ is collectively compact by the argument used in the proof of Lemma 5.2. Therefore, by [2, Prop. 2.2], (c) follows.

Without the hypothesis that $\{T_n - T\}$ is collectively compact, $T_n \to T$ would not imply even the existence of $f(T_n)$ for n sufficiently large, much less $f(T_n) \to f(T)$.

Essentially the same proof as for Theorem 6.1 yields the following somewhat similar result.

THEOREM 6.2. *Assume* $T_n \to T$ *and* $\{T_n - T\}$ *collectively compact. Let* $f_n, f \in \mathfrak{F}(T)$ *with* $\mathscr{D}(f_n) \supseteqq \mathscr{D}(f)$. *If* f_n *converges to* f *uniformly on* $\mathscr{D}(f)$, *then* $f_n(T_n)$ *is eventually defined and* $f_n(T_n) \to f(T)$.

Complementary spectral sets σ_1 and σ_2 associated with an operator T are disjoint, relatively closed subsets of $\sigma(T)$ with $\sigma(T) = \sigma_1 \cup \sigma_2$ (either σ_1 or σ_2 may be void). Given σ_1 and σ_2, there is a contour Γ in $\rho(T)$ with σ_1 inside Γ and σ_2 outside Γ. Conversely, every contour Γ in $\rho(T)$ separates $\sigma(T)$ into complementary spectral sets σ_1 and σ_2. With this notation, the operator.

$$E_\Gamma(T) = \frac{1}{2\pi i}\int_\Gamma (\lambda - T)^{-1} d\lambda$$

is the spectral projection and $E_\Gamma(T)X$ the spectral subspace corresponding to σ_1. Theorems 6.1 and 4.2 yield

PROPOSITION 6.3. Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. Let Γ be a contour in $\rho(T)$. Then there is an N such that for $n \geqq N$: $E_\Gamma(T_n)$ is defined, $E_\Gamma(T_n) \to E_\Gamma(T)$, $\{E_\Gamma(T_n) - E_\Gamma(T)\}$ is collectively compact, and $\dim E_\Gamma(T_n)X = \dim E_\Gamma(T)X$.

7. **Eigenvalues and eigenvectors.** Throughout this final section assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. The null space of an operator L will be denoted by $\mathscr{N}(L)$.

Let μ be an isolated eigenvalue of T with finite index (ascent and descent) ν and

$$P(\lambda) = (\mu - \lambda)^\nu$$

Let Γ be any contour in $\rho(T)$ with μ the only point of $\sigma(T)$ inside Γ, and $E = E_\Gamma(T)$. Then

$$EX = \mathscr{N}[P(T)] ,$$
$$EP(T) = P(ET) = 0 ,$$

and P divides every polynomial Q such that $Q(ET) = 0$. Thus, P is the minimal polynomial for ET. Assume $\dim EX < \infty$. Then E is compact.

By Propositions 2.1 and 6.3 there is an N such that for $n \geqq N$: $E_n = E_\Gamma(T_n)$ is defined, $E_n \to E$, $\{E_n\}$ is collectively compact, and $\dim E_nX = \dim EX$. Assume $n \geqq N$ henceforth. Since $\dim E_nX < \infty$, the part of $\sigma(T_n)$ inside Γ consists of a finite set of eigenvalues μ_{nk}

with finite indices ν_{nk}, $k = 1, \cdots, k_n$. Then Theorem 5.3(a) implies

PROPOSITION 7.1. $\max_{k=1,\cdots,k_n} |\mu_{nk} - \mu| \to 0$ as $n \to \infty$.

Let

$$P_n(\lambda) = \prod_{k=1}^{k_n}(\mu_{nk} - \lambda)^{\nu_{nk}} .$$

Then

$$E_n X = \mathscr{N}[P_n(T_n)] ,$$
$$E_n P_n(T_n) = P_n(E_n T_n) = 0 ,$$

and P_n is the minimal polynomial for $E_n T_n$. Let ν_n be the degree of P_n. Then

$$\nu_n = \sum_{k=1}^{k_n} \nu_{nk} .$$

THEOREM 7.2. *$\nu_n \geqq \nu$ for n sufficiently large.*

Proof. Suppose $\nu_{n_i} = \beta$ for some subsequence. By Theorem 6.1 $E_n T_n \to ET$ and $\{E_n T_n - ET\}$ is collectively compact. Hence

$$O = P_{n_i}(E_{n_i} T_{n_i}) \to (\mu - ET)^\beta$$

by Proposition 7.1 and Theorem 6.2. Therefore $\beta \geqq \nu$ and the conclusion follows.

The next theorem relates generalized eigenmanifolds of T_n and T. Let γ be any integer such that $0 \leqq \gamma \leqq \nu$. By Theorem 7.2, for n sufficiently large there exist integers γ_{nk} such that

$$0 \leqq \gamma_{nk} \leqq \nu_{nk} , \qquad \sum_{k=1}^{k_n} \gamma_{nk} = \gamma .$$

Let

$$Q(T) = (\mu - T)^\nu , \qquad Q_n(T_n) = \prod_{k=1}^{k_n} (\mu_{nk} - T_n)^{\nu_{nk}} ,$$
$$Z = \mathscr{N}[Q(T)] , \qquad Z_n = \mathscr{N}[Q_n(T_n)] .$$

THEOREM 7.3. *For n sufficiently large and every choice of the ν_{nk}, $\dim Z_n \leqq \dim Z$.*

Proof. From $T_n \to T$ and Proposition 7.1, $Q_n(T_n) \to Q(T)$. Hence, $x_n \to x$, $x_n \in Z_n \Rightarrow x \in Z$. By Theorem 4.3, $\dim Z_n \leqq \dim Z$ eventually.

The final result relates eigenmanifolds of T_n and T.

COROLLARY 7.4. *For n sufficiently large,*

$$\dim \mathscr{N}(\mu_{nk} - T_n) \leqq \dim \mathscr{N}(\mu - T) , \qquad k = 1, \cdots, k_n .$$

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