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A TAUBERIAN RELATION BETWEEN THE BOREL AND THE LOTOTSKY TRANSFORMS OF SERIES

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This paper is concerned with the equiconvergence of the Lototsky transform and the Borel (exponential) transform for a class of series satisfying the Tauberian condition $a_n = o(1)$.

If $s_n = a_0 + a_1 + \cdots + a_n$, the Borel (exponential) transform $f(x)$ of s_n is usually defined by

$$e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} .$$

Writing $s_n = a_1 + a_2 + \cdots + a_n$, the Lototsky transform σ_n of s_n introduced by A. V. Lototsky [8] is defined by

$$\sigma_n = \frac{1}{n!} \sum_{k=1}^{n} p_{n,k} s_k , \tag{1.1}$$

where $p_{n,k}$ is the coefficient of x^k in

$$p_n(x) = x(x+1)(x+2) \cdots (x+n-1) , \qquad (n = 1, 2, \cdots) .$$

Thus it is usual in considering Lototsky summability to take the first term of the series as a_1, and in considering Borel summability[1] to take it as a_0. In order to compare the methods without changing the customary notation we will therefore apply the Borel methods to the series $0 + a_1 + a_2 + \cdots$ and apply the Lototsky method to the series $a_1 + a_2 + \cdots$. We recall (Hardy [5] pp. 182-3) that the Borel summability of $a_1 + a_2 + \cdots$ implies the Borel summability $0 + a + a + \cdots$, but not conversely. The two methods are equivalent if (and only if) $a_n \to 0(B)$; this is true in particular if

$$a_n = o(1) , \tag{1.2}$$

and thus for the series considered in this paper.

Lototsky's transform is essentially a special case of a class of transformations introduced by J. Karamata [7]. It is the (f, d_n) transform defined by G. Smith [11], when $f(z) = z$, $d_n = n$, and the $[F, d_n]$ transform defined by A. Jakimorski [6], when $d_n = n - 1$ and $n \geqq 1$. It is also the σ^α method of summability introduced by Vučković [12], when $\alpha = 1$.

Numerous properties of this Lototsky transform and its relation

[1] "Borel summability" is throughout taken to refer to Borel's exponential method.

with some of the other transformations have been shown in Agnew ([1], [3]).

In § 2 of the present paper we shall show that, for the class of series satisfying the Tauberian condition (1.2), the Lototsky transform σ_n and the Borel transform $f(\log n)$ are equiconvergent. This includes the result that, under the condition (1.2), Lototsky summability implies Borel summability, and it should therefore be remarked that this result is essentially due to Agnew ([1], [3]). For we have, with Agnew's notation, (since for suitably restricted sequences the starred and unstarred methods are equivalent)

$$L \subset BI^* \sim BI \sim B .$$

The argument of § 2 depends on an asymptotic expression for p_{nk} for large n given by Moser and Wyman [10].

In § 3, we introduce a Tauberian constant for the Lototsky transform.

Agnew ([2] §'s 2, 3) has obtained a result of a similar nature to Theorem 3.1 of this paper but for the Borel transform instead.

We may observe that Theorem 3.1 is included in Theorem 2.1 of the present paper. Also, a "O" Tauberian theorem for the Lototsky transform is included in Theorem 2.1, but not in Theorem 3.1.

2. THEOREM 2.1. *Suppose that* (1.2) *holds. Then*

$$\sigma_n - f(\log n) \to 0 , \quad \text{as } n \to \infty . \tag{2.1}$$

For the proof of Theorem 2.1, we require the following lemmas.

LEMMA 2.1. *There is a* $K = K(n)$ *such that*

$$p_{n1} < p_{n2} < \cdots < p_{nK} \geqq p_{n,K+1} > p_{n,K+2} > \cdots > p_{nn}$$

and that for large n

$$K(n) = \log n + O(1) . \tag{2.2}$$

The result is due to Hammersley [4]. Hammersley gives a more precise result than (2.2), but this is enough for our purposes.

LEMMA 2.2. *Let* a, b *be constants with* $0 < a < 1 < b$. *Then for large* n *uniformly in*

$$a \log n \leqq k \leqq b \log n , \tag{2.3}$$

we have

(2.4) $$\frac{P_{nk}}{n!} = O\Big(\frac{1}{\sqrt{\log n}}\, n^{\phi(\theta)}\Big)$$

where we write

(2.5) $$\phi(\theta) = \theta - 1 - \theta\log\theta\ ; \quad \theta = \frac{k}{\log n}\ .$$

Proof. Write

$$f_n(t) = \sum_{\nu=0}^{n-1}\frac{t}{t+\nu}\ .$$

We note that, for fixed n, as t increases from 0 to ∞, $f_n(t)$ increases from 1 to n.

Now, it follows from Moser and Wyman ([10], equation (4.51) and the line below it) that, uniformly in a bigger range which includes (2.3)

(2.6) $$p_{nk} = \frac{\Gamma(n+R)}{(2\pi H)^{\frac{1}{2}}R^k\Gamma(R)}\Big(1 + o\Big(\frac{1}{H}\Big)\Big)$$

where R is the unique positive solution of the equation

(2.7) $$f_n(R) = k$$

and where

(2.8) $$H = k - \sum_{\nu=0}^{n-1}\frac{R^2}{(R+\nu)^2}\ .$$

Now, it clearly follows from the definition that for large n uniformly in $0 \leqq t \leqq c$ (c is a constant) we have

(2.9) $$f_n(t) = t\log n + O(1)\ .$$

Choose $c > b$; then it follows from (2.9) that, for sufficiently large n

$$f_n(c) > b\log n$$

and hence, for sufficiently large n, we have $R \leqq C$ for all k satisfying (2.3).

In the rest of the proof of this lemma, the symbol O is to be taken as applying for large n uniformly for k in the range (2.3). Thus, by what has just been said, $R = O(1)$. Also since (2.9) is valid for $t = R$ we deduce from (2.7) that

(2.10) $$R = \frac{k}{\log n} + O\Big(\frac{1}{\log n}\Big)\ .$$

We also note, that since R is bounded

$$H = k + O(1) \ . \tag{2.11}$$

Now, since R is bounded, it follows at once from Stirling's approximation that

$$\frac{\Gamma(n+R)}{n!} = n^{R-1}\left(1 + O\left(\frac{1}{n}\right)\right) . \tag{2.12}$$

However, if we consider $\log(n^{R-1})$ we find, by (2.10) that

$$\begin{cases} \log(n^{R-1}) = (R-1)\log n = k - \log n + O(1) \\ \qquad\qquad\; = (\theta - 1)\log n + O(1) \ . \end{cases} \tag{2.13}$$

Also, by (2.10)

$$\begin{cases} \log(R^k) = k \log R = k \log \theta + k \log\left(1 + O\left(\frac{1}{k}\right)\right) \\ \qquad\qquad = (\theta \log \theta)\log n + O(1) \ . \end{cases} \tag{2.14}$$

Also, since $R \geqq K > 0$, where K is a constant, we have

$$\frac{1}{\Gamma(R)} = O(1) \ ; \tag{2.15}$$

also by (2.11)

$$\frac{1}{\sqrt{2\Pi H}} = O\left(\frac{1}{\sqrt{\log n}}\right) . \tag{2.16}$$

Thus combining (2.6) and (2.12)-(2.16) the result (2.4) follows.

LEMMA 2.3. *Let λ be a constant so that*

$$\frac{1}{2} < \lambda < \frac{2}{3} \ . \tag{2.17}$$

Then for large n uniformly in the range

$$|k - \log n| \leqq (\log n)^{\lambda} \ , \tag{2.18}$$

we have

$$\begin{aligned} \frac{p_{nk}}{n!} &= \frac{1}{\sqrt{2\pi \log n}} \exp\left(-\frac{h^2}{2\log n}\right) \\ &\quad \times \left\{1 + O\left(\frac{|h|+1}{\log n}\right) + \left(\frac{|h|^3}{\log^2 n}\right)\right\} , \end{aligned} \tag{2.19}$$

where we write

$$k = \log n + h \ . \tag{2.20}$$

Proof. To prove (2.19) we need an improvement on (2.10). We have

$$f_n(1) = \log n + \nu + O\left(\frac{1}{n}\right),$$

where ν is Euler's constant. Hence by definition of R

$$f_n(R) - f_n(1) = h - \nu + O\left(\frac{1}{n}\right).$$

But for some t between 1 and R

$$f_n(R) - f_n(1) = (R-1)f_n'(t) .$$

Also for the *relevant* t we have, since $R = O(1)$

$$\begin{aligned} f_n'(t) &= \sum_{\nu=0}^{n-1} \frac{\nu}{(t+\nu)^2} = \sum_{\nu=1}^{n-1} \frac{1}{t+\nu} - t\sum_{\nu=1}^{n-1} \frac{1}{(t+\nu)^2} \\ &= \log n + O(1) . \end{aligned}$$

Thus

$$h - \gamma + O\left(\frac{1}{n}\right) = (R-1) \qquad (\log n + O(1)) ,$$

$$\begin{aligned} R - 1 &= \frac{\left(h - \gamma + O\left(\frac{1}{n}\right)\right)}{\log n} \qquad \left(1 + O\left(\frac{1}{\log n}\right)\right) \\ &= \frac{h-\gamma}{\log n} + O\left(\frac{|h|+1}{\log^2 n}\right). \end{aligned} \tag{2.21}$$

Since $\Gamma(1) = 1$ and since $d/dt(1/\Gamma(t))$ is bounded for t between 1 and R, we have

$$\frac{1}{\Gamma(R)} = 1 + O\left(\frac{|h|+1}{\log n}\right). \tag{2.22}$$

Also

$$\frac{1}{\sqrt{H}} = \frac{1}{\sqrt{k}}\left(1 + O\left(\frac{1}{k}\right)\right); \tag{2.23}$$

$$\frac{1}{\sqrt{k}} = \frac{1}{\sqrt{\log n}}\left(1+O\left(\frac{|h|}{\log n}\right)\right). \tag{2.24}$$

Also

$$\log n^{R-1} = (R-1)\log n = h - \gamma + O\left(\frac{|h|+1}{\log n}\right)$$

so that

$$n^{R-1} = e^{h-\nu}\left\{1 + O\left(\frac{|h|+1}{\log n}\right)\right\}. \tag{2.25}$$

Up to this point, results are valid in the whole range (2.3) of Lemma 2.2, though they give an improvement on (2.3) when $|h| = o(\log n)$. But from now on, we take "O" as applying for large n uniformly in k in the range (2.18) only.

Consider $\log(R^k)$. We have

$$\begin{aligned}\log(R^k) &= k\log R\\ &= (\log n + h)\log\left\{1 + \frac{h-\nu}{\log n} + O\Big(\frac{|h|+1}{\log^2 n}\Big)\right\}\\ &= (\log n + h)\left\{\frac{h-\nu}{\log n} - \frac{h^2}{2\log^2 n} + O\left(\frac{|h|+1}{\log^2 n}\right)\right.\\ &\quad \left. + O\Big(\frac{|h|^3}{\log^3 n}\Big)\right\}\\ &= h - \gamma + \frac{h^2}{2\log n} + O\Big(\frac{|h|+1}{\log n}\Big) + O\Big(\frac{|h|^3}{\log^2 n}\Big).\end{aligned}$$

Thus

$$R^k = \left\{\exp\Big(h - \gamma + \frac{h^2}{2\log n}\Big)\right\}\left\{1 + O\left(\frac{|h|+1}{\log n}\right) + O\Big(\frac{|h|^3}{\log^2 n}\Big)\right\}. \tag{2.26}$$

Combining (2.6), (2.12) and (2.22) – (2.26), the result (2.19) follows.

Proof of Theorem 2.1. Let N be the integer nearest to $\log n$. Then we have, for $x = \log n$.

$$\begin{aligned}f(x) &= e^{-x}\sum_{k=1}^{\infty} s_k\frac{x^k}{k!} = e^{-x}\sum_{k=1}^{\infty}\frac{x^k}{k!}(s_N + s_k - s_N)\\ &= s_N + e^{-x}\sum_{k=1}^{\infty}\frac{x^k}{k!}(s_k - s_N).\end{aligned}$$

Let λ be a constant such that (2.17) holds. Write

$$\mu(n) = \log n - (\log n)^\lambda,\quad \nu(n) = \log n + (\log n)^\lambda. \tag{2.27}$$

Since, by (1.2)

$$s_k - s_N = o(k) \tag{2.28}$$

uniformly for $k \geqq N$, it follows from Theorem 137 (6) of Hardy [5] that

$$e^{-x}\sum_{k\geqq\nu(n)}\frac{x^k}{k!}(s_k - s_N) = o(1).$$

Also, since

$$s_k - s_N = o(N) \tag{2.29}$$

uniformly in $k \leqq N$, it follows from Theorem 137 (3), loc. cit., that

$$e^{-k} \sum_{k \leqq \mu(n)} \frac{x^k}{k!}(s_k - s_N) = o(1) .$$

Thus

$$f(x) = s_N + e^{-x} \sum_{\nu(n) < k < \mu(n)} \frac{x^k}{k!}(s_k - s_N) + o(1) . \tag{2.30}$$

We also have

$$\sigma_n = \frac{1}{n!} \sum_{k=1}^{n} p_{nk}(s_N + (s_k - s_N)) .$$

But Agnew ([1], p. 106) has remarked that

$$\frac{1}{n!} \sum_{k=1}^{n} p_{nk} = \frac{1}{n!} p_n(1) = 1 .$$

Hence

$$\sigma_n = s_N + \frac{1}{n!} \sum_{k=1}^{n} p_{nk}(s_k - s_N) . \tag{2.31}$$

Let b be a constant such that $b \geqq 1$ and such that, with the notation of (2.5),

$$\phi(b) < -2 . \tag{2.32}$$

It is possible to choose such a constant, since

$$\phi(\theta) \longrightarrow -\infty \text{ as } \theta \longrightarrow \infty .$$

It follows from (2.30) and (2.31) that

$$\begin{aligned} \sigma_n - f(\log n) &= \Big(\sum_{1 \leqq k \leqq \mu(n)} + \sum_{\nu(n) \leqq k < b \log n} + \sum_{k \geqq b \log n} \Big) \frac{p_{nk}}{n!}(s_k - s_N) \\ &\quad + \sum_{\mu(n) < k < \nu(n)} \Big(\frac{p_{nk}}{n!} - e^{-x}\frac{x^k}{k!} \Big)(s_k - s_N) + o(1) \\ &= \sum_1 + \sum_2 + \sum_3 + \sum_4 + o(1) , \end{aligned}$$

say, where $x = \log n$.

It follows from Lemma 2.1 that, for all terms occuring in the sum $\sum_1$, the value of $p_{nk}/n!$ is less than the value it takes for the last term, and by Lemma 2.3 this is

$$O\Big\{ \frac{1}{\sqrt{\log n}} \exp\Big[-\frac{1}{2}(\log n)^{2\lambda - 1} \Big] \Big\} .$$

Since the number of terms in the sum is $O(\log n)$, it follows with the aid of (2.28) that

$$\sum_1 = o(1) .$$

We can deal with $\sum_2$ in a similar way. Again for all terms occuring in the sum $\sum_3$, the value of $p_{nk}/n!$ is less than the value it takes for the first term, and by Lemma 2.2 this is

$$O\Big(\frac{1}{\sqrt{\log n}} n^{\phi(b)}\Big) .$$

We have, for each individual term

$$s_k - s_N = o(n)$$

and the number of terms in the sum does not exceed n; hence it follows with the aid of (2.32) that

$$\sum_3 = o\Big\{\frac{n^{\phi(b)+2}}{\sqrt{\log n}}\Big\} = o(1) .$$

It follows from Lemma 2.3 and from Theorem 137 (5) of Hardy [5] that in the range of summation of $\sum_4$ we have, with $x = \log n$, $h = k - \log n$

$$\frac{p_{nk}}{n!} - e^{-x}\frac{x^k}{k!} = \frac{1}{\sqrt{\log n}}\Big[\exp\Big(\frac{-h^2}{2\log n}\Big)\Big]\Big[O\Big(\frac{|h|+1}{\log n}\Big) + O\Big(\frac{|h|^3}{\log^2 n}\Big)\Big] .$$

Further, in this range it follows from (1.2) that

$$s_k - s_N = o(h) .$$

Further,

$$|h| + 1 = o(|h|)$$

except for the term $k = N$, since $|h| \geqq \frac{1}{2}$; and, for this term $s_k - s_N$ vanishes. Hence

$$\sum_4 = o\Big\{\frac{1}{\sqrt{\log n}} \sum_{\mu(n)<k<\nu(n)} \chi(h)\Big\} \tag{2.33}$$

where

$$\chi(h) = \chi(h; n) = |h|\Big(\frac{|h|}{\log n} + \frac{|h|^3}{\log^2 n}\Big)\exp\Big(\frac{-h^2}{2\log n}\Big) .$$

It is easily verified that, for $h > 0$, $\chi(h)$ is increasing for $h < h_0 = h_0(n)$ (say) and decreasing for $h > h_0$. Thus for any integer k with

$$h = k - \log n \leqq h_0 - 1$$

we have

$$\chi(h) < \int_h^{h+1} \chi(t)dt \ , \tag{2.34}$$

and similarly for $h \geqq h_0 + 1$.

$$\chi(h) < \int_{h-1}^{h} \chi(t)dt \ . \tag{2.35}$$

There are at most two terms for which neither of the inequalities (2.34), (2.35) are valid; and these are $O(1)$ (uniformly in n) since $\chi(h; n)$ is bounded. We can deal with negative values of h in a similar way. It thus follows from (2.27) that expression in curly brackets in (2.33) does not exceed

$$\int_{-(\log n)^\lambda}^{(\log n)^\lambda} \chi(h)dh + O\Big(\frac{1}{\sqrt{\log n}}\Big) .$$

Using this in (2.33) it follows that

$$\begin{aligned} \sum_4 &= O\Big\{\frac{1}{\sqrt{\log n}}\int_{-(\log n)^\lambda}^{(\log n)^\lambda}\Big(\frac{h^2}{\log n} + \frac{h^4}{\log n}\Big)\exp\frac{-h^2}{2\log n}\Big)du\Big\} \\ &= O\Big\{\int_{-\infty}^{\infty}(u^2 + u^4)\exp\Big(\frac{-u^2}{2}\Big)du\Big\} . \end{aligned}$$

This is enough to establish (2.1).

3. THEOREM 3.1. *Suppose that*

$$a_k = O\Big(\frac{1}{k^{\frac{1}{2}}}\Big) . \tag{3.1}$$

Let m be an integer valued function of n such that

$$\limsup \ |(m - \log n)/\sqrt{\log n}\ | \leqq c \ , \tag{3.2}$$

where c is a constant. In other words

$$m = \log n + c\sqrt{\log n} + o(\sqrt{\log n}) . \tag{3.3}$$

Then

$$\limsup_{n\to\infty} \ |\sigma_n - s_m| \leqq \phi(c) \limsup_{k\to\infty} \ |k^{\frac{1}{2}} a_k| \ , \tag{3.4}$$

where $\phi(c)$ is a Tauberian constant defined by

$$\phi(c) = \sqrt{\frac{2}{\pi}}\Big\{\exp(-c^2/2) + c\int_0^c \exp(-u^2/2)du\Big\} . \tag{3.5}$$

The result is the best possible in the sense that equality can occur *in* (3.4).

The least possible value of $\phi(c)$ *occurs when* $c = 0$.

Theorem 3.1 follows at once from Agnew's result of ([2] §'s 2, 3) with the aid of Theorem 2.1. It also could be deducted from Theorem 1 of Meir[2] [9], since Lemma 2.3 satisfies Meir's conditions when Meir's q equals $\log n$.

Theorem 3.1 implies analogous results to Theorem 1.4, 1.5 of Agnew [2] but for the Lototsky transform instead. The analogue of Agnew's result of ([2] § 4) for the Lototsky transform can be deduced from Agnew's result of § 4 with the aid of Theorem 2.1. The only change in our results is that we have $\log n$ instead of Agnew's t.

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[2] Meir states in his Lemma B that the other conditions imply his Equation (3.4). This is obviously untrue, but if we assume his Equation (3.4) as an additional hypothesis, then Meir's theorems become correct.