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ON QF-1 ALGEBRAS

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Let A be a finite-dimensional associative algebra with identity over a field k, M an A-module which is finite-dimentional as a vector space over k, and $E = \mathrm{Hom}_k(M, M)$ the algebra of linear transformations on M. For $a \in A$. Let a_L denote the linear transformation of M given by $a_L(x) = ax$, for $x \in M$. Define the following subalgebras of E:

$$A_L = \{a_L: a \in A\}$$
$$C = \{f \in E: f(ax) = af(x) \text{ for each } a \in A, x \in M\}$$
$$D = \{f \in E: f(g(x)) = g(f(x)) \text{ for each } g \in C, x \in M\} .$$

Clearly, $A_L \subseteqq D$. Require M to be faithful. Then A is isomorphic to, and will be identified with, A_L. If $A = D$, it is said that the pair (A, M) has the double centralizer property.

A is called a QF-1 algebra if (A, M) has the double centralizer property for each faithful A-module M.

The following results in the theory of QF-1 algebras are obtained:

1. Let A be a commutative algebra over an arbitrary field. Then A is QF-1 if and only if A is Frobenius.

2. Let A be an algebra such that the simple left A-modules are one-dimensional. Suppose there exist distinct simple two-sided ideals A_1 and A_2 contained in the radical of A, and primitive idempotents e and f, such that $eA_kf \neq 0$, for $k = 1, 2$. Then A is not QF-1.

3. Let A be an algebra with the properties that the simple left A-modules are one-dimensional, and the two-sided ideal lattice of A is distributive. Then if A satisfies any one of the following conditions, it is not QF-1.

(a) There exist, for $r \geqq 2$, $2r$ distinct simple two-sided ideals A_{uv} contained in the radical, and primitive idempotents e_{i_u} and e_{j_v} for $1 \leqq u, v \leqq r$, satisfying $e_{i_u}A_{uv}E_{j_v} \neq 0$, where the index pair (u, v) ranges over the set

$$(1, 1),\ (2, 1),\ (2, 2),\ (3, 2),\ (3, 3),\ \cdots,\ (r, r-1),\ (r, r),\ (1, r) .$$

(b) There exist, for $r \geqq 1$, $2r + 2$ distinct simple two-sided ideals A_{uv} and A^{ρ}_{v}, for $(u, v) = (1, 1), (1, 2), \cdots, (r-1, r-1)$, $(r-1, r)$, and $(\rho, v) = (1, 1), (2, 1), (3, r)$, and $(4, r)$, and primitive idempotents $e_{,u}$, e_{j_v}, and e_{k_ρ} satisfying $e_{i_u}A_{uv}e_j \neq 0$ and $e_{k_\rho}A^{\rho}_{v}e_{j_v} \neq 0$, where (u, v) and (ρ, v) range over the index pairs indicated above.

It is to be noted that the condition given in 2b is but one of three conditions of that type which may be formulated. An algebra satisfying either of the other two conditions is also not QF-1.

A special case of (2b) is worth mentioning, namely the case where the set of index pairs (u, v) which occur in statement is empty. There are two variants of the case, rather than the usual three. This special case appears separately in the following form: let A be an algebra whose simple two-sided ideals are one-dimensional, and whose two-sided ideal lattice is distributive. Suppose that either (i) $e_kA_ke \neq 0$ or (ii) $eA_ke_k \neq 0$, for $k = 1, 2, 3, 4$, where the A_k are distinct simple two-sided ideals, and the e_k and e are primitive idempotents of A. Then A is not QF-1.

The results (2a) and (2b) appear in Chapter 3, and are stated there in terms which involve the notion of the graph associated with the zero ideal of an algebra. The notion of the graph associated with A_0, where A_0 is a two-sided ideal of an algebra A contained in the radical of A was first used by J. P. Jans in his dissertation. The results above was stated in more elementary terms for the sake of brevity.

Introduction. Throughout this paper, an algebra will be a finite-dimensional associative algebra with identity over an arbitrary field. All modules are finite dimensional over these fields.

In 1946, C. Nesbitt and R. M. Thrall showed [5] that if A is a Quasi-Frobenius algebra, then each faithful representation R of A is equal to its own second commutator algebra R''. In 1948 Thrall [6] initiated the study of the class of algebras A for which $R = R''$ for each faithful representation R of A. He called this class the QF-1 algebras, and showed by an example that it properly contains the Quasi-Frobenius algebras.

Although results in the theory of QF-1 algebras have been obtained since Thrall's original paper, notably by Morita [3], a problem whose solution was unknown to Thrall remains unsolved to this day. The problem, which may be posed in the form of a question, is this: Is the property "QF-1 ness" equivalent to one or more purely internal properties of an algebra? By the expression "internal properties" is meant properties of the algebra expressible in terms of the left, right, or two-sided ideals of the algebra, in terms of the structure constants associated with a basis, etc.

The main theorems of this paper, Theorems 1.1, 2.1, and 3.2, may be viewed as contributions to the solution of this problem. The first of these states that a commutative algebra is QF-1 if and only if it is Frobenius. The latter class of algebras has several characterizations which may be called "internal" in the sense of the previous paragraph. The other results apply to algebras A which are required to satisfy the first, or both of the following conditions:

(i) The simple left A-modules are one-dimensional.

(ii) The two-sided ideal lattice is distributive.

The second results below is stated in terms which involve the notion of the graph associated with the zero ideal of an algebra. This notion will be defined in § 3. The results are as follows:

1. Let A be an algebra satisfying property (i). Suppose there exist distinct simple two-sided ideals A_1 and A_2 contained in the radical of A, and primitive idempotents e and f, such that $eA_kf \neq 0$, for $k = 1, 2$. Then A is not QF-1.

2. Let A be an algebra which satisfies conditions (i) and (ii). If the graph associated with the zero ideal of A contains a cycle, a vertex of order greater than three or a chain which branches at both ends, then A is not QF-1.

1. This section is devoted to the following theorem which provides a nice characterization of commutative QF-1 algebras.

THEOREM. *A commutative algebra is QF-1 if and only if it is Frobenius.*

The "if" part of the theorem is true in general; each Frobenius algebra is Quasi Frobenius, as was first established by, Nakayama [4] and Quasi-Frobenius algebras are QF-1, as was shown by Nesbitt and Thrall.

The proof in the other direction is facilitated by the following lemma, the proof of which is omitted.

LEMMA. *Suppose A is an algebra, and A_i, $i = 1, 2, \cdots, n$ are two-sided ideals of A such that $A = A_1 + A_2 + \cdots + A_n$; where the sum is vector space direct. Let e denote the identity of A, and write $e = e_1 + e_2 + \cdots + e_n$, where $e_i \in A_i$. Then:*

(i) *Each e_i is the identity for A_i.*

(ii) *A is Frobenius if and only if each A_i is Frobenius.*

(iii) *A is QF-1 if and only if each A_i is QF-1.*

We proceed with the "only if" part of the theorem. Let A be an algebra which is not Frobenius. We shall construct a faithful representation of A which is smaller than its second commutator algebra.

By virtue of the preceding lemma, we may assume A to be indecomposable as a module over itself. Then $\bar{A} = A/N$ is a simple A module and each simple A-module is isomorphic to $\bar{A}$. For proofs of these statements, see [1]. Let $(\bar{A}: k) = t < \infty$.

The set $S(A) = \{x \in A: Nx = xN = 0\}$ is called the socle of A. It is well-known that $S(A)$ is the sum of the simple (two-sided) ideals of A. The hypothesis implies that $S(A)$ is not simple. For suppose $S(A)$ were simple. Let B be a basis for A containing $s \in S(A)$, and let $f: A \to k$ be that unique linear map satisfying $f(s) = 1, f(b) = 0$, $b \in B \sim \{s\}$. Clearly $S(A) \not\subset \ker f$. Therefore, since each nonzero ideal of A must contain $S(A)$, it follows that $\ker f$ can contain no such ideals. This implies that A is Frobenius, a contradiction. Thus $S(A)$ is not simple. Let A_1 and A_2 be distinct simple ideals of A.

Choose a composition series $0 = V_0 \subset V_1 \subset \cdots \subset V_n = A$ satisfying $V_j = \sum_{i=1}^{j} A_i$, for $j = 1, 2$, and $V_{n-1} = N$. By an inductive procedure, a single basis for A may be chosen which contains a basis for V_i for each i. We may assume that that part of the basis in V_2 but not in V_1 is chosen from A_2.

Let R denote the regular representation of A relative to the basis chosen above. Let X_i be the induced representation whose space is V_i/V_{i-1} for $i = 1, 2, \cdots, n$. Because the composition factors are all isomorphic, it may be assumed that for each $\alpha \in A$, the matrices $X_i(\alpha)$ are equal; denote their common value by $X(\alpha)$. $R(\alpha)$ is exhibited in the following block form:

$$R(\alpha) = \left\| \begin{matrix} X(\alpha) & & & \\ Q(\alpha) & P(\alpha) & & \\ Y_1(\alpha) & S_1(\alpha) & X(\alpha) & \\ Y_2(\alpha) & S_2(\alpha) & & X(\alpha) \end{matrix} \right\|$$

where $X(\alpha)$ and $Y_i(\alpha)$ are $t \times t$ matrices, $Q(\alpha)$ and $S_i(\alpha)$ are $(n - m - 1)t \times t$ and $t \times (n - m - 1)t$ matrices, respectively, and $P(\alpha)$ is an $(n - m - 1)tx(n - m - 1)t$ triangular matrix which has $n - m - 1$ copies of $X(\alpha)$ on its main diagonal.

For $i = 1, 2$, let y_i denote arbitrary nonzero elements of A_i. For $x \in N$, $y_i x = x y_i = 0$. In general, $y_i x \in A_i$. Thus, $X(y_i)$, $Q(y_i)$ and $P(y_i)$ are zero matrices and $Y_i(y_j) = 0$, $i \neq j$. The matrices $Y_i(y_i)$ are nonzero, since otherwise y_i would annihilate A, and hence $y_i 1 = y_i$ would be zero, contradicting the choice of y_i. Furthermore, $Y_i(y_i)$ are nonsingular, by the following argument: consider the nontrivial linear transformation $T_i: \bar{A} \to S_i$ defined by $T_i(x + N) = y_i x$. Because A is commutative, it is easily seen that the T_i are A-homomorphisms, and hence isomorphisms, since $\bar{A}$ and S_i are simple. As $Y_i(y_i)$ is the matrix of T_i relative to appropriate bases, it is nonsingular. Without loss of generality we may assume that $Y_i(y_i)$ is the $t \times t$ identity matrix, I_t, for $i = 1, 2$.

Let I be the 3×3 identity matrix, T the 3×3 matrix with 1's directly below the main diagonal and 0's elsewhere. We observe, by direct computation, that the matrix function

$$S(\alpha) = \begin{Vmatrix} I \times X(\alpha) & & \\ I \times Q(\alpha) & I \times P(\alpha) & \\ I \times Y_1(\alpha) + T \times Y_2(\alpha) & I \times S_1(\alpha) + T \times S_2(\alpha) & I \times X(\alpha) \end{Vmatrix}$$

is a representation for A.

Let C represent a matrix in the commuting algebra $\mathscr{C}(V)$ of V, the space on which S acts. Then C must commute with the two matrices representing y_1 and y_2

$$\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I \times I_t & 0 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ T \times I_t & 0 & 0 \end{Vmatrix} .$$

This implies that C must have the form

$$\begin{Vmatrix} C_{11} & 0 & 0 \\ * & * & 0 \\ * & 0 & C_{11} \end{Vmatrix} ,$$

where $C_{11}(T \times I_t) = (T \times I_t)C_{11}$. The matrix

$$D = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ T^2 \times I_t & 0 & 0 \end{Vmatrix}$$

commutes with each matrix C having the above form, yet $D \neq S(\alpha)$, for each $\alpha \in A$. Thus, A is not a QF-1 algebra. This concludes the proof.

2. From this point on, we consider algebras A over a fixed field k with radical N such that A/N is the ring-direct sum of simple ideals each of which is isomorphic to k.

Each algebra of this type admits a vector space decomposition

$$A = S + N \tag{1}$$

where S is the direct sum of ideals of dimension 1 over θ. That is, S has a basis of primitive orthogonal idempotents $\{e_i\} i = 1, 2, \cdots, n$, and $1 = \sum e_i$. We call the $\{e_i\}$ *the collection of idempotents associated with the decomposition* (1).

If I is a simple two-sided ideal of an algebra A it is not difficult to prove the following facts:

(i) I is one-dimensional

(ii) There exists exactly one pair of indices (i, j) such that $e_i I e_j \neq 0$. For this pair of indices, $e_i I e_j = I$.

Theorem 2.1 gives the first of four conditions which imply that A is not QF-1.

THEOREM 2.1. *Let A be an algebra, A_1 and A_2 distinct simple two-sided ideals of A contained in N, and suppose $e_iA_ke_j \neq 0$ for $k = 1, 2$, where e_i and e_j are (not necessarily distinct) primitive idempotents of A. Then A is not QF-1.*

Proof. It is clear that the condition $e_iA_ke_j \neq 0$ remains true if e_i and e_j are replaced by isomorphic idempotents. We may therefore assume that if e_i and e_j are distinct they are nonisomorphic. Similar assumptions will be made tacitly in the proof of Theorem 3.2. We may choose a decomposition of A of form (1) such that the associated collection of idempotent contains e_i and e_j.

We put the left regular representation R of A into triangular form relative to a basis containing the $\{e_r\}$, a basis for N, and a basis b_k for $A_k, k = 1, 2$. Specifically, let $x_k(\alpha)$ be the coefficient of e_k in the representation of an element α in A in terms of this basis, and let $y_k(\alpha)$ be the coefficient of b_k in the expansion of αe_j, for $k = 1, 2$. Since the A_k are annihilated on either side by the radical, it follows that R has the form

$$R(\alpha) = \begin{Vmatrix} x_j(\alpha) & & & \\ P(\alpha) & Q(\alpha) & & \\ y_1(\alpha) & S_1(\alpha) & x_i(\alpha) & \\ y_2(\alpha) & S_2(\alpha) & & x_i(\alpha) \end{Vmatrix},$$

and that there exist elements $\alpha_k \in A_k$, for $k = 1, 2$ such that $x_j(\alpha_k)$, $x_i(\alpha_k)$, $P(\alpha_k)$, $Q(\alpha_k)$ and $S_l(\alpha_k)$ are zero matrices, and such that $y_l(\alpha_k)$ is 0 for $l \neq k$ and 1 for $l = k$.

Let I and T be as in the proof of Theorem 1.1. We observe by direct computation, that the matrix function

$$S(\alpha) = \begin{Vmatrix} I \times x_j(\alpha) & & \\ I \times P(\alpha) & I \times Q(\alpha) & \\ I \times y_1(\alpha) + T \times y_2(\alpha) & I \times S_1(\alpha) + T \times S_2(\alpha) & I \times x_i(\alpha) \end{Vmatrix}$$

is a representation for A. A comparison of S with the regular representation R shows that S is faithful.

Let V be the module on which S acts. If C represents a matrix in the commuting algebra $C(V)$ of this representation, then C must commute with the two matrices

$$\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ T & 0 & 0 \end{Vmatrix}$$

which represent α_1 and α_2. This implies that C must have the form

$$(2) \qquad \begin{Vmatrix} C_{11} & 0 & 0 \\ * & * & 0 \\ * & * & C_{11} \end{Vmatrix},$$

where $C_{11}T = TC_{11}$. The matrix

$$D = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ T^2 & 0 & 0 \end{Vmatrix}$$

commutes with each matrix of form (2). However, $D \neq S(\alpha)$ for each $\alpha \in A$. Thus, S is strictly smaller than its second commutator algebra S''. This concludes the proof.

3. For an algebra A, the symbol L_A will denote the lattice of two-sided ideals of A. In this section we consider only algebras A for which L_A is distributive. For such algebras, we define a graph, G_0, associated with the zero ideal of A. (This is a special case of the notion of the graph associated with an arbitrary two-sided ideal contained in the radical of A-a notion which was first defined and investigated in [2]). Let $\{A_k\}_k$ be the collection of simple two-sided ideals of A. Let e_i be the collection of primitive orthogonal idempotents associated with a vector space decomposition of A of the form (1). The graph G_0 consists of a set of n symbols $P_1, \cdots, P_n$, called the vertices of the graph, and a relation R in this set defined by: P_iRP_j if and only if there exists k such that $e_iA_ke_j \neq 0$. If P_iRP_j obtains, then P_i and P_j are said to be connected by an (oriented) *edge*. It is clear that the definition of R does not depend upon the particular decomposition of A of form (1).

We shall say that the vertex P^i has right order τ (left order τ) if there exist distinct vertices $P_{i_1}, \cdots, P_{i_\tau}$, such that $P_iRP_{i_j}(P_{i_j}RP_i)$ hold for $j = 1, 2, \cdots, \tau$. The order of a vertex is the larger of the two orders. A chain C is a set of vertices and edges

$$(P_{i_1}, P_{i_1}RP_{i_2}), P_{i_2}, P_{i_3}RP_{i_2}, \cdots, P_{i_{\tau-1}}, P_{i_{\tau-1}}RP_{i_\tau}, P_{i_\tau}, (P_{i_{\tau+1}}RP_{i_\tau}, P_{i_{\tau+1}}),$$

such that successive edges are distinct, that is $i_\nu \neq i_{\nu+2}$, for $\nu = 1, 2, \cdots, \tau - 1$. The parentheses indicate that the first and last edges of the chain may have either orientation. The chain C_2 extends the chain C_1 on the right (and C_1 extends C_2 on the left) if the last vertex of C_1 is the first vertex of C_2 and identifying these equal vertices makes C_1 followed by C_2 a chain. The chain C is a cycle if it extends itself. Note that a cycle has an even number of edges.

A chain branches at one end if it can be extended by at least two distinct edges at that end.

The following lemma is crucial in the proof of the main theorem.

LEMMA 3.1. *Let* $P_{i_k}RP_{j_k}$, $k = 1, 2, \cdots, m$ *be a collection of distinct edges of* G_0. *Then for each* k *there exists special element* $\alpha_k \in A$, *and representation* R_k *such that*:

(1) R_k *is a triangularized representation of* A *with the form*

$$R_k = \begin{Vmatrix} x_{j_k} & & \\ P_k & Q_k & \\ y_k & S_k & x_{i_k} \end{Vmatrix},$$

where x_{j_k} *and* x_{i_k} *are one-dimensional representations of* A.

(2) $R_k(\alpha_j) = 0$ *for* $k \neq j$, *and* $R_k(\alpha_k)$ *has a* 1 *in the lower left hand corner and zeros elsewhere.*

(3) $+\sum_{k=1}^{m} R_k$ *is a faithful representation for* A.

Proof. Without loss of generality, we may assume that, for each k, $e_{i_k}A_k e_{j_k} \neq 0$. (Observe that condition (ii) listed immediately preceding Theorem 2.1 implies that to each simple two-sided ideal is associated exactly one edge of G_0). We choose a decomposition of A of form (1) such that the associated collection $\{e_i\}$ of primitive idempotents contains e_{i_k} and e_{j_k}, all k. Select $\beta_k \neq 0$ in A_k. We may triangularize the left regular representation R of A relative to a basis B containing the $\{e_i\}$, containing a basis for N, and containing β_k for each k (here, distributivity of L_A is used, in case $m \geqq 3$). Let $x_i(\alpha)$ denote the coefficient of e_i in the expansion of an element α in terms of the basis first selected, and let $y_k(\alpha)$ denote the coefficient of α_k in the expansion of αe_{j_k}. Put $A'_k = \sum \{A_i : i \neq k\}$, and $B_k = A/A'_k$. The representation R_k with space B_k has, relative to the basis of B_k induced by B, the form described in (1). The elements α_k described in (2) can be gotten as the appropriate scalar multiples of β_k for each k. Finally, that $+\sum_{k=1}^{m} R_k$ is faithful is a consequence of the distributivity of L_A.

In terms of the graph G_0, we can now state three other conditions for an algebra to be QF-1.

THEOREM 3.2. *If the graph* G_0 *of an algebra* A *has a cycle, a vertex of order greater than three, or a chain which branches at both ends, then the algebra is not* QF-1.

Proof. We consider separately the various cases. Assume that G_0 has a vertex of order greater than three. We may assume that the order of the vertex in this case is the left order, the other case being handled analogously. Then there exist four distinct edges $P_{i_k}RP_i$, for $k = 1, \cdots, 4$. By Lemma 3.1, there exist four representations

$$R_k = \left\| \begin{matrix} x_i & & \\ p_k & Q_k & \\ y_k & S_k & x_{i_k} \end{matrix} \right\|$$

and four special elements α_k, $k = 1, 2, 3, 4$, satisfying the conditions of that lemma.

Let I and T be as in the proof of Theorem 2.1, and let I_6 be the 6×6 identity matrix. Then the matrix function

$$S = \left\| \begin{matrix} I_6 \times x_i & & & & & & & & \\ (I, 0) \times p_1 & I \times Q_1 & & & & & & & \\ (0, I) \times p_2 & & I \times Q_2 & & & & & & \\ (I, I) \times p_3 & & & I \times Q_3 & & & & & \\ (I, T) \times p_4 & & & & I \times Q_4 & & & & \\ (I, 0) \times y_1 & I \times S_1 & & & & I \times x_{j_1} & & & \\ (0, I) \times y_2 & & I \times S_2 & & & & I \times x_{j_2} & & \\ (I, I) \times y_3 & & & I \times S_3 & & & & I \times x_{j_3} & \\ (I, T) \times y_4 & & & & I \times S_4 & & & & I \times x_{j_4} \end{matrix} \right\|$$

is seen, by direct computation, to be a representation of A. A comparison of S with the representation $+\sum_{k=1}^{4} R_k$ shows that S is faithful. Let C represent a matrix in the commuting algebra $\mathscr{C}(V)$ of V, the module on which S acts. In particular, B must commute with the four matrices which represent α_k, $k = 1, \cdots, 4$. By direct computation one sees that this condition forces C to have the form:

$$C = \left\| \begin{matrix} C' & 0 & \cdot & \cdot & \cdot & & & & & 0 \\ 0 & C' & 0 & \cdot & \cdot & \cdot & & & & 0 \\ * & & & * & & & \cdot & & & \\ & & & & & & \cdot & \cdot & & \\ & & & & & & \cdot & \cdot & \cdot & \\ & & & & & & 0 & \cdot & \cdot & \cdot \\ & & & & & & C' & 0 & \cdot & \cdot \\ & & & & & & 0 & C' & 0 & \cdot \\ & & & & & & 0 & 0 & C' & 0 \\ * & & & * & & & 0 & 0 & 0 & C' \end{matrix} \right\|$$

where C' is a 3×3 matrix satisfying $TC' = C'T$.

Let D be the matrix of the same dimensions as C having the submatrix $(0, T^2) \times 1$ in the lower left hand corner and 0's elsewhere. Then $DC = CD$ for each matrix C in $\mathscr{C}(V)$. However, $D \neq S(\alpha)$ for each $\alpha \in A$, and thus $S \subsetneqq S''$, concluding the proof for this case.

Now suppose that G_0 contains a cycle. It can be shown that if G_0 has a chain which has a repeated edge then G_0 has a cycle. Thus we may assume that G_0 has a cycle all of the edges of which are distinct. Let

$$P_{i_1}, P_{i_1}RP_{j_1}, P_{j_1}, P_{i_2}RP_{j_1}, \cdots P_{j_\tau}, P_{i_1}RP_{j_\tau}, P_{i_1}$$

be that cycle. Let $R_{11}, R_{21}, \cdots R_{\tau\tau}, R_{1\tau}$ be the representations associated with the edges of the cycle by Lemma 3.1.

$$R_{\mu\nu} = \begin{Vmatrix} x_{j_\nu} & & \\ P_{\mu\nu} & Q_{\mu\nu} & \\ y_{\mu\nu} & S_{\mu\nu} & x_{i_\mu} \end{Vmatrix} \qquad (\mu, \nu) = (1, 1), (2, 1), \cdots (1, \tau) .$$

From the submatrices of these $R_{\mu\nu}$ construct a matrix function R,

$$R = \begin{Vmatrix} X_1 & & \\ P & Q & \\ Y & S & X_2 \end{Vmatrix}$$

which has the following description in block form (it is to be understood that those portions of the matrix blocks not described are filled with 0's):

(i) X_1 is the direct sum of the representations $I \times x_{j_\nu}$, for $\nu = 1, 2, \cdots, \tau$; X_2 is the direct sum of $I \times x_{i_\mu}$, for $\mu = 1, 2, \cdots, \tau$; and Q is the direct sum of $I \times Q_{\mu\nu}$ for

$$(\mu, \nu) = (1, 1), (2, 1), (2, 2), \cdots (\tau, \tau), (1, \tau) .$$

(ii) P has $I \times P_{\mu\nu}$ directly below $I \times x_{j_\nu}$, and to the left of $I \times Q_{\mu\nu}$, for $(\mu, \nu) \in J \sim \{(1, \tau)\}$, and contains $T \times P_{1\tau}$ directly below $I \times x_{j_\tau}$ and to the left of $I \times Q_{1\tau}$

(iii) S has $I \times S_{\mu\nu}$ directly below $I \times Q_{\mu\nu}$, and to the left of $I \times x_{i_\mu}$, for $(\mu, \nu) \in J$.

(iv) Y has $I \times y_{\mu\nu}$ directly below $I \times x_{j_\nu}$, and to the left of $I \times x_{i_\mu}$, for $(\mu, \nu) \in J \sim \{(1, \tau)\}$, and contains $T \times y_{i\tau}$ directly below $I \times x_{j_\tau}$, and to the left of $I \times x_{i_1}$.

Note that Y has the form

$$\left\| \begin{array}{cccccc} I \times y_{11} & & & & & T \times y_{1\tau} \\ I \times y_{21} & I \times y_{22} & & & & \\ & I \times y_{32} & I \times y_{33} & & & \\ & & I \times y_{43} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & \cdot & \\ & & & & I \times y_{\tau-1,\tau-1} & \\ & & & & I \times y_{\tau,\tau-1} & I \times y_{\tau\tau} \end{array} \right\| .$$

One shows, by direct computation, that the matrix function constructed above is indeed a representation for A. Comparing A with the (faithful) representation

$$+\sum \{R_{\mu\nu}\colon (\mu, \nu) = (1, 1), (2, 1) \cdots (\tau, \tau), (1, \tau)$$

we see that R is faithful. Let C be the matrix representing an element of the commuting algebra. $\mathscr{C}(V)$ of V the module on which R acts. Let $\alpha_{\mu\nu}$ be the 2τ special elements associated with the representations $R_{\mu\nu}$ by Lemma 3.1. Then C must commute with R evaluated at each of these elements. Direct computation shows that this forces C to have the form:

$$C = \left\| \begin{array}{cc} C' & \\ * & C' \end{array} \right\|$$

where C' is the direct sum of r copies of a 3×3 matrix C_0 satisfying $C_0 T = TC_0$.

Now let D be the matrix of the same dimensions as R, which has 0's in all positions except the 3×3 position corresponding to the upper right hand corner of the matrix Y. Require that the matrix in the special position indicated by T^2. Then $DC = CD$, and yet $D \neq R(\alpha)$, for each $\alpha \in A$. Thus, $R \nsubseteq R''$, A is not QF-1, and the proof is complete for this case.

Finally, assume that the graph of A contains a chain which branches at both ends. It is enough to consider the case that all the edges involved are distinct, for if they are not distinct the graph has a cycle. Let the chain and its branches be as follows:

$$\begin{array}{ccc} P_{k_1} R P_{j_1} & & P_{k_3} R P_{j_\tau} \\ & P_{j_1}, P_{i_1} R P_{j_1}, P_{i_1}, \cdots P_{j_\tau} & \\ P_{k_2} R P_{j_1} & & P_{k_4} R P_{j_\tau} \end{array} .$$

There are two other cases to consider, depending on the orientation of the first and last edges of C. These cases are handled analogously.

Let $R_{\mu\nu}$, for $(\mu, \nu) = (1, 1), (1, 2), \cdots, (\tau - 1, \tau)$, and R^{ρ}_{ν}, for $(\rho, \nu) = (1, 1), (2, 1), (3, \tau), (4, \tau)$ be the $2\tau + 2$ representations associated with C and its edges as given by Lemma 3.1.

$$R_{\mu\nu} = \begin{Vmatrix} x_{j_\nu} & & \\ P_{\mu\nu} & Q_{\mu\nu} & \\ y_{\mu_\nu} & S_{\mu\nu} & x_{i_\mu} \end{Vmatrix} \qquad (\mu, \nu) = (1, 1), (1, 2), \cdots, (\tau - 1, \tau)$$

and

$$R^{\rho}_{\nu} = \begin{Vmatrix} x_{j_\nu} & & \\ P^{\rho}_{\nu} & Q^{\rho}_{\nu} & \\ y^{\rho}_{\nu} & S^{\rho}_{\nu} & x_{k_\rho} \end{Vmatrix} \qquad (\rho, \nu) = (1, 1), (2, 1), (3, \tau), (4, \tau) .$$

Form the matrix function R,

$$R = \begin{Vmatrix} X_1 & & \\ P & Q & \\ Y & S & X_2 \end{Vmatrix} \tag{3}$$

from the submatrices of the $R_{\mu\nu}$ and R^{ρ}_{ν}, as follows:

(i) X_1 is the direct sum of $I_6 \times x_{j_\nu}$ for $\nu = 1, 2, \cdots, \tau$; Q is the direct sum of $I_6 \times Q_{\mu\nu}$ for $(\mu, \nu) = (1, 1), (1, 2), \cdots, (\tau - 1, \tau)$, and $I_6 \times Q^{\rho}_{\nu}$, for $(\rho, \nu) = (1, 1), (2, 1), (3, \tau), (4, \tau)$; X_2 has $I \times x_{k_1} + I \times x_{k_2}$ in the upper left hand corner, $I \times x_{k_3} + I \times x_{k_4}$ in the lower right hand corner and the direct sum of $I_6 \times x_{i_\mu}$, for $\mu = 1, 2, \cdots, \tau - 1$ in the middle diagonal position.

(ii) P has $I_6 \times P_{\mu\nu}$ directly below $I_6 \times x_{j_\nu}$, and to the left of $I_6 \times Q_{\mu\nu}$, and $I_6 \times P^{\rho}_{\nu}$ directly below $I_6 \times x_{j_\nu}$, and to the left of $I_6 \times Q^{\rho}_{\nu}$.

(iii) S has $(I, 0) \times S^1_1$ directly below $I_6 \times Q^1_1$, and to the left of $I \times x_{k_1}$; $(0, I) \times S^2_1$ directly below $I_6 \times Q^2_1$ and to the left of $I \times x_{k_2}$; $(I, I) \times S^3_\tau$ directly below $I_6 \times Q^3_\tau$, and to the left of $I \times x_{k_3}$; $(I, T) \times S^4_\tau$ directly below $I_6 \times Q^4_\tau$, and to the left of $I \times x_{k_4}$, and $I_6 \times S_{\mu\nu}$ directly below $I_6 \times Q_{\mu\nu}$ and to the left of $I_6 \times x_{i_\mu}$, for

$$(\mu, \nu) = (1, 1), (1, 2), \cdots, (\tau - 1, \tau) .$$

(iv) Y has $I \times y^1_1 + I \times y^2_1$ directly below $I_6 \times x_{j_1}$ and to the left of $I \times x_{k_1} + I \times x_{k_2}$; $(I, I) \times y^3_\tau$ directly below $I_6 \times x_{j_\tau}$, and to the left of $I \times x_{k_3}$; $(I, T) \times y^4_\tau$ directly below $I_6 \times y_{j_\tau}$ and to the left of $I \times x_{k_4}$; and $I_6 \times y_{\mu\nu}$ directly below $I_6 \times x_{j_\nu}$ and to the left of $I_6 \times x_{i_\mu}$. Note that Y has the following form:

$$Y = \begin{Vmatrix} I \times y_1^1 & & & & & & \\ & I \times y_1^2 & & & & & \\ I_6 \times y_{11} & & I_6 \times y_{12} & & & & \\ & & I_6 \times y_{22} & I_6 \times y_{23} & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & I_6 \times y_{\tau-1,\tau} \\ & & & & & & (I, I) \times y_\tau^3 \\ & & & & & & (I, T) \times y_\tau^4 \end{Vmatrix}$$

One shows by direct computation that R is a representation of A. Moreover, a comparison of R with the faithful representation

$$+\sum \{R_{\mu\nu}(\mu, \nu) = (1, 1), (1, 2), \cdots, (\tau - 1, \tau)\} + \sum \{R_\nu^\rho(\rho, \nu) \\ = (1, 1), (2, 1), (3, \tau), (4, \tau)\}$$

shows that R is faithful.

Now let C represent an element of the commuting algebra $\mathscr{C}(V)$ of V the module on which R acts. For convenience, we write $C = (C_{ij})i, j = 1, 2, 3$, where the dimensions of the C_{ij}'s correspond in an obvious way to the dimension of the submatrices of R as exhibited in (3). Let $\alpha_{\mu\nu}$, for $(\mu, \nu) = (1, 1), (1, 2), \cdots, (\tau - 1, \tau)$, and α_ν^ρ, for $(\rho, \nu) = (1, 1), (2, 1), (3, \tau), (4, \tau)$ be the special elements of A associated with the representations $R_{\mu\nu}$ and R_ν^ρ, as described in the lemma. C must commute with the matrices representing these special elements. This implies that C_{12}, C_{13} and C_{23} are zero matrices, and C_{11} and C_{33} are 2τ and $2\tau + 2$ copies, respectively, of a 3×3 matrix C_0 satisfying $C_0T = TC_0$.

Now let D be a matrix of the same dimensions as $R(\alpha)$ which has 0's in all positions except the 3×3 position corresponding to the lower right corner of the matrix Y. Require that the matrix in the special position indicated by T^2. Then it is easily checked that $DC = CD$; however, $D \neq R(\alpha)$, for each $\alpha \in A$; hence $R \nsubseteq R''$; this concludes the proof for the final case.

We conclude this paper with an example which illustrates the incompleteness of the theory. Let k be an arbitrary field. The algebra A of dimension 3 over k consisting of 2×2 matrices of the form

$$\begin{Vmatrix} a & 0 \\ b & c \end{Vmatrix}$$

where a, b, and c are in k, satisfies the conditions that its irreducible representations are one-dimensional, and its two sided ideal lattice is distributive. The representation of A consisting precisely of the above matrices is not equal to its second commutator algebra. However, A does not satisfy any of the conditions given in the hypotheses of Theorems 2.1 and 3.2.

Thus, "QF-1 ness" is not a consequence of the negation of any, or all, of the conditions stated in the hypotheses of these theorems. A precise characterization of the class of QF-1 algebras, given in terms of "internal properties" as defined in the introduction, has yet to be found.

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