This paper deals primarily with a characterization of the tensor products of a family of $W^*$-algebras (abstract von Neumann algebras). It is especially concerned with infinite tensor products; the results, however, apply and have interest in the finite case.

A tensor product for a family $(\mathcal{A}_i)_{i \in I}$ of $W^*$-algebras is defined to be a $W^*$-algebra $\mathcal{A}$ together with injections $\alpha_i$ of $\mathcal{A}_i$ into $\mathcal{A}$ satisfying four conditions: the first two are that the $\alpha_i(\mathcal{A}_i)$ commute and generate $\mathcal{A}$; the last two are conditions on the set of positive normal functionals of $\mathcal{A}$ which are products with respect to the $\alpha_i(\mathcal{A}_i)$. A local tensor product is defined to be a tensor product satisfying a fifth condition—that its tail reduce to the scalars. It is shown that the local tensor products of $(\mathcal{A}_i)$ are precisely the incomplete direct products $\bigotimes_{i \in I} \mathcal{A}_i$, and that every tensor product is a direct sum of local tensor products which are not product isomorphic.

Suppose that $(\mathcal{A}_i)_{i \in I}$ is a family of $W^*$-algebras. We call $(\mathcal{A}, (\alpha_i)_{i \in I})$ a product for the family $(\mathcal{A}_i)_{i \in I}$ if $\mathcal{A}$ is a $W^*$-algebra if, for each $i \in I$, $\alpha_i$ is an injection of $\mathcal{A}_i$ into $\mathcal{A}$ with $\alpha_i(1) = 1$ and if the following conditions hold:

(I). $\alpha_i(\mathcal{A}_i)$ commutes with $\alpha_j(\mathcal{A}_j)$ for all $i, j \in I$ with $i \neq j$.
(II). $\mathcal{B}(\alpha_i(\mathcal{A}_i)) = \mathcal{A}$: that is, $\mathcal{A}$ is the smallest $W^*$ subalgebra of $\mathcal{A}$ which contains all $\mathcal{A}_i$ for $i \in I$.

By a product functional for $(\mathcal{A}, (\alpha_i))$ we mean a nonzero norma positive functional $\mu$ on $\mathcal{A}$ for which there exist normal positive functionals $\mu_i$ on $\mathcal{A}_i$ for each $i \in I$ such that:

$$\mu \left( \prod_{i \in I} \alpha_i(A_i) \right) = \prod_{i \in I} \mu_i(A_i)$$

whenever each $A_i \in \mathcal{A}_i$ and $A_i = 1$ for a.a. $i \in I$. (a.a. $i \in I$ means—here and throughout the paper—all but a finite number of $i \in I$)
Because of (II), it is evident that the $\mu_i$ determine $\mu$ uniquely, and we write $\mu = \bigotimes_{i \in I} \mu_i$. We will denote the set of product functional for $(\mathcal{A}, (\alpha_i))$ by $\Sigma_p$.

We call $(\mathcal{A}, (\alpha_i))$ a tensor product for $(\mathcal{A}_i)$ if it is a product for $(\mathcal{A}_i)$ (i.e., if (I) and (II) hold) and if the following conditions hold
(III). $\Sigma_p$ is separating: i.e., if $A \in \mathcal{A}^+$ and $\mu(A) = 0$ for all $\mu \in \Sigma_p$, then $A = 0$. 

13
(IV). For all \( \mu \in \Sigma_p \), (IV-\( \mu \)) holds.

(IV-\( \mu \)). \( \mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p \), and, if \( \nu_i \) is a nonzero normal positive functional on \( \mathcal{A}_i \) with \( \nu_i = \mu_i \) for a.a. \( i \in I \), then \( \bigotimes_{i \in I} \nu_i \) exists in \( \Sigma_p \).

We define the tail \( \mathcal{F} \) of a product \( (\mathcal{A}_i(\alpha_i)) \) to be the intersection over all finite subsets \( F \) of \( I \) of the algebras

\[ \mathcal{A}_{i \in F} = \mathcal{B}\{\alpha_i(\mathcal{A}_i); i \in I - F\}. \]

We call \( (\mathcal{A}_i(\alpha_i)) \) a local tensor product if it is a tensor product and the following condition holds:

(V). The tail \( \mathcal{F} \) of the product \( (\mathcal{A}_i(\alpha_i)) \) consists of the scalars only.

A local tensor product will be called a \((\mu_i)\)-local tensor product if \( \bigotimes \mu_i \in \Sigma_p \).

We show (Theorem 4.7) that, for every family \( (\mathcal{A}_i, \mu_i)_{i \in I} \) with \( \mu_i \) a normal positive functional on the \( W^* \)-algebra \( \mathcal{A}_i \) and

\[ 0 < \prod_{i \in I} \mu_i(1) < \infty, \]

a \((\mu_i)\)-local tensor product exists and is unique up to isomorphism. (An isomorphism of a product \( (\mathcal{A}_i(\alpha_i)) \) with a product \( (\mathcal{A}_i(\beta_i)) \) is an isomorphism \( \psi \) of \( \mathcal{B} \) onto \( \mathcal{B} \) such that \( \psi \circ \alpha_i = \beta_i \) for all \( i \in I \).)

In fact, a \((\mu_i)\)-local tensor product for \( (\mathcal{A}_i) \) can be constructed as follows. For each \( i \in I \) let \( \phi_i \) be an isomorphism of \( \mathcal{A}_i \) onto a von Neumann algebra on the Hilbert space \( H_i \) and let \( x_i \in H_i \) induce \( \mu_i \):

\[ \mu_i(A_i) = (\phi_i(A_i)x_i | x_i) \quad \text{for all } A_i \in \mathcal{A}_i. \]

Let \( \mathcal{A} \) be \( \bigotimes_{i \in I} (\phi_i(\mathcal{A}_i), x_i) \), i.e., von Neumann's incomplete direct product of \( (\phi(\mathcal{A}_i))_{i \in I} \) with respect to the \( C_0 \)-sequence \( \{x_i\} \) (see [7], [1], [2], or §2 below); and for each \( i \in I \) let \( \alpha_i = \gamma_i \circ \phi_i \), where \( \gamma_i \) is the natural injection of \( \phi_i(\mathcal{A}_i) \) into \( \mathcal{A} \). Then \( (\mathcal{A}_i(\alpha_i)) \) is a \((\mu_i)\)-local tensor product for \( (\mathcal{A}_i)_{i \in I} \). A special consequence of the uniqueness of \((\mu_i)\)-local tensor products is, then, roughly that the tensor product of a family of von Neumann algebras depends on their algebraic structure only (see Corollary 3.5, below, for a proper statement). This is an easy result which can be proved also from [9] or directly (see remark in [2, §3]). For finite \( I \), it is a result due to Misonou [4].

If \( I \) is finite, all tensor products of \( (\mathcal{A}_i)_{i \in I} \) are local and all are isomorphic. Thus properties (I), (II), (III), and (IV) characterize the finite tensor product. A special case of this result was proved by Nakamura [6]: he showed that (I) and (II) characterize the finite tensor product of finite factors. A stronger result of this kind was proved by Takesaki [10]: he showed that (I), (II) and the existence
of a nonzero ultraweakly continuous (not necessarily positive) product functional characterize the finite tensor product of factors (c.f. Lemma 6.2, below).

In §5, we determine all possible tensor products for \((\mathcal{A}_i)_{i \in I}\). Let \(\mathcal{A}\) be the set of all families \((\mu_i)_{i \in I}\) where each \(\mu_i\) is a normal positive functional on \(\mathcal{A}_i\) and \(0 < \prod_{i \in I} \mu_i(1) < \infty\). Define an equivalence relation \(R\) on \(\mathcal{A}\) by writing \((\mu_i) \sim (\nu_i)\) when a \((\mu_i)\)-local tensor product is necessarily a \((\nu_i)\)-local tensor product. Denote \(\mathcal{A}/R\) by \(\Delta\) and the natural quotient map \(\mathcal{A} \to \mathcal{A}/R\) by \(\varphi\). If \(\Gamma\) is a subset of \(\mathcal{A}\), we call \((\mathcal{A}_i, (\alpha_i))\) a \(\Gamma\)-tensor product for \((\mathcal{A}_i)_{i \in I}\) if \((\mathcal{A}_i, (\alpha_i))\) is a tensor product for \((\mathcal{A}_i)_{i \in I}\) and if

\[
\{(\mu_i) \in \mathcal{A} : \bigotimes \mu_i \text{ exists on } \mathcal{A}\} = \varphi^{-1}(\Gamma).
\]

Then:

1. Every tensor product for \((\mathcal{A}_i)_{i \in I}\) is a \(\Gamma\)-tensor product for some subset \(\Gamma\) of \(\mathcal{A}\).
2. For every nonempty subset \(\Gamma\) of \(\mathcal{A}\) a \(\Gamma\)-tensor product exists for \((\mathcal{A}_i)_{i \in I}\).
3. A \(\Gamma_1\)-tensor product is isomorphic (as a product) to a \(\Gamma_2\)-tensor product if and only if \(\Gamma_1 = \Gamma_2\).
4. A \(\Gamma\)-tensor product is a local tensor product if and only if \(\Gamma\) consists of only one point.
5. A \(\Gamma\)-tensor product is the direct sum of \(\{\alpha\}\)-tensor products as \(\alpha\) runs through \(\Gamma\).

In case each \(\mathcal{A}_i\) is semi-finite, the equivalence relation \(R\) may be defined explicitly by using the Kakutani product theorem for \(W^*\)-algebras [2]. We obtain \((\mu_i) \sim (\nu_i)\) if and only if

\[
\sum_{i \in I} [d(\mu_i, \nu_i)]^2 < \infty,
\]

where \(d(\mu, \nu)\) is roughly the infimum of \(||x - y||\) over all representations of \(\mathcal{A}\) as a von Neumann algebra and all \(x, y\) inducing \(\mu\) and \(\nu\) respectively.

It is not difficult to see that Takeda’s infinite direct product of \((\mathcal{A}_i)_{i \in I}\) (see [9]) is a \(\Delta\)-tensor product for \((\mathcal{A}_i)_{i \in I}\).

Section 6 contains some special results on tensor products of factors. Section 7 contains a few simple counterexamples which demonstrate that conditions (III) and (IV) are necessary.

1. Products and factorizations. If \(\mu\) is a normal positive functional on a \(W^*\)-algebra \(\mathcal{A}\), we denote the support of \(\mu\) by \(S(\mu)\), and the central support (the smallest projection of the center of \(\mathcal{A}\) larger than \(S(\mu)\)) by \(Z(\mu)\).

Throughout this section \((\mathcal{A}_i)_{i \in I}\) will be a factorization of the
$W^*$-algebra $\mathcal{A}$. By this we mean that each $\mathcal{A}_i$ is a $W^*$-subalgebra of $\mathcal{A}$ and that, if $\lambda_i$ denotes the inclusion mapping of $\mathcal{A}_i$ into $\mathcal{A}$, $(\mathcal{A}_i, (\lambda_i))$ is a product for $(\mathcal{A}_i)_{i \in I}$. $\mathcal{K}$ will denote the center of $\mathcal{A}$ and $\mathcal{K}_i$ the center of $\mathcal{A}_i$. For $J$ a subset of $I$ we let $\mathcal{A}_J = \mathcal{B}(\mathcal{A}_i; i \in J)$. We call an element of $\mathcal{A}_i$ tail if it is in $\mathcal{I} = \bigcap_{F} \mathcal{A}_{i-F}$. For $\mu \in \Sigma_p$, $T(\mu)$ will denote the smallest tail projection larger than $S(\mu)$.

**Lemma 1.1.** (i). If $\mu \in \Sigma_p$ and $x > 0$, then $x\mu \in \Sigma_p$, where $(x\mu)(A) = x(\mu(A))$ for all $A \in \mathcal{A}$.

(ii). Suppose that $\mu$ is a normal positive functional on $\mathcal{A}$ with $\mu(1) = 1$. Then $\mu \in \Sigma_p$ if and only if the family $(\mathcal{A}_i)_{i \in I}$ is independent with respect to $\mu$; i.e., if and only if

\[ S(\mu) = \prod_{i \in F} S(\mu_i) \]

for all $A_i \in \mathcal{A}_i$ and all finite subsets $F$ of $I$.

**Proof.** (i) is obvious. Suppose that $\mu$ is a normal positive functional on $\mathcal{A}$ with $\mu(1) = 1$. If (1.1) holds let $\mu_i$ be the restriction of $\mu$ to $\mathcal{A}_i$; then $\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p$. Suppose, on the other hand, that $\mu \in \Sigma_p$. Then $\mu = \bigotimes_{i \in I} \mu_i$ for normal positive functionals $\mu_i$ on $\mathcal{A}_i$. We have $\mu(1) = \prod_{i \in I} \mu_i(1)$, so that $\mu = \bigotimes_{i \in I} \mu'_i$ where $\mu'_i = (\mu_i(1))^{-1} \mu_i$ and $\mu'_i(1) = 1$. Evidently $\mu'_i$ is the restriction of $\mu$ to $\mathcal{A}_i$, and (1.1) follows.

**Lemma 1.2.** (i) $\mathcal{I} \subset \mathcal{K}$.

(ii) $Z(\mu) \leq T(\mu)$ for all $\mu \in \Sigma_p$.

**Proof.** $\mathcal{I}$ commutes with each $\mathcal{A}_i$ because $\mathcal{I} \subset \mathcal{A}_{i-\{i\}}$; therefore $\mathcal{I}$ commutes with $\mathcal{A}' = \mathcal{B}(\mathcal{A}_i; i \in I)$.

**Lemma 1.3.** (i) $\mathcal{K} \supset \mathcal{K}_i$ for each $i \in I$

(ii) If $\mathcal{A}$ is a factor then each $\mathcal{A}_i$ is a factor.

**Lemma 1.4.** For all $\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p$:

\[ S(\mu) \leq \prod_{i \in I} S(\mu_i) \]

and

\[ Z(\mu) \leq \prod_{i \in I} Z(\mu_i) \]

**Proof.**
\[
\mu \left( \prod_{i \in I} S(\mu_i) \right) = \prod_{i \in I} \mu_i(S(\mu_i)) = \prod_{i \in I} \mu_i(1) = \mu(1).
\]

Therefore (1.2) holds. (1.3) holds because \( \prod_{i \in I} Z(\mu_i) \) is a projection of \( \mathfrak{A} \) larger than \( \prod_{i \in I} S(\mu_i) \) and, hence, by (1.2), larger than \( S(\mu) \).

**Remark.** The two propositions which follow are stated now for convenience in referring to them later. For the moment, we need only parts (i) and (ii) of Proposition 1.6.

**Proposition 1.5.** Suppose that \( J \) is a subset of \( I \). Then:

(i). \( (\mathcal{A}_i)_{i \in I} \) is a factorization of \( \mathcal{A}_J \).

(ii). If \( \mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p \), then the restriction \( \mu' \) of \( \mu \) to \( \mathcal{A}_J \) is a product functional on \( \mathcal{A}_J \) for the factorization \( (\mathcal{A}_i)_{i \in J} \), and \( \mu' \) is a scalar multiple of \( \mu_J = \bigotimes_{i \in J} \mu_i \).

(iii). If \( \Sigma \) is a separating subset of \( \Sigma_p \), then \( \{ \mu_J : \bigotimes_{i \in J} \mu_i \in \Sigma \} \) is separating on \( \mathcal{A}_J \).

(iv). If (III) holds for \( (\mathcal{A}_i)_{i \in I} \) then (III) holds for \( (\mathcal{A}_i)_{i \in J} \).

(v). If (III) and (IV) hold for \( (\mathcal{A}_i)_{i \in I} \), then (IV-\( \mu \)) holds for \( (\mathcal{A}_i)_{i \in J} \) for \( \mu \) in a separating subset of product functionals on \( \mathcal{A}_J \) for \( (\mathcal{A}_i)_{i \in J} \).

(vi). If (V) holds for \( (\mathcal{A}_i)_{i \in I} \) then (V) holds for \( (\mathcal{A}_i)_{i \in J} \).

**Proof.** (i) and (ii) are obvious, (iii) follows from (ii) and (iv) from (iii). To prove (v) observe that (IV-\( \mu \)) clearly holds for all \( \mu \in \Sigma_p \). To prove (vi) let \( \mathcal{T}_J \) be the tail of the factorization \( (\mathcal{A}_i)_{i \in J} \). For every finite subset \( F \) of \( I \):

\[ \mathcal{A}_{J-F} \subseteq \mathcal{A}_{J-F}. \]

Taking the intersection as \( F \) runs over all finite subsets of \( I \), since \( F \cap J \) runs over all finite subsets of \( J \), we obtain \( \mathcal{T}_J \subseteq \mathcal{T} \).

**Proposition 1.6.** Suppose that \( (I(j))_{j \in J} \) is a mutually disjoint family of subsets of \( I \) whose union is \( I \). Then:

(i). \( (\mathcal{A}_{I(j)})_{j \in J} \) is a factorization of \( \mathcal{A} \).

(ii). If \( \mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p \), then \( \mu \) is a product functional for the factorization \( (\mathcal{A}_{I(j)})_{j \in J} \) and \( \mu = \bigotimes_{j \in J} (\bigotimes_{i \in I(j)} \mu_i) \).

(iii). If (III) holds for the factorization \( (\mathcal{A}_i)_{i \in I} \) then (III) holds for the factorization \( (\mathcal{A}_{I(j)})_{j \in J} \).

(iv). If (V) holds for the factorization \( (\mathcal{A}_i)_{i \in I} \) then (V) holds for the factorization \( (\mathcal{A}_{I(j)})_{j \in J} \).

**Remark.** (IV) holding for \( (\mathcal{A}_i)_{i \in I} \) does not necessarily mean that (IV) holds for \( (\mathcal{A}_{I(j)})_{j \in J} \); see Example 7.3.
PROPOSITION 1.7. (Zero-one law). For all \( \mu \in \Sigma_p \) with \( \mu(1) = 1 \) and all tail projections \( T \):
\[
\mu(T) = 0 \quad \text{or} \quad 1 .
\]

Proof. Let \( F \) be a finite subset of \( I \). Then \( \mu \) is a product functional for the factorization \( \{ \mathcal{A}_F, \mathcal{A}_{-F} \} \) of \( \mathcal{A} \) (Proposition 1.6. (i)) and \( T \in \mathcal{A}_{-F} \); therefore (Lemma 1.1), for all \( A \in \mathcal{A}_F \):

\[
(1.4) \quad \mu(AT) = \mu(A)\mu(T) .
\]
Now \( \bigcup_F \mathcal{A}_F \) is ultraweakly dense in \( \mathcal{A} \), so (1.4) holds for all \( A \in \mathcal{A} \). Putting \( A = T \in \mathcal{A} \), we obtain:
\[
\mu(T) = (\mu(T))^2 .
\]

COROLLARY 1.8. If \( \mu \in \Sigma_p \) and \( T \) is a tail projection:
\[
\mu(T) \neq 0 \quad \text{implies} \quad S(\mu) \leq T .
\]

PROPOSITION 1.9. For every \( \mu \in \Sigma_p \), \( T(\mu) \) is an atomic projection of \( \mathcal{T} \).

Proof. Suppose that \( T \) is a projection of \( \mathcal{T} \) with \( 0 \leq T \leq T(\mu) \). Then either \( \mu(T) = 0 \) or \( S(\mu) \leq T \), by Corollary 1.8. If \( S(\mu) \leq T \) then \( T = T(\mu) \) by definition. If \( \mu(T) = 0 \) then \( T \leq 1 - S(\mu) \) and \( T(\mu) - T \geq S(\mu) \); that implies \( T = 0 \).

COROLLARY 1.10. For all \( \mu, \nu \in \Sigma_p \):
\[
either \quad T(\mu) = T(\nu) \quad \text{or} \quad [T(\mu)]T(\nu)] = 0 .
\]

COROLLARY 1.11. If condition (III) holds, then \( \mathcal{T} \) is an atomic \( W^* \)-algebra.

LEMMA 1.12. Suppose that conditions (III) and (IV) hold and that \( i \in I \). For all \( A_i \in \mathcal{A}_i^+ \) and all \( T \in \mathcal{T}^+ \):
\[
A_i T = 0 \quad \text{implies} \quad A_i = 0 \quad \text{or} \quad T = 0 .
\]

Proof. Suppose that \( T \neq 0 \). Then because of (III), there exists \( \mu \in \Sigma_p \) with \( \mu(T) \neq 0 \). By Proposition 1.6, \( \{ \mathcal{A}_i, \mathcal{A}_{-i} \} \) is a factorization for \( \mathcal{A} \) and \( \mu = \mu_i \otimes \mu' \) is a product functional for this factorization. We have \( T \in \mathcal{A}_{-i} \) and \( \mu'(T) \neq 0 \). Now for every nonzero normal positive functional \( \nu_i \) on \( \mathcal{A}_i \), \( \nu_i \otimes \mu' \) exists on \( \mathcal{A} \) by (IV). Hence \( A_i T = 0 \) implies that \( (\nu_i \otimes \mu')(A_i T) = 0 \) or that \( \nu_i(A_i) = 0 \) for each \( \nu_i \). Therefore \( A_i T = 0 \) implies \( A_i = 0 \).
DEFINITION 1.13. Suppose that \((\mathcal{A}, (\alpha_i))\) is a product for \((\mathcal{A}_i)_{i \in I}\) and that \(\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p\). Let \(E(\mu) = \sup \{S(\nu); \nu \in \Sigma_p\text{ and } \nu = \bigotimes_{i \in I} \nu_i\text{ with } \nu_i = \mu_i \text{ for a.a. } i \in I\} \).

REMARK. It is clear that \(E(\mu)\) is well defined: i.e., \(E(\mu)\) does not depend on how \(\mu\) is expressed as \(\bigotimes \mu_i\).

DEFINITION 1.14. A product \((\mathcal{A}, (\alpha_i))\) for \((\mathcal{A}_i)_{i \in I}\) will be said to satisfy (VI-(\mu_i)), where each \(\mu_i\) is a normal positive functional on \(\mathcal{A}_i\), if the following conditions hold:

(i). \(\mu = \bigotimes_{i \in I} \mu_i\) exists on \(\mathcal{A}\).

(ii). (IV-\mu) holds.

(iii). \(E(\mu) = 1\).

PROPOSITION 1.15. For all \(\mu \in \Sigma_p\):

\[ E(\mu) \leq T(\mu) . \]

Proof. Suppose that \(\nu = \bigotimes_{i \in I} \nu_i \in \Sigma_p\) with \(\nu_i = \mu_i\) for a.a. \(i \in I\). Let \(F = \{i \in I; \nu_i = \mu_i\}\). Then \(F\) is finite so that \(T(\mu) \in \mathcal{M}_F\). By Proposition 1.6, \(\{\mathcal{A}, \mathcal{M}_F\}\) is a factorization of \(\mathcal{A}\) for which \(\mu\) and \(\nu\) are product functionals: \(\mu = \mu_F \bigotimes \mu'\) and \(\nu = \nu_F \bigotimes \nu'\). Clearly \(\mu' = \nu'\). We have \(0 \neq \mu(T(\mu)) = \mu_F(1)\mu'(T(\mu))\), so that \(\nu'(T(\mu)) \neq 0\). Hence \(\nu(T(\mu)) = \nu_F(1)\nu'(T(\mu)) \neq 0\) and by Corollary 1.8, \(S(\nu) \leq T(\mu)\).

Since \(E(\mu)\) is the supremum of such \(S(\nu)\), \(E(\mu) \leq T(\mu)\).

PROPOSITION 1.16. Condition (VI-(\mu_i)) implies conditions (III), (V) and (IV-\nu) for \(\nu\) in a separating subset of \(\Sigma_p\).

Proof. Evidently (VI-(\mu_i)) implies (IV-\nu) for \(\nu\) in a separating subset of \(\Sigma_p\), and hence (III). That it implies (V) is a consequence of Proposition 1.9 and Proposition 1.15.

LEMMA 1.17. Suppose that \(Z\) is a projection of \(\mathcal{A}\). Let \(\alpha_i; \mathcal{A}_i \to Z \mathcal{A}_i\) be defined, for each \(i \in I\), by:

\[ \alpha_i(A_i) = ZA_i \text{ for all } A_i \in \mathcal{A}_i . \]

Let \(Z_i\) be the support of \(\alpha_i\). Then \((Z, \mathcal{A}_i, (\alpha_i'))\) is a product for \((Z, \mathcal{A}_i, (\alpha_i))\), where \(\alpha_i'\) denotes the restriction of \(\alpha_i\) to \(Z_i \mathcal{A}_i\). Suppose that \(\mu' = \bigotimes \mu_i'\) is a product functional for \((Z, \mathcal{A}_i, (\alpha_i))\). Define \(\mu\) on \(\mathcal{A}\) and \(\mu_i\) on \(\mathcal{A}_i\) by:

\[ \mu(A) = \mu'(ZA) \text{ for all } A \in \mathcal{A} \]
\[ \mu_i(A_i) = \mu_i'(Z_iA_i) \text{ for all } A_i \in \mathcal{A}_i . \]
Then \( \mu \) is in \( \Sigma_p \), \( S(\mu) = S(\mu') \), and \( \mu = \bigotimes \mu_i \).

Proof. Obviously, since \( \alpha_i'(Z_i, \mathcal{A}_i) = \alpha_i(\mathcal{A}_i) = Z, \mathcal{A}_i, (Z, \mathcal{A}, (\alpha_i)) \) is a product for \( (Z, \mathcal{A}, (\alpha_i)) \). Suppose \( \mu', \mu, \mu_i \), and \( \mu_i \) are as in the lemma. Then whenever each \( A_i \in \mathcal{A}_i \) and \( A_i = 1 \) for a.a. \( i \in I \):

\[
\mu \left( \prod_{i \in I} A_i \right) = \mu' \left( Z \prod_{i \in I} A_i \right) = \mu' \left( \prod_{i \in I} \alpha_i(A_i) \right)
= \mu' \left( \prod_{i \in I} \alpha_i'(Z_i A_i) \right) = \prod_{i \in I} \mu_i'(Z_i A_i) = \prod_{i \in I} \mu_i(A_i) .
\]

PROPOSITION 1.18. Suppose that the factorization \((\mathcal{A}_i)_{i \in I}\) satisfies (III) and (IV), and suppose that \( T \) is a nonzero tail projection. Let \( \alpha_i : \mathcal{A}_i \to T, \mathcal{A} \) be defined, for each \( i \in I \), by:

\[
\alpha_i(A_i) = TA_i \quad \text{for all } A_i \in \mathcal{A}_i .
\]

Then:

(i). Each \( \alpha_i \) is an isomorphism and \((T, \mathcal{A}, (\alpha_i))\) is a tensor product for \((\mathcal{A}_i)\): i.e., \((T, \mathcal{A}, (\alpha_i))\) is a product for \((\mathcal{A}_i)\) satisfying (III) and (IV).

(ii). \((T, \mathcal{A}, (\alpha_i))\) is a local tensor product if and only if \( T \) is atomic in \( \mathcal{F} \).

(iii). There is a one-to-one correspondence \( \mu' \leftrightarrow \mu \) between product functionals \( \mu' \) for \((T, \mathcal{A}, (\alpha_i))\) and product functionals \( \mu \) on \( \mathcal{A} \) for \((\mathcal{A}_i)\) with \( S(\mu) \leq T \), where \( \mu' \) is the restriction of \( \mu \) to \( T, \mathcal{A} \) and \( \mu(A) = \mu'(TA) \) for all \( A \in \mathcal{A} \). We have \( S(\mu) = S(\mu') \) and \( \mu = \bigotimes \mu_i \) if and only if \( \mu' = \bigotimes \mu_i \).

Proof. Lemma 1.12 shows that each \( \alpha_i \) is an isomorphism. Then Lemma 1.17 shows both that \((T, \mathcal{A}, (\alpha_i))\) is a product for \((\mathcal{A}_i)\), and also that, if \( \mu' = \bigotimes \mu_i \) is a product functional for \((T, \mathcal{A}, (\alpha_i))\), then the \( \mu \) corresponding to \( \mu' \) is in \( \Sigma_p \), \( \mu = \bigotimes \mu_i \), and \( S(\mu) = S(\mu') \). Suppose that \( \mu = \bigotimes \mu_i \) is in \( \Sigma_p \) with \( S(\mu) \leq T \), and let \( \mu' \) be the restriction of \( \mu \) to \( T, \mathcal{A} \). Suppose \( A_i \in \mathcal{A}_i \) and \( A_i = 1 \) for all \( i \in I - F \) for a finite subset \( F \) of \( I \). Then:

\[
(1.5) \quad \mu' \left( \prod_{i \in I} \alpha_i(A_i) \right) = \mu \left( \prod_{i \in I} TA_i \right) = \left[ \prod_{i \in F} \mu_i(A_i) \right] \mu_{I-F}(T) ;
\]

because \( T \in \mathcal{F}_{I-F} \). Now \( \mu(T) = \mu_F(1) \mu_{I-F}(T) \), and, since \( S(\mu) \leq T \), \( \mu(T) = \mu(1) = \mu_F(1) \mu_{I-F}(1) \). Therefore:

\[
(1.6) \quad \mu_{I-F}(T) = \mu_{I-F}(1) = \prod_{i \in I-F} \mu_i(1) .
\]

Combining (1.5) and (1.6), we conclude that \( \mu = \bigotimes \mu_i \in \Sigma_p \). That completes the proof of (iii).
Since (III) holds for the factorization \( (\mathcal{A}) \), evidently Corollary 1.8 and (iii) demonstrate that (III) holds for \((T, \mathcal{A}, (\alpha_i))\). To prove (IV) for \((T, \mathcal{A}, (\alpha_i))\), let us assume that \( \mu' = \bigotimes \mu_i \) is a product functional for \((T, \mathcal{A}, (\alpha_i))\) and that \( \nu_i \) is a non-zero normal positive functional on \( \mathcal{A} \) with \( \nu_i = \mu_i \) for a.a. \( i \in I \). Let \( \mu \) correspond to \( \mu' \) as in (iii) so that \( \mu = \bigotimes \mu_i \) for \( (\mathcal{A}) \) and \( S(\mu) \leq T \). Now (IV) holds for \((\mathcal{A})\), so that \( \nu = \bigotimes \nu_i \) exists on \( \mathcal{A} \). We have \( \nu(\nu) \leq \nu(\mu) \leq T(\mu) \leq T \) by Propositions 1.15 and 1.9. \( \nu' = \bigotimes \nu_i \) exists as a product functional for \((T, \mathcal{A}, (\alpha_i))\) by (iii). That demonstrates (IV) and thus (i).

Since \( T \) is in \( \mathcal{T} \) and each \( \mathcal{A}_{-i} \), a direct calculation shows that the tail of the product \((T, \mathcal{A}, (\alpha_i))\) is precisely \( T\mathcal{T} \). Hence (V) holds for \((T, \mathcal{A}, (\alpha_i))\) if and only if \( T \) is atomic in \( \mathcal{T} \). That proves (ii).

2. Direct products of von Neumann algebras. We summarize here the definition and same basic properties of the direct product of a family of von Neumann algebras. For details and omitted proofs, see [7] or [1].

Let \( I \) be an arbitrary indexing set. Suppose that \((H_i)_{i \in I}\) is a family of Hilbert spaces and that, for each \( i \in I \), \( x_i \) is in \( H_i \) with \( 0 < \prod_{i \in I} ||x_i|| < \infty \). Then we denote by \( \bigotimes_{i \in I} (H_i, x_i) \) von Neumann's incomplete direct product of the family \((H_i)\) with respect to the \( C_0 \)-sequence \((x_i)\), (see [7]). Let \( \Lambda = \{(y_i)\}: \) each \( y_i \in H_i, \sum |1 - (x_i | y_i)| < \infty \) and \( \sum |1 - ||y_i||| < \infty \). Then there is a natural multilinear mapping \( (y_i) \mapsto \bigotimes y_i \) from \( \Lambda \) into a dense subset of \( H \) with:

\[
(\bigotimes y_i | \bigotimes z_i) = \prod (y_i | z_i) \quad \text{for all } (y_i), (z_i) \in \Lambda. 
\]

**Lemma 2.1.** Suppose that \( x_i, y_i \in H_i \) with \( 0 < \prod ||x_i||, \prod ||y_i|| < \infty \) and that \( \sum |1 - (x_i | y_i)| < \infty \). Then \( \bigotimes (H_i, x_i) = \bigotimes (H_i, y_i) \).

**Lemma 2.2.** Suppose that, for each \( i \in I \), \( L_i \) is a dense linear subset of \( H_i \) with \( x_i \in L_i \), and suppose that

\[
0 < \prod ||x_i|| < \infty. \quad \text{Let } L = \{\bigotimes_{i \in I} y_i : y_i \in L_i \text{ for all } i \in I \text{ and } y_i = x_i \text{ for a.a. } i \in I\}
\]

Then \( L \) is dense in \( \bigotimes_{i \in I} (H_i, x_i) \).

**Lemma 2.3.** Let \( H = \bigotimes_{i \in I} (H_i, x_i) \). Then, for each \( j \in I \), there exists a normal isomorphism \( \alpha_j \) of \( \mathcal{L}(H_j) \) into \( \mathcal{L}(H) \) such that, for all \( A_j \in \mathcal{A}_j \) and all \( (y_i) \in \Lambda \):

\[
(\alpha_j(A_j))(\bigotimes y_i) = \bigotimes y'_i
\]

where \( y'_i = y_i \) for \( i \neq j \) and \( y'_j = A_j y_j \). We call \( \alpha_j \) the natural injection of \( \mathcal{L}(H_j) \) into \( \mathcal{L}(H) \).
DEFINITION 2.4. Suppose that, for each \( i \in I \), \( \mathcal{A}_i \) is a von Neumann algebra on \( H_i \) and \( x_i \in H_i \), and suppose that \( 0 < \prod \| x_i \| < \infty \). Then by \( \bigotimes_{i \in I} (\mathcal{A}_i, x_i) \) we will mean \( \mathcal{B}(\mathcal{A}_i; i \in I) \), where \( \alpha_i \) is the natural injection of \( \mathcal{L}(H_i) \) into \( \mathcal{L}(\bigotimes_{i \in I} (H_i, x_i)) \).

**Lemma 2.5.** (i). \( \bigotimes_{i \in I} (\mathcal{A}_i, x_i) = \mathcal{L}(\bigotimes_{i \in I} (H_i, x_i)) \).

(ii). \( \bigotimes_{i \in I} (\mathcal{A}_i, x_i) \) is a factor if and only if each \( \mathcal{A}_i \) is a factor.

**Proposition 2.6.** Suppose that, for each \( i \in I \), \( \mathcal{A}_i \) is a von Neumann algebra on \( H_i \) and \( x_i \in H_i \), and suppose that \( 0 < \prod \| x_i \| < \infty \).

Let \( \mu_i(A_i) = (A_i x_i | x_i) \). Let \( \mathcal{N} \) be \( \bigotimes_{i \in I} (\mathcal{A}_i, x_i) \), and let \( \alpha_i \) be the natural injection of \( \mathcal{A}_i \) into \( \mathcal{N} \) for each \( i \in I \). Then \( (\mathcal{A}_i, \alpha_i) \) is a product for \( (\mathcal{A}_i)_{i \in I} \) which satisfies (VI-(\( \mu_i \))). Furthermore, if \( \mu = \bigotimes_{i \in I} \mu_i \), then

\[
S(\mu) = \prod_{i \in I} \alpha_i(S(\mu_i)).
\]

**Remark.** (IV) also holds, of course, and is easily proved directly. See Proposition 4.2.

**Proof.** Obviously \( (\mathcal{N}, \alpha_i) \) is a product for \( (\mathcal{A}_i) \) and \( \mu = \bigotimes_{i \in I} \mu_i \) exists in \( \Sigma \); in fact, if \( x = \bigotimes_{i \in I} x_i \) then \( \mu(A) = (Ax | x) \) for all \( A \in \mathcal{N} \).

By Lemma 1.4,

\[
S(\mu) \leq \prod_{i \in I} \alpha_i(S(\mu_i)).
\]

Now \( S(\mu) = \text{pr}[\mathcal{N}]x \) (By \([L]\) we mean the closure of \( L \); by \( \text{pr}[L] \) we mean the orthogonal projection onto \([L]\)). Because \( \mathcal{N} \) contains each \( \alpha_i(\mathcal{A}_i) \), \([\mathcal{N}]x \) contains the closure of

\( \bigotimes A_i' x_i \) each \( A_i' \in \mathcal{A}_i \) and \( A_i' = 1 \) for a.a. \( i \in I \).

Thus \( (\mathcal{N}, x) \) contains \( \bigotimes (\mathcal{A}_i', x_i) \). The projection onto this last subspace of \( H = \bigotimes (H_i, x_i) \) is \( \prod \alpha_i(S(\mu_i)) \). Hence \( S(\mu) \geq \prod \alpha_i(S(\mu_i)) \) and (2.1) follows from (2.2).

To prove (VI-(\( \mu_i \))), let us assume first that every normal positive functional on \( \mathcal{A}_i \) is induced by a vector of \( H_i \). Let

\( L = \{ \bigotimes y_i; y_i \in H_i, y_i = x_i \text{ for a.a. } i \in I \} \).

Then \( L \) is dense in \( H \) by Lemma 2.2. For each nonzero \( y \in L \), let \( \nu_y \) be the functional induced by \( y \):

\( \nu_y(A) = (Ay | y) \) for all \( A \in \mathcal{A} \).

Then a direct calculation shows that \( \nu_y = \bigotimes \nu_i \) where \( \nu_i \) is induced by \( y \) and \( \nu_i = \mu_i \) for a.a. \( i \in I \). We have \( (\nu_y)y = y \). Since every
normal positive functional on $\mathcal{A}$ is induced by a vector, as $y$ runs through $L$, $\nu_y$ runs through 

$$\Sigma = \{ \otimes \nu_y : \nu_y = \mu_i \text{ for a.a. } i \in I \}.$$ 

Thus (IV-$\mu$) holds, and 

$$E(\mu) = \sup \{ S(\nu) : \nu \in \Sigma \} \geq \operatorname{pr} [L] = 1.$$ 

To prove (VI-$\mu_i$) in the general case we will show that there exist von Neumann algebras $\mathcal{B}_i$ on $G_i$ and vectors $z_i \in G_i$, and that there exists an isomorphism $\psi$ of $\mathcal{A}$ onto $\bigotimes (\mathcal{B}_i, z_i)$ such that:

(2.3) Every normal positive functional on $\mathcal{B}_i$ is induced by a vector.

(2.4) $\psi(\alpha_i(\mathcal{A}_i)) = \beta_i(\mathcal{B}_i)$ where $\beta_i$ is the natural injection of $\mathcal{L}(G_i)$ into $\mathcal{L}(G)$ and $G = \bigotimes (G_i, z_i)$.

(2.5) If $z = \otimes z_i$ then 

$$\langle \psi(A)z, z \rangle = \mu(A) \text{ for all } A \in \mathcal{A}.$$ 

Then by the preceding paragraph (VI-$\mu_i$) will hold for the product $(\mathcal{B}, (\beta_i))$ and thus for the product $(\mathcal{A}, (\alpha_i))$.

For each $i \in I$, let $H_i'$ be a Hilbert space of infinite dimension, let $x_i \in H_i'$ with $\| x_i \| = 1$, and let $\mathcal{C}_i$ be the algebra of scalars on $H_i'$. Let $\mathcal{B}_i = \mathcal{A}_i \otimes \mathcal{C}_i$ on $G_i = H_i \otimes H_i'$ and let $z_i = x_i \otimes x_i$. Let $G = \bigotimes (G_i, z_i) = \bigotimes (H_i \otimes H_i', x_i \otimes x_i')$ and let $H' = \bigotimes (H_i', x_i)$. Then [7] it is easy to construct a natural isometry $\phi$ from $H \otimes H'$ onto $G$ such that:

$$\phi(\alpha_i(T_i) \otimes 1_{H_i'}) \phi^{-1} = \beta_i(T_i \otimes 1_{H_i'})$$

for all $T_i \in \mathcal{L}(H_i)$ and all $i \in I$. Define $\psi : \mathcal{A} \to \mathcal{L}(G)$ by:

$$\psi(A) = \phi(A \otimes 1_{H_i'}) \phi^{-1} \text{ for all } A \in \mathcal{A}.$$ 

Then (2.4), (2.5), and $\psi(\mathcal{A}) = \mathcal{B}$ follow immediately.

**Corollary 2.7.** Suppose that $(\mathcal{A}_i)_{i \in F}$ is a finite family of von Neumann algebras. Let $\mathcal{A} = \bigotimes_{i \in F} \mathcal{A}_i$ and let $\alpha_i$ be the natural injection of $\mathcal{A}_i$ into $\mathcal{A}$. Then $(\mathcal{A}, (\alpha_i))$ is a tensor product for $(\mathcal{A}_i)_{i \in F}$ which satisfies (V). In particular $\bigotimes \mu_i$ exists in $\Sigma_p$ for every nonzero normal positive functional $\mu_i$ of $\mathcal{A}_i$.

**Lemma 2.8.** Suppose that $(H_i)_{i \in I}$ is a family of Hilbert spaces and that, for each $i \in I$, $H_i = \bigoplus_{j \in J(i)} H_i^j$ where $0 \in J(i)$ (by $H_i = \bigoplus_{j \in J(i)} H_i^j$ we mean that the $H_i^j$ are mutually orthogonal subspaces of $H_i$ which span $H_i$). Suppose that, for each $i \in I$ and $j \in J(i)$, $x_i^j$
is a nonzero vector of \( H_i \), and suppose that \( 0 < \prod_{\hat{i} \in I} \| x_{\hat{i}} \| < \infty \). Denote by \( J \) the set of families \( (j(\hat{i}))_{\hat{i} \in I} \) with each \( j(\hat{i}) \in J(\hat{i}) \) and \( j(\hat{i}) = 0 \) for a.a. \( i \in I \). If \( j = (j(\hat{i})) \in J \) let \( H^j = \bigotimes_{\hat{i} \in I} (H^j_{\hat{i}}, x^j_{\hat{i}}) \).

Then each \( H^j \) is a subspace of \( H = \bigoplus_{j \in J} H^j \). Furthermore, if \( \alpha^j \) denotes, for each \( j = (j(\hat{i})) \in J \), the natural injection of \( \mathcal{L}(H^j) \) into \( \mathcal{L}(H^j) \), then:

\[
(2.6) \quad \alpha^j(\bigoplus_{T \in J(\hat{i})} T^j) = \bigoplus_{T \in J(\hat{i})} [\alpha^j(T^j)] \text{ for all } (T^j)_{j \in J(\hat{i})} \text{ with each } T^j \in \mathcal{L}(H^j). \quad \text{(Here } \bigoplus T^j: \bigoplus x^j \to \bigoplus T^j x^j).\]

**Proof.** The \( H^j \) are clearly mutually orthogonal, and \( [H^j; j \in J] \) is \( H \) by Lemma 2.2. Formula (2.6) can be confirmed by a direct calculation.

3. The basic isomorphism theorems. By a representation \( \psi \) of a \( W^* \)-algebra \( \mathcal{A} \) on a Hilbert space \( H \) we mean a normal homomorphism \( \psi \) onto a von Neumann algebra on \( H \). (Notice that \( \psi(1) \) is the identity on \( H \).) If \( \psi \) is a representation of \( \mathcal{A} \) on \( H \) and \( \mu \) is a normal positive functional on \( \mathcal{A} \), a vector \( x \in H \) will be called a \( \mu \)-cyclic vector for \( \psi \) if \( [\psi(A)x] = H \) and

\[
\mu(A) = (\psi(A)x | x) \quad \text{for all } A \in \mathcal{A}. \]

Given \( \mathcal{A} \) and \( \mu \) it is well known (see [3, p. 51], for example) that a representation \( \psi \) with a \( \mu \)-cyclic vector exists (and is essentially unique), and that such a \( \psi \) acts isomorphically on \( (Z(\mu))_{\mathcal{A}} \) and takes \( (1 - Z(\mu))_{\mathcal{A}} \) into 0.

**Proposition 3.1.** Suppose that \( (\mathcal{A}_i)_{i \in I} \) is a factorization of the \( W^* \)-algebra \( \mathcal{A} \) and that \( \mu = \bigotimes_{i \in I} \mu_i \) is a product functional for this factorization. Suppose that \( \psi \) is a representation of \( \mathcal{A} \) on \( H \) with \( \mu \)-cyclic vector \( x \). Suppose that, for each \( i \in I \), \( \psi_i \) is a representation of \( \mathcal{A}_i \) on \( H_i \) with \( \mu_i \)-cyclic vector \( x_i \). Then there exists an isometry \( \phi \) of \( H \) onto \( \bigotimes_{i \in I} (H_i, x_i) \) such that:

(i). \( \phi(x) = \bigotimes_{i \in I} x_i. \)

(ii). \( \phi(\psi(\mathcal{A}))\phi^{-1} = \bigotimes_{i \in I} (\psi_i(\mathcal{A}_i), x_i). \)

(iii). For all \( A_i \in \mathcal{A}_i \) and each \( i \in I: \)

\[
\phi(\psi(A_i))\phi^{-1} = \alpha_i(\psi(A_i))
\]

where \( \alpha_i \) denotes the natural injection of \( \mathcal{L}(H_i) \) into \( \mathcal{L}(\bigotimes_{i \in I} (H_i, x_i)). \)

**Proof.** Let \( \mathcal{H} \) denote the set of families \( (A_i)_{i \in I} \) with each \( A_i \in \mathcal{A}_i \) and \( A_i = 1 \) for a.a. \( i \in I \). Let

\[
M = \left\{ \psi\left(\prod_{i \in I} A_i\right)x: (A_i) \in \mathcal{H} \right\}
\]
and

\[ N = \left\{ \bigotimes_{i \in I} [(\psi_i(A_i)x_i)]: (A_i) \in \mathcal{A} \right\}. \]

First we claim that \( M \) is a dense subset of \( H \). For \( \mathcal{A} \), the *-algebra \( \{\prod_{i \in I} A_i: (A_i) \in \mathcal{A}\} \), is ultrastrongly dense in \( \mathcal{A} \) (a corollary of the double-commutant theorem); hence \( \psi(\mathcal{A}) \) is strongly dense in \( \psi(\mathcal{A}) \) and \( [\psi(\mathcal{A})] = [\psi(\mathcal{A})] = H \) because \( x \) is a cyclic vector for \( \psi(\mathcal{A}) \).

Secondly, \( N \) is a dense subset of \( \bigotimes_{i \in I} (H_i, x_i) \) by Lemma 2.2, since \( x_i \in [\psi_i(\mathcal{A}_i)x_i] = H_i \) for each \( i \in I \).

Fix \( (A_i) \in \mathcal{A} \). Then:

\[
\left\| \left( \psi\left( \prod_{i \in I} A_i \right) \right) x \right\|^2 = \mu\left( \prod_{i \in I} A_i^* A_i \right) = \prod_{i \in I} \mu_i(A_i^* A_i) \left\| \bigotimes_{i \in I} [(\psi_i(A_i))x_i] \right\|^2
\]

\[ = \prod_{i \in I} \| (\psi_i(A_i))x_i \|^2 = \prod_{i \in I} \mu_i(A_i^* A_i). \]

Therefore, since \( M \) is dense in \( H \) and \( N \) is dense in \( \bigotimes_{i \in I} (H_i, x_i) \), there exists a (unique) isometry \( \phi \) of \( H \) onto \( \bigotimes_{i \in I} (H_i, x_i) \) such that, for all \( (A_i) \in \mathcal{A} \):

\[ \phi\left[ \left( \psi\left( \prod_{i \in I} A_i \right) \right) x \right] = \bigotimes_{i \in I} [(\psi_i(A_i))x_i]. \]

Now (i) follows immediately, (iii) by a direct calculation, and (ii) from (iii).

**Theorem 3.2.** Suppose that \((\mathcal{A}_i)_{i \in I}\) is a factorization of the \( W^* \)-algebra \( \mathcal{A} \). Suppose that \( \mu = \bigotimes_{i \in I} \mu_i \) is a product functional for this factorization, and suppose that (IV-\( \mu \)) holds. Then there exist, for each \( i \in I \), a faithful representation \( \mathcal{A}_i \) of \( \mathcal{A} \) on \( H_i \) and a vector \( x_i \in H_i \), and there exists a representation \( \psi \) of \( \mathcal{A} \) on \( H = \bigotimes_{i \in I} (H_i, x_i) \) such that:

(i) \( \psi \) maps \( (1 - E(\mu))\mathcal{A} \) into 0 and maps \( (E(\mu))\mathcal{A} \) isomorphically onto \( \psi(\mathcal{A}) = \bigotimes_{i \in I} (\mathcal{A}_i(\mathcal{A}_i), x_i) \).

(ii) For each \( i \in I \) and all \( A_i \in \mathcal{A}_i \):

\[ \psi(A_i) = \alpha_i(\mathcal{A}_i(A_i)), \]

where \( \alpha_i \) denotes the natural injection of \( \mathcal{A}(H_i) \) into \( \mathcal{A}(H) \).

(iii). For each \( i \in I \) and all \( A_i \in \mathcal{A}_i \):

\[ ((\mathcal{A}_i(A_i)x_i | x_i) = \mu_i(A_i). \]

(iv). If \( x \) denotes \( \bigotimes_{i \in I} x_i \), then, for all \( A \in \mathcal{A} \):

\[ ((\psi(A)x | x) = \mu(A). \]
Proof. For each \( i \in I \), select (by Zorn's lemma) a family \( (\mu^i_j)_{j \in J(i)} \) of normal nonzero positive functionals on \( \mathcal{A} \) such that \( \sum_{j \in J(i)} Z(\mu^i_j) = 1 \) and \( 0 \in J(i) \) with \( \mu^i_i = \mu_i \). Let \( J \) be the subset of \( \prod_{i \in I} J(i) \) consisting of \( (j(i)) \) with \( j(i) = 0 \) for a.a. \( i \in I \). Since \( (IV-\mu) \) holds, each \( j = (j(i)) \in J \) the product functional \( \mu^j = \bigotimes_{i \in I} \mu^i_{j(i)} \) exists on \( \mathcal{A} \). We have \( Z(\mu^j) \leq \prod_{i \in I} Z(\mu^i_{j(i)}) \) by Lemma 1.4, so that \( (Z(\mu^i))_{i \in I} \) is a mutually orthogonal family of central projections of \( \mathcal{A} \). Let \( Z = \sum_{j \in J} Z(\mu^j) \).

For each \( j \in J \) let \( \Gamma^j \) be a representation of \( \mathcal{A} \) on \( G^j \) with a \( \mu^j \)-cyclic vector \( y^j \). Let \( \Gamma \) be the direct sum representation \( \bigoplus_{j \in J} \Gamma^j \) of \( \mathcal{A} \) on \( G = \bigoplus_{j \in J} G^j \):

\[
\Gamma(A) = \bigoplus_{j \in J} \Gamma^j(A) \quad \text{for all } A \in \mathcal{A}.
\]

Then \( \Gamma \) maps \( (1 - Z)\mathcal{A} \) into 0 and maps \( Z\mathcal{A} \) isomorphically onto \( \Gamma(\mathcal{A}) \).

For each \( i \in I \) and each \( j \in J(i) \), let \( \mathcal{J}^i_j \) be a representation of \( \mathcal{A}_i \) on \( H^i_j \) with \( \mu^i_j \)-cyclic vector \( x^i_j \). Let \( \mathcal{J}^i \) be the direct sum representation \( \bigoplus_{j \in J(i)} \mathcal{J}^i_j \) of \( \mathcal{A}_i \) on \( H_i = \bigoplus_{j \in J(i)} H^i_j \):

\[
\mathcal{J}^i(A_i) = \bigoplus_{j \in J(i)} \mathcal{J}^i_j(A_i) \quad \text{for all } A_i \in \mathcal{A}_i.
\]

Then each \( \mathcal{J}^i \) is faithful.

Fix \( j = (j(i)) \) in \( J \). We know that \( \mu^j = \bigotimes_{i \in I} \mu^i_{j(i)} \), that \( \Gamma^j \) is a representation of \( \mathcal{A} \) on \( G^j \) with \( \mu^j \)-cyclic vector \( y^j \), and that \( \mathcal{J}^j \) for each \( i \in I \), is a representation of \( \mathcal{A}_i \) on \( H^i \)-cyclic vector \( x^i \). Therefore Proposition 3.1 demonstrates the existence of an isometry \( \phi^j \) from \( G^j \) onto \( H^j = \bigotimes_{i \in I} (H^i_j, x^i_j) \) such that:

\[
\phi^j(y^j) = \bigotimes_{i \in I} x^i_j
\]

and

\[
\phi^j(\Gamma^j(A_i))(\phi^j)^{-1} = \alpha_i(\mathcal{J}^i(A_i)) \quad \text{for all } A_i \in \mathcal{A}_i,
\]

where \( \alpha_i \) denotes the natural injection of \( \mathcal{L}(H^i_j) \) into \( \mathcal{H}(H^j) \).

Let \( x^i \) denote \( x^i_j \) for each \( i \in I \). Let \( H = \bigoplus_{i \in I} (H_i, x_i) \), and denote by \( \alpha_i \) the natural injection of \( \mathcal{L}(H_i) \) into \( \mathcal{H}(H) \). Then (Lemma 2.8), \( H = \bigoplus_{i \in I} H^i_j \), and, for each \( i \in I \) and all operators \( T_i \in \mathcal{L}(H_i) \) with \( T_i = \bigoplus_{j \in J(i)} T^i_j \) and with each \( T^i_j \in \mathcal{L}(H^i_j) \):

\[
\alpha_i(T_i) = \bigoplus_{j = (j(i)) \in J} (\alpha_i(T^i_j)) \ .
\]

Define the isometry \( \phi \) of \( G \) onto \( H \) by:

\[
\phi\left( \bigoplus_{j \in J} f^j \right) = \bigoplus_{j \in J} \phi^j(f^j) \quad \text{for all } f^j \in G^j.
\]

Let \( \psi \) be the representation of \( \mathcal{A} \) on \( H \) defined by:
\[ \psi(A) = \phi(\Gamma(A))\phi^{-1} \] for all \( A \in \mathcal{A} \).

Evidently \( \psi \) has the same kernel as \( \Gamma \): \( \psi \) maps \((1 - Z)\mathcal{A} \) into 0 and \( Z\mathcal{A} \) isomorphically onto \( \psi(\mathcal{A}) \).

Now fix \( i \in I \) and \( A_i \in \mathcal{A} \). In view of (3.2), applying (3.5) to \( \mathcal{A}(A_i) \) we obtain:

\[ (3.6) \quad \alpha_i(\mathcal{A}(A_i)) = \bigoplus_{j=(j;i) \in J} [\alpha_i^j(\mathcal{A}(A_i))] . \]

Using (3.1), the definitions of \( \psi \) and \( \phi \), and (3.4), we get:

\[ (3.7) \quad \psi(A_i) = \phi\left[ \bigoplus_{j=(j;i) \in J} \Gamma^i(A_i) \right] \phi^{-1} = \bigoplus_{j=(j;i) \in J} \phi^i(\Gamma^i(A_i))(\phi^j)^{-1} = \bigoplus_{j=(j;i) \in J} [\alpha_i^j(\mathcal{A}(A_i))] . \]

We conclude, from (3.6) and (3.7), that:

\[ (3.8) \quad \psi(A_i) = \alpha_i(\mathcal{A}(A_i)) \] for all \( A_i \in \mathcal{A}_i \) and all \( i \in I \).

Hence \( \psi \) maps \( \mathcal{A} = \mathcal{B}(\mathcal{A}_i; i \in I) \) onto

\[ \mathcal{B}(\mathcal{A}_i(\mathcal{A}_i)), i \in I) = \bigotimes_{i \in I} (\mathcal{A}_i(\mathcal{A}_i), x_i) . \]

Assertion (ii) of the theorem is precisely (3.8). (iii) holds because \( x_i = x_i^x \) is a \( \mu \)-cyclic vector for \( \mathcal{A}_i \). (iv) holds because of (3.4) and the choice of \( \eta^* \) to be a \( \mu \)-cyclic vector for \( \Gamma^\mu \). To complete the proof of the theorem, then, we need to show only that \( Z = E(\mu) \).

Evidently \( Z \leq E(\mu) \). Let \( \beta_i: \mathcal{A}_i \to Z\mathcal{A} \) be defined by \( \beta_i(A_i) = ZA_i \) for all \( A_i \in \mathcal{A}_i \). Then we have just proved that \( (Z\mathcal{A}_i(\beta_i)) \) is isomorphic to the product \( (\bigotimes (\mathcal{A}_i(\mathcal{A}_i), x_i), (\alpha_i \cdot \mathcal{A}_i)) \), which satisfies (VI-\((\mu_i)\)) by Proposition 2.6. Hence \( (Z\mathcal{A}_i(\beta_i)) \) is a product for \( (\mathcal{A}_i) \), which satisfies (VI-\((\mu_i)\)). Now suppose that each \( \nu_i \) is a nonzero normal positive functional on \( \mathcal{A}_i \) and that \( \nu_i = \mu_i \) for a.a. \( i \in I \). Then \( \nu = \bigotimes \nu_i \) exists as a product functional for \( (Z\mathcal{A}_i(\beta_i)) \). Hence, by Lemma 1.17, \( \nu = \bigotimes \nu_i \) exists in \( \Sigma_\rho \) with \( S(\nu) = S(\nu') \leq Z \). Since \( E(\mu) \) is the supremum of such \( S(\nu), E(\mu) \leq Z \). This completes the proof.

**Corollary 3.3.** Suppose that \( (\mathcal{A}_i)_{i \in I} \) is a factorization of the \( W^* \)-algebra \( \mathcal{A} \), that \( \mu = \bigotimes_{i \in I} \mu_i \) is a product functional for this factorization, and that (IV-\( \mu \)) holds. Then

\[ S(\nu) = [E(\mu)] \prod_{i \in I} S(\mu_i) . \]

**Proof.** Use Theorem 3.2 and (2.1) of Proposition 2.6.

**Corollary 3.4.** Suppose that \( (\mathcal{A}_i)_{i \in I} \) is a family of \( W^* \)-algebras,
and that, for each $i \in I$, $\mu$ is a normal positive functional of $\mathcal{A}$. Suppose that $(\mathcal{A}, (\alpha_i))$ and $(\mathcal{B}, (\beta_i))$ are products for $(\mathcal{A})$ which satisfy (VI-(\mu)). Then $(\mathcal{A}, (\alpha_i))$ and $(\mathcal{B}, (\beta_i))$ are isomorphic: i.e., there exists an isomorphism $\psi$ of $\mathcal{A}$ onto $\mathcal{B}$ such that $\psi \circ \alpha_i = \beta_i$ for all $i \in I$.

**COROLLARY 3.5.** Suppose that, for each $i \in I$, $\mathcal{A}$ and $\mathcal{B}_i$ are von Neumann algebras on $H_i$ and $G_i$ respectively, that $x_i \in H_i$ and $y_i \in G_i$ with

$$0 < \prod \|x_i\|, \prod \|y_i\| < \infty,$$

and that $\psi_i$ is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}_i$ such that:

$$((\psi_i(A_i))x_i \mid y_i) = (A_i x_i \mid x_i) \quad \text{for all } A_i \in \mathcal{A}_i.$$

Then there exists an isomorphism $\psi$ of $\mathcal{A} = \bigotimes (\mathcal{A}_i, x_i)$ onto $\mathcal{B} = \bigotimes (\mathcal{B}_i, y_i)$ such that $\psi \circ \alpha_i = \beta_i \circ \psi_i$ for each $i \in I$, where $\alpha_i$ is the natural injection of $\mathcal{A}_i$ into $\mathcal{A}$ and $\beta_i$ is the natural injection of $\mathcal{B}_i$ into $\mathcal{B}$.

**Proof.** Use Corollary 3.4 and Proposition 2.6.

**THEOREM 3.6.** Suppose that $(\mathcal{A}_i)_{i \in F}$ is a finite family of $W^*$-algebras. Suppose that $(\mathcal{A}, (\alpha_i))$ is a product for $(\mathcal{A}_i)_{i \in F}$ satisfying (III) and (IV-(\mu)) for some product functional $\mu$. Then there exists an isomorphism $\psi$ of $\mathcal{A}$ onto $\bigotimes_{i \in F} \mathcal{A}_i$ such that:

$$\psi \left( \prod_{i \in F} \alpha_i(A_i) \right) = \bigotimes_{i \in F} A_i \quad \text{for all } A_i \in \mathcal{A}_i.$$

(We write $\bigotimes_{i \in F} A_i$ for $\prod_{i \in F} \lambda_i(A_i)$, where $\lambda_i$ is the natural injection of $\mathcal{A}_i$ into $\bigotimes_{i \in F} \mathcal{A}_i$.) Furthermore, for every product functional $\nu = \bigotimes \nu_i$ for $(\mathcal{A}_i, (\alpha_i))$:

$$S(\nu) = \prod_{i \in F} \alpha_i(S(\nu_i)).$$

**Proof.** If $\mu = \bigotimes_{i \in F} \mu_i$, $E(\mu) = 1$ because (III) holds and $F$ is finite. Hence (VI-(\mu)) holds, and Corollary 3.4 and Proposition 2.6 complete the proof.

4. Local tensor products.

**LEMMA 4.1.** Suppose that $(\mathcal{A}_i)_{i \in I}$ is a factorization of the $W^*$-algebra $\mathcal{A}$, and that $\mu_i$ is a normal positive functional on $\mathcal{A}_i$. Let $\Sigma_i$ be the set of normal positive functionals $\mu_i$ on $\mathcal{A}_i$ such that
\( \mu_1 \otimes \mu_2 \) exists as a product functional on \( \mathcal{A} \) for the factorization \( (\mathcal{A}_i) \). Then:

(i). If \( \mu_1 \in \Sigma \) and \( x > 0 \), then \( x\mu_1 \in \Sigma \).

(ii). If \( \mu_1^* \in \Sigma \) and \( \sum_n \mu_1^*(1) < \infty \), then \( \sum_n \mu_1^* \in \Sigma \).

(iii). If \( \mu_1 \in \Sigma \) and \( A_1 \in \mathcal{A}_i \), with \( \mu_1(A_1^*A_i) \neq 0 \), then \( (\mu_1)_A \in \Sigma \). \((\mu_1)_A \) is defined by \( ((\mu_1)_A)(B_i) = \mu_1(A_i^*B_i) \) for all \( B_i \in \mathcal{A}_i \).

(iv). If \( \nu_i \) is a nonzero normal positive functional on \( \mathcal{A}_i \) with \( S(\nu_i) \leq Z(\mu_i) \) and \( \mu_1 \in \Sigma \), then \( \nu_i \in \Sigma \).

(v). If \( \Sigma \) is separating then \( \Sigma \) is the set of all nonzero normal positive functionals on \( \mathcal{A} \).

Proof. (i), (ii), and (iii) are obvious by direct calculation. To prove (iv) suppose that \( \nu_i \) is a normal positive functional on \( \mathcal{A}_i \) and that \( S(\nu_i) \leq Z(\mu_i) \) with \( \mu_1 \in \Sigma \). Then, by Proposition 3.1, there exists a normal homomorphism \( \psi \) from \( \mathcal{A}_i \) onto \( (Z(\mu_i))_A \) \( \otimes (Z(\nu_i))_A \) such that:

\[
\psi(A_1A_2) = (Z(\mu_i)A_1) \otimes (Z(\nu_i)A_2)
\]

for all \( A_1 \in \mathcal{A}_i \) and \( A_2 \in \mathcal{A}_2 \).

Now, since \( S(\nu_i) \leq Z(\mu_i) \), by Corollary 2.7 there exists a normal positive functional \( \omega = \nu_i \otimes \mu_2^* \) on \( (Z(\mu_i))_A \otimes (Z(\nu_i))_A \) such that

\[
\omega((Z(\mu_i)A_1) \otimes (Z(\nu_i)A_2)) = (\nu_i(A_1))(\mu_2(A_2))
\]

for all \( A_1 \in \mathcal{A}_i \) and \( A_2 \in \mathcal{A}_2 \). Evidently \( \omega \circ \psi \) equals \( \nu_i \otimes \mu_2^* \in \Sigma_\phi \) and \( \nu_i \in \Sigma \).

To prove (v) assume that \( \Sigma \) is separating. Then, using (iii) and Zorn's lemma, we can choose a family \( (\mu_i^j)_{j \in J} \) with each \( \mu_i^j \in \Sigma \) and \( \sum_{j \in J} Z(\mu_i^j) = 1 \). Suppose that \( \nu_i \) is a normal state of \( \mathcal{A}_i \). Then \( \sum_{j \in J} \nu_i(Z(\mu_i^j)) < \infty \) and therefore \( \nu_i(Z(\mu_i^j)) = 0 \) for all but a countable number of \( j \in J \). Hence a suitable countable linear combination \( \mu_i \) of the \( \mu_i^j \) satisfies \( S(\nu_i) \leq Z(\mu_i) \) and \( \mu_1 \in \Sigma \) by (ii). Then \( \nu_i \in \Sigma \) by (iv).

Remark. Lemma 4.1 may be proved directly (without using Proposition 3.1 or properties of the tensor product) by using Sakai's Radon-Nikodým theorem [8] and the weak Radon-Nikodým type result of [5, p. 211].

Proposition 4.2. Suppose that \( (\mathcal{A}_i)_{i \in I} \) is a factorization of the \( W^* \)-algebra \( \mathcal{A} \). Let \( \Sigma_{iv} \) be the set of product functionals on \( \mathcal{A} \) for which \( (IV-\mu) \) holds, and suppose that \( \Sigma_{iv} \) is separating. Suppose that \( F \) is a finite subset of \( I \). Then there exists an isomorphism \( \psi \) of \( \mathcal{A} \) onto \( \mathcal{A}_F \otimes \mathcal{A}_{-F} \) such that:

\[
\psi(AB) = A \otimes B \quad \text{for all} \quad A \in \mathcal{A}_F \quad \text{and} \quad B \in \mathcal{A}_{-F}.
\]
Proof. By Proposition 1.6 \{\mathcal{A}_F, \mathcal{A}_{1-F}\} is a factorization of \mathcal{A} and each \mu \in \Sigma_{iv} is a product functional for this factorization: \mu = \mu_F \otimes \mu_{1-F}. Let \Sigma_2 = \{\mu_{1-F}: \mu \in \Sigma_{iv}\} and let \Sigma_1 be the set of product functionals on \mathcal{A}_F for the factorization \{\mathcal{A}_i\}_{i \in F}. By Proposition 1.5 (iii), \Sigma_1 is separating on \mathcal{A}_F and \Sigma_2 is separating on \mathcal{A}_{1-F}. Because (IV-\mu) holds for all \mu \in \Sigma_{iv}, \nu \otimes \mu_{1-F} exists on \mathcal{A} for all \nu \in \Sigma_1 and all \mu_{1-F} \in \Sigma_2. Hence, by Lemma 4.1 (v), \nu \otimes \omega exists on \mathcal{A} for all \nu \in \Sigma_1 and all nonzero normal positive functionals \omega on \mathcal{A}_{1-F}; and, from there, by the same lemma, \nu \otimes \omega exists on \mathcal{A} for all nonzero normal positive functionals \nu of \mathcal{A}_F and \omega of \mathcal{A}_{1-F}. Thus (IV) holds for the factorization \{\mathcal{A}_F, \mathcal{A}_{1-F}\}. (III) obviously holds for \{\mathcal{A}_F, \mathcal{A}_{1-F}\} (Proposition 1.6 (iii)). Now Theorem 3.6 completes the proof.

Remark. Proposition 4.2 is false if the hypothesis that \mathcal{F} be finite is omitted (see Example 7.3).

Corollary 4.3. If \Sigma_{iv} is separating then (III) and (IV) hold.

Proof. That (III) holds is obvious. To prove (IV) use Proposition 4.2 and Corollary 2.7.

Corollary 4.4. If a product \{\mathcal{A}, (\alpha_i)\} satisfies (VI-\mu_i), then it satisfies (III), (IV), and (V): i.e., it is a (\mu_i)-local tensor product.

Proof. Use Proposition 1.16 and Proposition 4.2.

Proposition 4.5. Suppose that \{\mathcal{A}, (\alpha_i)\} is a tensor product for \{\mathcal{A}_i\}_{i \in I}; i.e., that (III) and (IV) hold. Then, for all \mu = \bigotimes_{i \in I} \mu_i \in \Sigma:\n
\begin{align*}
(4.1) \quad E(\mu) &= T(\mu) \\
(4.2) \quad S(\mu) &= \prod_{i \in I} \alpha_i(S(\mu_i)).
\end{align*}

Proof. \(E(\mu) \leq T(\mu)\) by Proposition 1.15. To prove (4.1), then, it suffices to prove that \(E(\mu)\) is tail. Let \(\Sigma = \{\nu \in \Sigma_F: \nu = \bigotimes \nu_i\) with \(\nu_i = \mu_i\) for a.a. \(i \in I\}. Then \(E(\mu) = \sup \{S(\nu): \nu \in \Sigma\}\). Suppose that \(F\) is a finite subset of \(I\). Then \{\mathcal{A}_F, \mathcal{A}_{1-F}\} is a factorization of \(\mathcal{A}\) for which each \(\nu \in \Sigma\) is a product functional: \(\nu = \nu_F \otimes \nu_{1-F}\). By Proposition 4.2 and (2.1) of Proposition 2.8, for all \(\nu \in \Sigma:\n
\begin{align*}
(4.3) \quad S(\nu) &= [S(\nu_F)][S(\nu_{1-F})].
\end{align*}

Thus:
Because of (IV), for fixed $\nu_{i-F}$, $\nu_F$ runs through all product functionals for $(\mathcal{A}_F, (\alpha_i)_{i \in F})$ and therefore (Proposition 1.5. (iv)):

$$\sup \{S(\nu_F) : \nu \in \Sigma, \nu_{i-F} \text{ fixed} \} = 1.$$ 

Using (4.3), we obtain:

$$E(\mu) \leq \sup \{S(\nu_{i-F}) : \nu \in \Sigma\}.$$ 

Now (4.4) and (4.5) show that $E(\mu) \in \mathcal{A}_{i \in F}$. Since $F$ was an arbitrary finite subset of $I$, we have shown that $E(\mu)$ is tail. That proves (4.1). (4.2) follows from (4.1) and Corollary 3.3.

COROLLARY 4.6. A product $(\mathcal{A}, (\alpha_i))$ for $(\mathcal{A})$ is a $(\mu_i)$-local tensor product for $(\mathcal{A})$ if and only if (VI-(\mu)) holds.

Proof. Corollary 4.4 shows that (VI-(\mu)) is sufficient. Suppose that $(\mathcal{A}, (\alpha_i))$ is a $(\mu_i)$-local tensor product for $(\mathcal{A})$. Let $\mu = \bigotimes \mu_i$. Then (IV-$\mu$) holds because (IV) does, and, using Proposition 4.5 and (V), we see that $E(\mu) = T(\mu) = 1$.

THEOREM 4.7. Suppose that, for each $i \in I$, $\mathcal{A}_i$ is a $W^*$-algebra and $\mu_i$ is a normal positive functional on $\mathcal{A}_i$. Suppose that $0 < \prod_{i \in I} \mu_i(1) < \infty$. Then a $(\mu_i)$-local tensor product exists and is unique up to isomorphism.

Proof. Proposition 2.6, Corollary 3.4, and Corollary 4.6.

5. Tensor products. Throughout this section we suppose that $(\mathcal{A}_i)_{i \in I}$ is a family of $W^*$-algebras. Let $A$ be the set of families $(\mu_i)_{i \in I}$, each $\mu_i$ a normal positive functional on $\mathcal{A}_i$ and

$$0 < \prod_{i \in I} \mu_i(1) < \infty.$$ 

Let the relation $R$ on $A$ be defined by writing $(\mu_i) \sim (\nu_i)$ (mod $R$) to mean that a $(\mu_i)$-local tensor product for $(\mathcal{A}_i)_{i \in I}$ is necessarily a $(\nu_i)$-local tensor product for $(\mathcal{A}_i)_{i \in I}$. $R$ is a well defined equivalence relation because a $(\mu_i)$-local tensor product exists and is unique up to isomorphism. The following lemma is a trivial consequence of the definition of $(\mu_i)$-local tensor product.

LEMMA 5.1. Let $(\mathcal{A}, (\alpha_i))$ be a $(\mu_i)$-local tensor product and let $(\nu_i) \in A$. Then $(\mu_i) \sim (\nu_i)$ (mod $R$) if and only if $\bigotimes \nu_i$ exists on $(\mathcal{A}, (\alpha_i))$. 

REMARK. If $\sum_{i \in I} |d(\mu_i, \nu_i)|^2 < \infty$ then $(\mu_i) \sim (\nu_i)$, and the converse holds provided each $\mathcal{A}_i$ is semi-finite [2].

**Lemma 5.2.** If $(\mu_i)$ and $(c_i \mu_i)$ are in $A$ (where the $c_i$ are positive real numbers), then $(\mu_i) \sim (c_i \mu_i)$.

*Proof.* Since $\prod \mu_i(1)$ and $\prod c_i \mu_i(1)$ both converge to a nonzero number, so must $\prod c_i$ converge to $c \neq 0$. If $\mu = \bigotimes \mu_i$ exists as a product for $(\mathcal{A}_i)$, $c \mu$ is a product state equal to $\bigotimes (c_i \mu_i)$ by direct calculation.

REMARK. This lemma shows that we could, without loss of generality, confine ourselves to $(\mu_i)$ with each $\mu_i(1) = 1$.

Define $\Delta$ to be the quotient set $A/R$ and let $\varphi$ be the quotient map $A \to A/R = \Delta$.

**Definition 5.3.** A tensor product $(\mathcal{A}, (\alpha_i))$ for $(\mathcal{A}_i)$ will be called a $\Gamma$-tensor product for $(\mathcal{A}_i)$ when:

$$\{(\mu_i) \in A: \bigotimes \mu_i \text{ exists on } (\mathcal{A}_i, (\alpha_i)) \} = \varphi^{-1}(\Gamma).$$

**Lemma 5.4.** Let $\gamma = \varphi((\mu_i))$. Then a $(\mu_i)$-local tensor product is a $\{\gamma\}$-tensor product.

**Theorem 5.5.** Suppose that $(\mathcal{A}, (\alpha_i))$ is a tensor product for $(\mathcal{A}_i)_{i \in I}$. Let

$$\Gamma = \varphi((\mu_i) \in A: \bigotimes \mu_i \text{ exists on } (\mathcal{A}_i, (\alpha_i)) \}.$$

Then:

(i) $(\mathcal{A}, (\alpha_i))$ is a $\Gamma$-tensor product for $(\mathcal{A}_i)$.

(ii) If $\mu = \bigotimes \mu_i$ and $\nu = \bigotimes \nu_i$ are product functionals for $(\mathcal{A}_i, (\alpha_i))$ then:

$$T(\mu) = T(\nu) \text{ if and only if } (\mu_i) \sim (\nu_i)$$

and

$$[T(\mu)][T(\nu)] = 0$$

otherwise.

(iii) If $\mu = \bigotimes \mu_i$ is a product functional for $(\mathcal{A}, (\alpha_i))$ then:

$$(5.1) \quad S(\mu) = [T(\mu)] \prod \alpha_i(S(\mu_i))$$

$$(5.2) \quad T(\mu) = \sup \{S(\bigotimes \nu_i): (\nu_i) \sim (\mu_i)\}.$$
(iv). \((\mathcal{A}, (\alpha_i))\) is a \((\mu_i)\)-local tensor product if and only if \(\Gamma = \{\varphi((\mu_i))\}\).

(v). For each \(\gamma \in \Gamma\), define \(T(\gamma)\) to be \(T(\mu)\) for \(\mu = \bigotimes \mu_i\) and \(\varphi((\mu_i)) = \gamma\). Let \(\mathcal{A}(\gamma) = [T(\gamma)]_{\mathcal{A}}\) and let \(\alpha_i(\gamma)\) be defined by:

\[(\alpha_i(\gamma))(A_i) = [T(\gamma)]_{\mathcal{A}(\gamma)}(\alpha_i(A_i))\quad \text{for all } A_i \in \mathcal{A}_i\text{ and all } i \in I.

Then, for each \(\gamma \in \Gamma\), \((\mathcal{A}(\gamma), (\alpha_i(\gamma)))\) is a \((\mu_i)\)-local tensor product for \((\mathcal{A}_i)\) provided that \(\gamma = \varphi((\mu_i))\).

Furthermore:

\[\mathcal{A} = \bigoplus_{\gamma \in \Gamma} \mathcal{A}(\gamma)\quad \text{and}\quad \alpha_i = \bigoplus_{\gamma \in \Gamma} \alpha_i(\gamma)\quad \text{for all } i \in I,

with respect to the same direct sum decomposition of \(\mathcal{A}\).

\textbf{Proof.} Suppose that \(\mu = \bigotimes \mu_i\) is a product functional on \((\mathcal{A}, (\alpha_i))\). For each \(i \in I\), define \(\beta_i : \mathcal{A} \rightarrow \mathcal{A}_i\) by:

\[\beta_i(A_i) = [T(\mu)]_{\mathcal{A}(\gamma)}(\alpha_i(A_i))\quad \text{for all } A_i \in \mathcal{A}_i.

Then, by Proposition 1.18:

(5.3) \([(T(\mu)]_{\mathcal{A}_i}(\beta_i))\) is a \((\mu_i)\)-local tensor product for \((\mathcal{A}_i)\).

By Lemma 5.1, therefore, for all \((\nu_i) \in A, (\nu_i) \sim (\mu_i)\) if and only if \(\bigotimes \nu_i\) exists on \([(T(\mu)]_{\mathcal{A}_i}(\beta_i))\). According to Proposition 1.18. (iii), however, this happens precisely when \(\bigotimes \nu_i\) exists on \((\mathcal{A}_i, (\alpha_i))\) and \(S(\bigotimes \nu_i) \leq T(\mu)\). We have shown that, for all \((\nu_i) \in A\), and for all product functionals \(\mu = \bigotimes \mu_i\) for \((\mathcal{A}_i, (\alpha_i))\):

(5.4) \((\nu_i) \sim (\mu_i)\) if and only if \(\bigotimes \nu_i\) exists on \((\mathcal{A}_i, (\alpha_i))\) and \(S(\bigotimes \nu_i) \leq T(\mu)\).

(5.4) shows that, if \((\nu_i) \sim (\mu_i)\) and if \(\bigotimes \mu_i\) exists on \((\mathcal{A}_i, (\alpha_i))\), then \(\bigotimes \nu_i\) exists on \((\mathcal{A}_i, (\alpha_i))\); (i) follows. (ii) is an immediate consequence of (5.4) and the fact that \(\mathcal{T}\) is atomic (Proposition 1.9). (5.1) of (iii) is just (4.2) of Proposition 4.5, and (5.2) is a consequence of (4.1) of Proposition 4.5 and (5.4). (iv) follows from (ii). (5.3), together with (ii), proves (v).

\textbf{Theorem 5.6.} Suppose that \((\mathcal{A}_i)_{i \in I}\) is a family of \(W^*-\)algebras and that \(A\) is as defined above. Then:

(i). If \(A\) is a nonempty subset of \(A\), a \(\Gamma\)-tensor product for \((\mathcal{A})\) exists and is unique up to isomorphism.

(ii). Suppose that \((\mathcal{A}_i, (\alpha_i))\) is a \(\Gamma_i\)-tensor product for \((\mathcal{A}_i)\) and that \((\mathcal{B}_i, (\beta_i))\) is a \(\Gamma_i\)-tensor product for \((\mathcal{B}_i)\). Then \(\Gamma_i = \Gamma_i\) if and only if \((\mathcal{A}_i, (\alpha_i))\) and \((\mathcal{B}_i, (\beta_i))\) are isomorphic: i.e., if and only if
there exists an isomorphism $\psi$ of $\mathcal{A}$ onto $\mathcal{B}$ such that $\psi \circ \alpha_i = \beta_i$ for all $i \in I$.

**Proof.** Everything but the existence of a $\Gamma$-tensor product for $(\mathcal{A}_i)$ follows from Theorem 5.5. For each $\gamma \in \Delta$, a $\{\gamma\}$-tensor product exists by Theorem 4.7. Hence the existence is a result of the following proposition.

**Proposition 5.7.** Suppose that $\Gamma$ is a subset of $\Delta$ and that, for each $\gamma \in \Gamma$, $(\mathcal{A}(\gamma), \alpha(\gamma))$ is a $\{\gamma\}$-tensor product for $(\mathcal{A}_i)$. Let $(\mathcal{A}, \alpha_i)$ is a $\Gamma$-tensor product for $(\mathcal{A}_i)$.

**Proof.** Let $(\mathcal{A}, \alpha_i)$ be defined as above and let $E(\gamma)$ be the projection of $\mathcal{A}$ with $\mathcal{A}(\gamma) = [E(\gamma)].$ Let $\mathcal{B} = \mathcal{B}(\alpha_i, \mathcal{A}_i)$ for $i \in I$. Then $(\mathcal{B}, \alpha_i)$ is a product for $(\mathcal{A}_i)$. If $\phi((\mu_i)) = \gamma \in \Gamma$, then $\mu' = \bigotimes \mu_i$ exists on $(\mathcal{A}(\gamma), (\alpha(\gamma)))$, and, if $\mu$ is defined by $\mu(B) = \mu'(E(\gamma)B)$ for all $B \in \mathcal{B}$, we can see by direct calculation that $\mu = \bigotimes \mu_i$ on $(\mathcal{B}, \alpha_i)$ with (5.5)

$$S(\mu) \leq E(\gamma) \quad \text{where} \quad \gamma = \phi((\mu_i)).$$

It is clear that such $\mu$ form a separating subset $\Sigma$ of the normal positive functionals on $\mathcal{B}$, and that—since $\nu_i = \mu_i$ for a.a. $i \in I$ implies $(\nu_i) \sim (\mu_i)$—(IV-$\mu$) holds for each $\mu \in \Sigma$. Therefore (Corollary 4.3), $(\mathcal{B}, \alpha_i)$ is a tensor product for $(\mathcal{A}_i)$. By (5.2) of Theorem 5.5 (iii), and by (5.5):

$$T(\gamma) = E(\gamma) \quad \text{for all} \quad \gamma \in \Gamma.$$

Hence each $E(\gamma) \in \mathcal{B}$ and $\mathcal{B} = \mathcal{A}$. Furthermore $\sum_{\gamma \in \Gamma} T(\gamma) = 1$, so that, if $\mu = \bigotimes \mu_i$ is a product functional for $(\mathcal{A}_i, (\alpha_i))$, then $T(\mu) = T(\gamma)$ for some $\gamma \in \Gamma$ (Proposition 1.9) and $\phi((\mu_i)) = \gamma \in \Gamma$ by Theorem 5.5. (ii). Therefore $(\mathcal{A}_i, (\alpha_i))$ is a $\Gamma$-tensor product for $(\mathcal{A}_i)$.

**Proposition 5.8.** Suppose that, for each $i \in I$, $\mathcal{A}_i$ is a von Neumann algebra on $H_i$ and every normal positive functional on $\mathcal{A}_i$ is induced by a vector. Let $H$ denote von Neumann's complete direct product $\mathcal{L}(H_i)_{i \in I}$ and let $\alpha_i$ be the natural injection of $\mathcal{L}(H_i)$ into $\mathcal{L}(H)$ for each $i \in I$. Let $(\mathcal{A}_i, (\alpha_i))$ be a $J$-tensor product for $(\mathcal{A}_i)_{i \in I}$. Furthermore, for every nonempty subset $\Gamma$ of $\Delta$, there exists a projection $T(\Gamma)$ in the tail of $(\mathcal{A}_i, (\alpha_i))$ such that $(T(\Gamma), (\alpha_i))$ is a $\Gamma$-tensor product for $(\mathcal{A}_i)_{i \in I}$, where $\beta_i(A_i) = [T(\Gamma)]A_i$ for all $A_i \in \mathcal{A}_i$. 

The proof is easy and is omitted.

6. Tensor products of factors.

**Lemma 6.1.** If \((\mathcal{A}, (\alpha_i))\) is a product for \((\mathcal{A})\) and if each \(\mathcal{A}_i\) is a factor, then (IV) holds for \((\mathcal{A}, (\alpha_i))\).

*Proof.* Use Lemma 4.1. (iv) and mathematical induction.

**Lemma 6.2.** Suppose that \((\mathcal{A}, (\alpha_i))\) is a product for \((\mathcal{A})\) and that \(\mathcal{A}\) is a factor. Then \((\mathcal{A}, (\alpha_i))\) is a local tensor product for \((\mathcal{A})\) if and only if there exists a product state of \((\mathcal{A}, (\alpha_i))\).

*Proof.* Each \(\mathcal{A}_i\) is a factor by Lemma 1.3, and therefore (IV) holds by Lemma 6.1. (V) holds because \(\mathcal{J} \subset \mathcal{K}\). If \(\mu\) is a product functional on \((\mathcal{A}, (\alpha_i))\), then \(E(\mu) = 1\), for \(E(\mu)\) is central by Theorem 3.2. Thus (III) holds if and only if a product state \(\mu\) exists.

**Proposition 6.3.** Suppose that \((\mathcal{A}, (\alpha_i))\) is a tensor product for \((\mathcal{A})\) and that each \(\mathcal{A}_i\) is a factor. Then \(\mathcal{J} = \mathcal{K}\): the tail of \((\mathcal{A}, (\alpha_i))\) equals the center of \(\mathcal{A}\).

*Proof.* By Theorem 5.5, the family \((T(\gamma))_{\gamma \in \Gamma}\) of atomic projections of \(\mathcal{J}\) is such that each \([T(\gamma), \mathcal{A}]\) is a local tensor product for \((\mathcal{A})\). By Lemma 2.5, each \([T(\gamma), \mathcal{A}]\) is a factor. Hence the center of \(\mathcal{A} = \bigoplus [T(\gamma), \mathcal{A}]\) is \(\mathcal{B}(T(\gamma); \gamma \in \Gamma) = \mathcal{J}\).

**Corollary 6.4.** Suppose that \((\mathcal{A}, (\alpha_i))\) is a tensor product for \((\mathcal{A})\) and that each \(\mathcal{A}_i\) is a factor. Then \(\mathcal{A}\) is a factor if and only if \((\mathcal{A}, (\alpha_i))\) is a local tensor product: i.e., if and only if (V) holds.

**Proposition 6.5.** Suppose that \(\mathcal{A}\) is a finite factor and that \((\mathcal{A})_{i \in I}\) is a factorization of \(\mathcal{A}\). Let \(\mu_i\) be the restriction of the normalized trace on \(\mathcal{A}\) to \(\mathcal{A}_i\). Let \((\varnothing, (\alpha_i))\) be a \((\mu_i)\)-local tensor product for \((\mathcal{A})_{i \in I}\). Then there exists an isomorphism \(\psi\) of \(\mathcal{A}\) onto \(\mathcal{B}\) such that, for each \(i \in I\):

\[
\psi(A_i) = \alpha_i(A_i) \quad \text{for all } A_i \in \mathcal{A}_i.
\]

*Proof.* (c.f. [6]). If \(\mu\) is the normalized trace on \(\mathcal{A}\), a direct calculation (see the proof of Theorem 4.3 in [2]) demonstrates that \(\mu = \bigotimes \mu_i\) for \((\mathcal{A})\). From there Lemma 6.2, Theorem 4.7, Corollary 4.4 and Proposition 2.6 complete the proof.
7. Some simple counterexamples.

**EXAMPLE 7.1.** Let \( \mathcal{A} \) be a factor of Type II, on the Hilbert space \( H \). Then \( \{\mathcal{A}, \mathcal{A}'\} \) is a factorization of \( \mathcal{F}(H) \) which satisfies (IV) and (V), and for which no product functional exists.

See Lemmas 6.1, 6.2 and Theorem 3.6.

**EXAMPLE 7.2.** For \( i = 1 \) and 2, let \( \mathcal{A}_i \) be a \( W^* \)-algebra with central projection \( Z_i \neq 0 \) or 1. Let

\[
Z = (Z_i \otimes Z_2) + (1 - Z_i) \otimes (1 - Z_2)
\]

in \( \mathcal{A} \otimes \mathcal{A}_2 \), and let \( \mathcal{A} = Z(\mathcal{A}_1 \otimes \mathcal{A}_2) \). Let \( \alpha_i: \mathcal{A}_i \to \mathcal{A} \) be defined by

\[
\alpha_i(A_i) = Z(A_i \otimes 1) \quad \text{for all } A_i \in \mathcal{A}_i
\]

\[
\alpha_2(A_2) = Z(1 \otimes A_2) \quad \text{for all } A_2 \in \mathcal{A}_2.
\]

Then \( (\mathcal{A}, (\alpha_i)) \) is a product for \( (\mathcal{A}_i)^{i=1,2} \) which satisfies (III) and (V) but not (IV).

**EXAMPLE 7.3.** Let \( I = \{1, 2\} \times J \) where \( J \) is infinite, and, for each \( i \in I \), let \( \mathcal{A}_i \) be an abelian \( W^* \)-algebra generated by its two atomic projection \( E_i \) and \( 1 - E_i \). Let the states \( \mu_i \) and \( \nu_i \) of \( \mathcal{A}_i \) be defined by \( \mu_i(1) = \nu_i(1) = 1 \) and \( \mu_i(E_i) = 1 \) and \( \nu_i(E_i) = 1/2 \). Let \( \Gamma = \varphi(\{(\mu_i), (\nu_i)\}) \) and let \( (\mathcal{A}, (\alpha_i)) \) be a \( \Gamma \)-tensor product for \( (\mathcal{A}_i)^{j=1,2} \). Let

\[
\mathcal{B}_i = \mathcal{B}(\alpha_i(\mathcal{A}_i): i \in \{\delta\} \times J)
\]

and let \( \lambda_\delta: \mathcal{B}_\delta \to \mathcal{A} \) be the inclusion map, for \( \delta = 1 \) and 2. Then \( (\mathcal{A}, (\lambda_\delta)) \) is a product for \( (\mathcal{A}_\delta)^{\delta=1,2} \) which satisfies (III) and (V) but not (IV). In particular, \( (\mathcal{A}, (\lambda_\delta)) \) is not isomorphic to \( \mathcal{A} \otimes \mathcal{A} \).

To make this clearer, let \( \mathcal{B} = \mathcal{A}_1 \otimes \mathcal{A}_2 \) with \( \lambda_\delta \) the natural injection of \( \mathcal{A}_\delta \) into \( \mathcal{B} \). Let \( \beta_i: \mathcal{A}_i \to \mathcal{B} \) be defined for each \( i = (\delta, j) \in I \) by \( \beta_i = \lambda_\delta \circ \alpha_i \). Then \( (\mathcal{B}, (\beta_i)) \) is a \( \Gamma'^* \)-tensor product for \( (\mathcal{A}_i) \) where \( \Gamma'' \) contains four points. In fact

\[
\Gamma'' = \varphi(\{(\mu_i), (\nu_i), (\omega_i), (\rho_i)\})
\]

where:

\[
\omega_i = \mu_i \quad \text{and} \quad \rho_i = \nu_i \quad \text{for } i \in \{1\} \times J
\]

and

\[
\omega_i = \nu_i \quad \text{and} \quad \rho_i = \mu_i \quad \text{for } i \in \{2\} \times J.
\]
EXAMPLE 7.4. Let $I = \{1, 2\} \times J$ with $J$ infinite. For each $i \in I$, let $H_i$ be a Hilbert space (of arbitrary dimension $\geq 2$) and let $\varphi_i$ and $\psi_i$ be orthogonal unit vectors in $H_i$. For each $j \in J$, let $H_j = H_{(1,j)} \otimes H_{(2,j)}$ and let $x_j = [\varphi_{(1,j)} \otimes \omega_{(2,j)} + \psi_{(1,j)} \otimes \psi_{(2,j)}] / \sqrt{2}$. Let $H = \bigotimes_{j \in J} (H_j, x_j)$ and let $\beta_j$ be the natural injection of $L(H_j)$ into $L(H)$. Let $\gamma_{(i,j)}$ be the natural injection of $L(H_{(i,j)})$ into $L(H_j)$. Let $\alpha_{(i,j)} = \beta_j \circ \gamma_{(i,j)}$ for all $(i,j) \in I$. Then $(L(H), (\alpha_i))$ is a product for $(L(H_i))_{i \in I}$, and there exist no product functionals for $(L(H), (\alpha_i))$.

See [2] or the remark which follows Lemma 5.1.

REFERENCES


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