PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 2, 1968

UNCOUNTABLY MANY ALMOST POLYHEDRAL WILD $(k-2)$-CELLS IN E^k FOR $k \geqq 4$

LESLIE C. GLASER

In [1] infinitely many almost polyhedral wild arcs were constructed in E^3 so as to have an end point as the "bad ' point. In [5] uncountably many almost polyhedral wild arcs were constructed in E^3 with an interior point as the "bad" point. In [4] Doyle and Hocking constructed an almost polyhedral wild disk in E^4 with the property that the proof of the nontameness is perhaps the most elementary possible. They state that essentially the same construction yields a wild $(n-2)$-disk in E^n for $n \geqq 4$. Here, making use of the construction given in [4], we prove that for each $k \geqq 4$, there exist uncountably many almost polyhedral wild $(k-2)$-cells in E^k. To obtain the above result we also prove that for each $k \geqq 3$, there exist countably many polyhedral locally flat $(k-2)$-spheres in E^k so that the fundamental groups of the complements of these spheres are all distinct and given any two of these groups, one is not the surjective image of the other.

A set S in E^k is polyhedral if it can be covered by a finite rectilinear subcomplex of E^k. A $(k-2)$-cell D in E^k is almost polyhedral if for some point $q \in D$, $D - \{q\}$ can be covered by an infinite locally finite rectilinear subcomplex of $E^k - \{q\}$. The $(k-2)$-cells constructed here all have $q \in \mathrm{Bd}\, D$. D is wild if there does not exist a homeomorphism h of E^k onto itself such that $h(D)$ is a finite rectilinear subcomplex of E^k. An n-manifold $M^n \subset E^k$ is locally flat if each $p \in \mathrm{int}\, M (p \in \mathrm{Bd}\, M)$ has a neighborhood U in E^k such that the pair $(U, U \cap M)$ is homeomorphic as pairs to (E^k, E^n) (to (E^k, E^n_+)).

THEOREM 1. *There exist countably many polyhedral simple closed curves $\{J_n\}$ $(n = 1, 2, 3, \cdots)$ in E^3 so that if $G_n \cong \pi_1(E^3 - J_n)$, then for all positive integers n and m $(n \neq m)$, $G_n \not\cong Z$ and $G_n \not\cong G_m$. Furthermore, if $m > n$, then there is no surjection of G_m onto G_n.*

Proof. Expressing points of E^3 in terms of cylindrical coordinates (θ, r, z), let T be the "unknotted" torus $(r-2)^2 + z^2 = 1$. Let $K_{p,q}$ denote the torus knot of type p, q, where p and q are relatively prime nonnegative integers and $K_{p,q}$ is a curve on the surface T that cuts a merdian in p points and a longitude in q points. More precisely, $K_{p,q}$ is defined by the equations $r = 2 + \cos(q\theta/p)$ and $z = \sin(q\theta/p)$.

A presentation for $\pi_1(E^3 - K_{p,q})$ is $P_{p,q} = \{x, y \mid x^p = y^q\}$ [3].

Suppose q is an odd integer >1, p is a prime $>q$, and $G_{p,q}$ denotes a group having presentation $P_{p,q}$. Then $G_{p,q}$ has a nontrivial representation in the symmetric group S_p by sending $x \to (1, 2, 3, \cdots, p)$ and $y \to (1, 2, 3, \cdots, q)$. Let $\hat{S}_p$ denote the subgroup of S_p generated by $(1, 2, 3, \cdots, p)$ and $(1, 2, 3, \cdots, q)$. Then we have a surjection $\varphi_{p,q}: G_{p,q} \to \hat{S}_p$.

Since

$$(1, 2, 3, \cdots, q)(1, 2, 3, \cdots, q, \cdots, p) \\ = (1, 3, \cdots, q-2, q, 2, 4, \cdots, q-1, q+1, q+2, \cdots, p)$$

and

$$(1, 2, 3, \cdots, q, \cdots, p)(1, 2, 3, \cdots, q) \\ = (1, 3, \cdots, q-2, q, q+1, q+2, \cdots, p, 2, 4, \cdots, q-3, q-1)\ ,$$

$\hat{S}_p$ is not commutative and hence $G_{p,q} \not\cong Z$.

Let $\{(p_n, q_n)\}$ $(n = 1, 2, 3, \cdots)$ be a sequence of pairs of positive odd integers, where

$$q_1 = 3 < p_1 < q_2 = p_1! + 1 < p_2 < \cdots < p_{n-1} < q_n \\ = p_{n-1}! + 1 < p_n < \cdots$$

and the p_n's are all distinct primes. Let $\{J_n\}$ $(n = 1, 2, 3, \cdots)$ be a sequence of polyhedral simple closed curves in E^3, so that for each n, we have a homeomorphism h_n of E^3 onto itself carrying J_n onto K_{p_n, q_n}. Then $\pi_1(E^3 - J_n) \cong G_n \cong G_{p_n, q_n} \not\cong Z$. Suppose for some $m > n$ there is a surjection ψ carrying G_m onto G_n. Since $G_m \cong G_{p_m, q_m}$ and $G_n \cong G_{p_n, q_n}$ we can suppose we have a surjection, which we also denote by ψ, carrying G_{p_m, q_n} onto G_{p_n, q_n}. Then $\rho = \varphi \circ \psi$ is a surjection carrying G_{p_m, q_m} onto $\hat{S}_{p_n}$. Since x and y generate G_{p_m, q_m}, $u = \rho(x)$ and $v = \rho(y)$ generate $\hat{S}_{p_n}$. But in considering the relation defining G_{p_m, q_m} we get that $u^{p_m} = v^{q_m}$. Since the order of S_{p_n} is $p_n!$ and since $q_m = p_{m-1}! + 1$ and $p_{m-1} \geqq p_n$, it follows that $v^{q_m} = v$ and hence $u^{p_m} = v$. This gives the contradiction that the noncommutative group $\hat{S}_{p_n}$ is generated by two commuting elements u and y. Therefore, for all $m > n$ there is no surjection of G_m onto G_n and hence $G_m \not\cong G_n$.

THEOREM 2. *For each $k \geqq 3$, there exist countably many polyhedral locally flat $(k-2)$-spheres $\{S_n^{k-2}\}$ $(n = 1, 2, 3, \cdots)$ in E^k so that if $G_n \cong \pi_1(E^k - S_n^{k-2})$, then for all positive integers n and m $(n \neq m)$, $G_n \not\cong Z$ and $G_n \not\cong G_m$. Furthermore, if $m > n$, then there is no surjection of G_m onto G_n.*

Proof. We could easily obtain the desired result if we omit the local flatness from the conclusion by taking repeated suspensions of the sequence $\{J_n\}$ of Theorem 1. This follows since the fundamental group of the complement of a $(k-2)$-sphere S^{k-2} in E^k is isomorphic to the fundamental group of the complement of the suspension of S^{k-2} in E^{k+1}.

The proof will be by induction on k. For $k = 3$ the result follows by taking the sequence of polyhedral locally flat 1-spheres $\{S_n^1\}$ to be the $\{J_n\}$ of Theorem 1. Suppose inductively for each $k, 3 \leqq k \leqq m$, there exist countably many polyhedral locally flat $(k-2)$-spheres $\{S_n^{k-2}\}$ $(n = 1, 2, 3, \cdots)$ in E^k having the desired properties.

We now consider the collection $\{S_n^{m-2}\}$ of polyhedral locally flat $(m-2)$-spheres in E^m. Let $S \in \{S_n^{m-2}\}$ be an arbitrary $(m-2)$-sphere from our given collection. Since S is polyhedral we can assume that S lies in $E^m \subset E^{m+1}$ so that we have

$$S \subset E_+^m = \{(x_1, x_2, \cdots, x_m, x_{m+1}) \in E^{m+1} \mid x_m \geqq 0, x_{m+1} = 0\}$$

and so the $S \cap E^{m-1}$ is a $(m-2)$-simplex $\varDelta \in S$, where

$$E^{m-1} = \{(x_1, x_2, \cdots, x_m, x_{m+1}) \mid x_m = 0 = x_{m+1}\} = \operatorname{Bd} E_+^m .$$

Let D be the closure of $S - \varDelta$. Let $\alpha_t \colon E_+^m \to E^{m+1}$ be the rigid rotation in $E^{m+1} = \{(y_1, y_2, \cdots, y_m, y_{m+1})\}$ of $E_+^m = \{(x_1, \cdots, x_m, 0)\}$ defined by the equations

$$\begin{aligned} y_i &= x_i && i \leqq m-1 , \\ y_m &= x_m \cos t , \\ y_{m+1} &= x_m \sin t . \end{aligned}$$

Then the set $\hat{K} = \{\alpha_t(r) \in E^{m+1} \mid r \in D \text{ and } t \in [0, 2\pi]\}$ is clearly an $(m-1)$-sphere in E^{m+1}. By the proof given in [2], if follows that $\pi_1(E^{m+1} - \hat{K}) \cong \pi_1(E^m - S)$. Since S is locally flat in E^m, it follows that $\hat{K}$ is locally flat in E^{m+1}. Hence using the sequence $\{S_n^{m-2}\}$ and constructing a $\hat{K}_n$ as above for each S_n, we obtain countably many locally flat $(m-1)$-spheres in E^{m+1} having all the desired properties except that of being polyhedral.

Now for each $S \in \{S_n^{m-2}\}$ we have a continuous family of functions $\{\alpha_t \colon E_+^m \to E^{m+1} \mid t \in [0, 2\pi]\}$ and a locally flat $(m-1)$-sphere $\hat{K}$ containing $D = \overline{S - \varDelta}$ so that

$$\pi_1(E^{m+1} - \hat{K}) \cong \pi_1(E^m - S) .$$

For each $r \in E_+^m - E^{m-1}$, let $\hat{C}_r$ be the circle in E^{m+1} determined by the point set $\{\alpha_t(r) \in E^{m+1} \mid t \in [0, 2\pi]\}$ and let C_r be the polyhedral simple closed curve in E^{m+1} consisting of the union of the four segments $[\alpha_0(r), \alpha_{\pi/2}(r)]$, $[\alpha_{\pi/2}(r), \alpha_\pi(r)]$, $[\alpha_\pi(r), \alpha_{(3\pi)/2}(r)]$, and $[\alpha_{(3\pi)/2}(r), \alpha_{2\pi}(r)]$. Let K denote the point set $\bigcup_r \{C_r \mid r \in D - E^{m-1}\} \cup D \cap E^{m-1}$. Then K is a polyhedral $(m-1)$-sphere containing $D = \overline{S - \Delta} \subset E^m_+$. The claim is that there is a homeomorphism h carrying E^{m+1} onto itself so that $h(\hat{K}) = K$. It would follow then that K is also locally flat and $\pi_1(E^{m+1} - K) \cong \pi_1(E^{m+1} - \hat{K})$ and hence we could obtain the desired result.

To see that such an h exists, let E^m_{+t} denote $\alpha_t(E^m_+)$. For each $r \in E^m_+ - E^{m+1}$ we define h sending E^m_{+t} onto itself by defining

$$h(\alpha_t(r)) = h(\hat{C}_r \cap E^m_{+t})$$

to be the point $C_r \cap E^m_{+t}$ and for $r \in E^m_{+t} \cap E^{m-1} = E^{m-1}$ we let $h(r) = r$. It is clear then that $h(\hat{K}) = K$. h can also be defined explicitly as follows. Let $s\colon [0, 2\pi] \to [0, 1]$ be defined as follows.

$$s(t) = \begin{cases} \sqrt{2}\big/2 \sin\left(\dfrac{3\pi}{4} - t\right); & 0 \leqq t \leqq \pi/2\,, \\ \sqrt{2}\big/2 \sin\left(t - \dfrac{\pi}{4}\right); & \pi/2 \leqq t \leqq \pi\,, \\ \sqrt{2}\big/2 \sin\left(\dfrac{7\pi}{4} - t\right); & \pi \leqq t \leqq \dfrac{3\pi}{2}\,, \\ \sqrt{2}\big/2 \sin\left(t - \dfrac{5\pi}{4}\right), & \dfrac{3\pi}{2} \leqq t \leqq 2\pi\,. \end{cases}$$

If $r_0 = (x_1, x_2, \cdots, x_{m-1}, 1, 0) \in E^m_+$, then $s(t)$ is merely the distance of the point $C_{r_0} \cap E^m_{+t}$ to the origin of E^{m+1}. h is then defined by sending $(x_1, x_2, \cdots, x_{m-1}, x_m \cos t, x_m \sin t)$ to

$$(x_1, x_2, \cdots, x_{m-1}, s(t)x_m \cos t, s(t)x_m \sin t)\,.$$

Suppose S_1 and S_2 are two polyhedral $(k-2)$-spheres in E^k with $G_i \cong \pi_1(E^k - S_i)$ $(i = 1, 2)$ so that there exists no surjection $\varphi\colon G_1 \to G_2$. Let D_1 be the polyhedral $(k-1)$-cell in E^{k+1} obtained by taking the cone over S_1. That is,

$$D_1 = p_1 * S_1 \subset E^{k+1}_+ \subset E^{k+1}$$

where $p_1 \in E^{k+1}_+ - E^k$ "above" S_1. Similarly let $D_2 = p_2 * S_2 \subset E^{k+1}_+ \subset E^{k+1}$. Let x_{ik+1} $(i = 1, 2)$ denote the $(k+1)$-coordinate of p_i and P_{ij} denote the horizontal k-plane in E^{k+1}_+ parallel to E^k given by

$$x_{ijk+1} = x_{ik+1} - \frac{1}{j} x_{ik+1}\,, \qquad j = 1, 2, 3, \cdots; i = 1, 2\,.$$

We note each P_{ij} lies below p_i $(i = 1, 2)$ and $P_{11} = E^k = P_{21}$. Let

$\{N_{ij}\}$ $(i = 1, 2; j = 1, 2, 3, \cdots)$ denote two sequences of $(k+1)$-cells obtained as follows. Each N_{ij} is to be "centered" at p_i having its "bottom" face B_{ij} in P_{ij} so that $\operatorname{int} B_{ij} \supset P_{ij} \cup D_i$, so that the part of D_i lying on or above P_{ij} lies in $(\operatorname{int} N_{ij}) \cup B_{ij}$, and so that the following properties hold for $i = 1, 2$:

(a) $N_{i1} \supset \operatorname{int} N_{i1} \supset N_{i2} \supset \operatorname{int} N_{i2} \supset N_{i3} \supset \cdots$,

(b) $\bigcap_{j=1}^{\infty} N_{ij} = p_i$,

(c) $\pi_1(N_{i1} - D_i)$ is isomorphic to $\pi_1(E^k - S_i)$, and

(d) the injection $\pi_1(N_{ij} - D_i) \to \pi_1(N_{i1} - D_i)$ is an isomorphism onto for each j.

THEOREM 3. *Suppose F_1 and F_2 are two $(k-1)$-cells in E^{k+1} so that if D_1 and D_2 are the polyhedral $(k-1)$-cells as given above, then there exist homeomorphisms f_1, f_2 taking E^{k+1} onto itself so that $f_1(D_1) \subset F_1$ and $f_2(D_2) \subset F_2$. Let $q_1 = f_1(p_1) \in F_1$ and $q_2 = f_2(p_2) \in F_2$. Then there exists no homeomorphism $h: E^{k+1} \to E^{k+1}$ carrying F_1 onto F_2 with $h(q_1) = q_2$.*

Proof. Suppose there exists a homeomorphism h taking E^{k+1} onto itself carrying F_1 onto F_2 with $h(q_1) = q_2$. We now consider the sequences $\{N_{1j}\}, \{N_{2j}\}$ given above. There exists an N_{2m} so that

$$f_2(N_{2m}) \cap F_2 = f_2(N_{2m}) \cap f_2(D_2) .$$

Let N_{1n} be chosen so that $f_1(N_{1n}) \cap f_1(D_1) = f_1(N_{1n}) \cap F_1$ and

$$hf_1(N_{1n}) \subset \operatorname{int} f_2(N_{2m}) .$$

Finally, let N_{2r} be chosen so that $f_2(N_{2r}) \subset \operatorname{int} hf_1(N_{1n})$. Since

$$f_2(N_{2r}) \subset \operatorname{int} f_2(N_{2m}), f_2(N_{2r}) \cap f_2(D_2) = f_2(N_{2r}) \cap F_2 .$$

The commutativity of the inclusion diagram

$$\begin{array}{ccc} f_2(N_{2r}) & \longrightarrow & hf_1(N_{1n}) \\ & {}_{i}\searrow \quad \swarrow_{j} & \\ & f_2(N_{2m}) & \end{array}$$

implies the commutativity of the induced injection diagram

$$\begin{array}{ccc} \pi_1(f_2(N_{2r} - D_2)) & \longrightarrow & \pi_1(hf(N_{1n} - D_1)) \\ & {}_{i_*}\searrow \quad \swarrow_{j_*} & \\ & \pi_1(f_2(N_{2m} - D_2)) . & \end{array}$$

Since i_* is onto, j_* must be onto. But

$$\pi_1(hf(N_{1n} - D_1)) \cong \pi_1(N_{1n} - D_1) \cong \pi_1(N_{11} - D_1) \cong \pi_1(E^k - S_1) \cong G_1$$

and

$$\pi_1(f_2(N_{2m} - D_2)) \cong \pi_1(N_{2m} - D_2) \cong \pi_1(N_{21} - D_1) \cong \pi_1(E^k - S_2) \cong G_2 .$$

It follows then that there would be a surjection φ of G_1 onto G_2, which by assumption is impossible and hence the result follows.

Given any fixed integer $k \geqq 3$, let $\{S_n\}$ $(n = 1, 2, 3, \cdots)$ be the countable collection of polyhedral locally flat $(k-2)$-sheres in E^k given by Theorem 2. For any subsequence $\alpha = (n_1, n_2, n_3, \cdots)$ of positive integers we will define an almost polyhedral wild $(k-1)$-cell in E^{k+1} using the construction given in [4]. That is, in E^k let $\{B_i\}$ be a sequence of disjoint k-balls converging to a point q. For each $i = 1, 2, 3, \cdots$, we suppose that S_{n_i} is embedded in $\operatorname{int} B_i$ by "shrinking" and translating each S_{n_i} in an appropriate manner. In E_+^{k+1}, let $\{p_i\}$ be the sequence of distinct points converging to q where p_i lies above the "center" of B_i and is a distance $1/i$ from E^k. If $p_i * S_{n_i}$ is the cone over S_{n_i} with vertex p_i, then the polyhedral $(k-1)$-cells $\{p_i * S_{n_i}\}$ are disjoint in pairs and each $p_i * S_{n_i}$ is locally flat except for p_i. The fact that $p_i * S_{n_i}$ is locally flat at points other than p_i follows since S_{n_i} is locally flat in E^k. The fact that $p_i * S_{n_i}$ is not locally flat at p_i follows in a manner similar to that used in the proof of Theorem 3. That is, there are arbitrarily small neighborhoods N about p_i in E^{k+1} such that $\pi_1(N - (p_i * S_{n_i})) \cong G_{n_i}$. If $p_i * S_{n_i}$ were locally flat at p_i then there would be arbitrarily small neighborhoods M about p_i such that $\pi_1(M - (p_i * S_{n_i})) \cong Z$. Hence we would be able to obtain a surjection of Z onto G_{n_i}, which would allow us to obtain a surjection of Z onto $\hat{S}_{n_i}$ which is noncommutative.

Now in E^k join $p_1 * S_{n_1}$ and $p_2 * S_{n_2}$ by a polyhedral $(k-1)$-cell D_1 so that $p_1 * S_{n_1} \cup D_1 \cup p_2 * S_{n_2}$ is a polyhedral $(k-1)$-cell disjoint from $(\bigcup_{i=3}^{\infty} p_i * S_{n_i}) \cup q$ that is locally flat except at p_1 and p_2. Next we join $p_2 * S_{n_2}$ and $p_3 * S_{n_3}$ by a polyhedral $(k-1)$-cell D_2 in E^k so that $p_1 * S_{n_1} \cup D_1 \cup p_2 * S_{n_2} \cup D_2 \cup p_3 * S_{n_3}$ is a polyhedral $(k-1)$-cell disjoint from $(\bigcup_{i=4}^{\infty} p_i * S_{n_i}) \cup q$ that is locally flat except at p_1, p_2 and p_3. This process is continued so that as $i \to \infty$ the diameter of D_i tends to zero and the desired $(k-1)$-cell D_α is $(\bigcup_{i=1}^{\infty} p_i * S_{n_i} \cup D_i) \cup q$. As a subset of E^{k+1}, D_α is almost polyhedral except perhaps at q. Also D_α is locally flat except at the points q and p_i $(i = 1, 2, 3, \cdots)$. By [4], D_α is wild. That is, if there is a homeomorphism h of E^{k+1} onto itself such that $h(D_\alpha)$ is the union of a finite number of $(k-1)$-simplexes, then some point of $\{h(p_j)\}$ lies in the interior of a $(k-1)$-cell formed by the union of two $(k-1)$-simplexes of $h(D_\alpha)$. Then by rotating one of these $(k-1)$-simplexes (if necessary) keeping the other fixed so that the union of the two lies in a $(k-1)$-plane in E^k, it would follow that $h(D_\alpha)$ is locally flat at this point. This contradicts the fact that D_α is not locally flat at the preimage of the given point.

THEOREM 4. *For each $k \geqq 4$, there exist uncountably many almost polyhedral wild $(k-2)$-cells in E^k.*

Proof. Let $\{\alpha\}$ be an uncountable collection of sequences of positive integers such that in two different ones some integer occurs more in one than in the other. For any fixed integer $k \geqq 3$, let $\{D_\alpha\}$ be the corresponding uncountable sequence of almost polyhedral wild $(k-1)$-cells in E^{k+1} constructed as above. Suppose for some

$$\alpha = \{n_1, n_2, n_3, \cdots\} \neq \alpha' = \{n'_1, n'_2, n'_2, \cdots\}$$

there exists a homeomorphism h of E^{k+1} onto itself such that $h(D_\alpha) = D_{\alpha'}$. Since each of D_α and $D_{\alpha'}$ is locally flat except at $\{q_\alpha \cup \bigcup p_{n_i}\}$ and $\{q_{\alpha'} \cup \bigcup p_{n'_i}\}$, respectively, and q_α and $q_{\alpha'}$ are limit points of the nonlocally flat points, it follows that $h(q_\alpha) = q_{\alpha'}$ and for each $i = 1, 2, 3, \cdots, h(p_{n_i}) = p_{n'_j}$ for some j. Since some integer in α occurs more in α than it does in α', there is an integer n_i such that $h(p_{n_i}) = p_{n'_j}$ and $n_i \neq n'_j$. But by Theorem 3, this is impossible and hence the result follows.

REFERENCES

1. W. R. Alford and B. J. Ball, *Some almost polyhedral wild arcs,* Duke Math. J. **30** (1963) 33-38.
2. J. J. Andrews and M. L. Curtis, *Knotted 2-spheres in the 4-sphere,* Ann. of Math. **70** (1959), 565-571.
3. R. H. Crowell and R. H. Fox, *Introduction to Knot Theory,* Ginn and Company, 1963.
4. P. H. Doyle and J. G. Hocking, *Proving that wild cells exist,* (to appear).
5. R. H. Fox and O. G. Harrold, *Topology of 3-Manifolds and Related Topics,* M. K. Fort, Jr., Editor, Prentice Hall 1962.

Received January 30, 1968. Work on this paper was supported by NSF Contract GP 6776.

RICE UNIVERSITY
AND
UNIVERSITY OF UTAH