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A NOTE ON PROPOSITION OBSERVABLES

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We consider some questions which were brought up in a previous paper: (1) If the product of two proposition observables is a proposition observable, do the corresponding propositions split? (2) What is the relationship, if any, between the concepts of compatibility and simultaneity of proposition observables? It is shown that the answer to (1) is yes and as a corollary we find the partial answer to (2) that compatibility implies simultaneity. It is also proved that the sum of two proposition observables is a proposition observable if and only if the corresponding propositions are orthogonal and also that $(x_a \circ x_b)^n$ converges weakly to $x_{a \wedge b}$ as $n \to \infty$.

We assume that all observables are defined on a quite full logic satisfying conditions U and E. (See [1] for definitions and notation.) Recall that an observable x is a *proposition observable* if $\sigma(x) \subset \{0, 1\}$. (These observables are also called *questions*, cf. [3].) If $x(\{1\}) = a$ we denote x by x_a. It is clear that proposition observables behave, in some respects, like orthogonal (self-adjoint) projections on a Hilbert space and for this reason it is natural to consider certain properties of projections and ask whether these properties are retained by proposition observables. For example, it is easy to show that if A and B are orthogonal projections then $A + B$ is an orthogonal projection if and only if $AB = BA = 0$ and AB is an orthogonal projection if and only if $AB = BA$ ([4], Th. 13.4). Also if C is the orthogonal projection on the range of A intersected with the range of B then C is the strong operator limit of the sequence $A, BA, ABA, BABA, \cdots$ ([4] Th. 13.7). We show that these results generalize to proposition observables. Recall that the *product* $x \circ y$ of two bounded observables x, y is $x \circ y = \frac{1}{2}[(x + y)^2 - x^2 - y^2]$ and that x and y are *compatible* if $x \circ (z \circ y) = (x \circ z) \circ y = (x \circ y) \circ z$ for all bounded observables z.

2. The theorems.

LEMMA 1. *The following statements are equivalent.* (i) $a \leftrightarrow b$. (ii) $m(a) = 1$, $m(a \wedge b) = 0$ *implies* $m(b) = 0$ *for any state* m. (iii) $\sigma(x_a + x_b) \subset \{0, 1, 2\}$.

Proof. Since $a = (a - a \wedge b) + (a \wedge b)$ and $b = (b - a \wedge b) + (a \wedge b)$, $a \leftrightarrow b$ if and only if $(a - a \wedge b) \perp (b - a \wedge b)$. But the latter holds if and only if $m(a - a \wedge b) = 1$ implies $m(b - a \wedge b) = 0$. This last condition is equivalent to (ii) and hence (i) and (ii) are equivalent.

Since $\sigma(x_a + x_b) \subset [0, 2]$ applying Lemma 6.2 [1] we have

$$m(a) + m(b) = m(x_a + x_b) = \int_{[0,2]} \lambda m[(x_a + x_b)(d\lambda)]$$
$$= m[x_a + x_b)(\{1\})] + m(a \wedge b) + m[(x_a + x_b)(\{0, 1, 2\}')] .$$

If $\sigma(x_a + x_b) \subset \{0, 1, 2\}$ and $m(a) = 1$, $m(a \wedge b) = 0$ then

$$m(b) = m[(x_a + x_b)(\{1\})] - 1 = 0 ,$$

and hence $a \leftrightarrow b$. Conversely if $a \leftrightarrow b$ it easily follows that

$$m[(x_a + x_b)(\{1\})] = m(a \wedge b') + m(b \wedge a')$$

for every state. It then follows that $(x_a + x_b)(\{0, 1, 2\}') = 0$ and hence $\sigma(x_a + x_b) \subset \{0, 1, 2\}$.

THEOREM 2. *The following statements are equivalent.* (i) $x_a \circ x_b$ *is a proposition observable.* (ii) $x_a \circ x_b = x_{a \wedge b}$. (iii) $a \leftrightarrow b$.

Proof. It follows from Lemma 6.6 [1] that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). To show (i) $\Rightarrow$ (iii) suppose $x_a \circ x_b$ is a proposition observable. Define the continuous function $f(\lambda) = \frac{1}{2}(\lambda^2 - \lambda)$. Since a proposition observable is its own square we have

$$\{0, 1\} \supset \sigma(x_a \circ x_b) = \sigma[\tfrac{1}{2}((x_a + x_b)^2 - (x_a + x_b))] = \sigma[f(x_a + x_b)] .$$

Applying the spectral mapping theorem (Theorem 4.2 [2]) we have $f(\sigma(x_a + x_b)) \subset \{0, 1\}$. We then conclude that $\sigma(x_a + x_b) \subset \{0, 1, 2\}$ and using Lemma 1 we have that $a \leftrightarrow b$.

COROLLARY 3. *If x_a and x_b are compatible, then $x_a \leftrightarrow x_b$.*

Proof. If x_a and x_b are compatible, then

$$(x_a \circ x_b)^2 = (x_a \circ x_b) \circ (x_a \circ x_b) = ((x_a \circ x_b) \circ x_a) \circ x_b = (x_a \circ x_b) \circ x_b = x_a \circ x_b$$

and hence $x_a \circ x_b$ is a proposition observable. Applying Theorem 2, $a \leftrightarrow b$ and hence $x_a \leftrightarrow x_b$.

COROLLARY 4. *The following statements are equivalent.* (i) $x_a + x_b$ *is a proposition observable.* (ii) $a \perp b$. (iii) $\sigma(x_a + x_b) \subset [0, 1]$. (iv) $x_a \circ x_b = 0$.

Proof. From Theorem 2 and Theorem 6.7 [1] we deduce that (iv) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) and it is trivial that (i) $\Rightarrow$ (iii). Now (iii) is equivalent to $0 \leqq m(a) + m(b) \leqq 1$ or $m(a) \leqq 1 - m(b)$ for every state m. But this last statement is equivalent to (ii) and thus (iii) and (ii) are equivalent.

COROLLARY 5. *The following statements are equivalent.* (i) $x_b - x_a$ *is a proposition observable.* (ii) $a \leqq b$. (iii) $\sigma(x_b - x_a) \subset [0, 1]$. (iv) $x_a \circ x_b = x_a$.

Proof. Replace b by b' in Corollary 4.

We say that a sequence of observables x_n *converges weakly* to an observable x if $\lim_{n\to\infty} m(x_n) = m(x)$ for every state m.

THEOREM 6. *The sequence of observables* $(x_a \circ x_b)^n$ *converges weakly to the proposition observable* $x_{a\wedge b}$.

Proof. Let $z = x_a \circ x_b$ and again let $f(\lambda) = \frac{1}{2}(\lambda^2 - \lambda)$. Then using the spectral mapping theorem and the fact that $\sigma(x_a + x_b) \subset [0, 2]$ we have

$$\sigma(z) = \sigma(f(x_a + x_b)) = f(\sigma(x_a + x_b)) \subset f([0, 2]) \subset [-1/8, 1] .$$

Since

$$m(z^n) = \int_{[-1/8,1)} \lambda^n m[z(d\lambda)] + m[z(\{1\})]$$

we have

$$|m(z^n) - m[z(\{1\})]| \leqq \int_{[-1/8,1)} |\lambda|^n m[z(d\lambda)] .$$

Applying the dominated convergence theorem, the integral approaches zero as $n \to \infty$ and hence $\lim_{n\to\infty} m(z^n) = m[z(\{1\})]$. Applying Lemma 6.2 [1] we have

$$\begin{aligned} z(\{1\}) &= f(x_a + x_b)(\{1\}) = (x_a + x_b)[f^{-1}(\{1\})] \\ &= (x_a + x_b)(\{-1, 2\}) = (x_a + x_b)(\{2\}) = a \wedge b \end{aligned}$$

and since $m(x_{a\wedge b}) = m(a \wedge b)$, the proof is complete.

COROLLARY 7. *The sequence of observables* $I - [I - x_a) \circ (I - x_b)]^n$ *converges weakly to the proposition observable* $x_{a\vee b}$.

REMARKS. Strictly speaking Theorem 6 is not exactly a generalization of Von Neumann's Theorem 13.7 [4], however it is probably the most natural form that a corresponding result would take under these more general circumstances. Notice that our definition of compatibility is stronger than that given in [1].

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