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THE MAXIMAL SET OF CONSTANT WIDTH IN A LATTICE

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A new construction for sets of constant width is employed to determine the largest such set which will fit inside a square lattice.

A set W in E^2 is said to have *constant width* λ (denoted $\omega(W) = \lambda$) if the distance between each pair of parallel supporting lines of W is λ. If $x \in \text{bd } W$ we will denote all points *opposite* x (that is, at a distance λ from x) in W by $0(x)$.

In what follows we will be most concerned with *Reuleaux polygons*, which are sets of constant width λ whose boundaries consist of an odd number of arcs of radius λ centered at other boundary points (see [2], p. 128, for a more complete description).

We say a set S *avoids* another set X if int $S \cap X = \varnothing$.

THEOREM 1. *Let L be a square planar unit lattice. Then the unique set of maximal constant width which avoids L is a Reuleaux triangle T having width $\omega(T) > 1.545$. An axis of symmetry of T parallels one of the major axex of L and is midway between two parallel rows of the lattice.*

The proof depends upon a variational method for altering Reuleaux polygons which will be described in § 2. A useful lemma is also proved there. In § 3 the proof of the theorem is given, while various generalizations are discussed in § 4.

The construction described in the next section was also found independently by Mr. Dale Peterson.

2. Variants of sets of constant width.

Let P be a set of constant width λ and p_0 a point near P but exterior to it. Suppose that q and r are the two points on the boundary of P which are at a distance λ from p_0. Let Q be the convex set whose boundary is following: the shorter arc of the circle $C(p_0, \lambda)$ [the circle of radius λ centered at p_0] between q and r, the boundary of P from r to q' (a point opposite q), an arc of $C(q, \lambda)$ between q' and p_0, an arc of $C(r, \lambda)$ between p_0 and r', and the boundary of P from r' to q [see Figure 1]. We call Q the *p_0-variant* of P. It is easy to see that Q is a set of constant width λ. In order for the construction to work p_0 must be close enough to P so that the boundary arc of P between q and

r on the side nearer p_0 contains two opposite points. It is also possible to determine the variant by prescribing the two points q and r. When this is done, we will refer to Q as the (q, r)-variant of P.

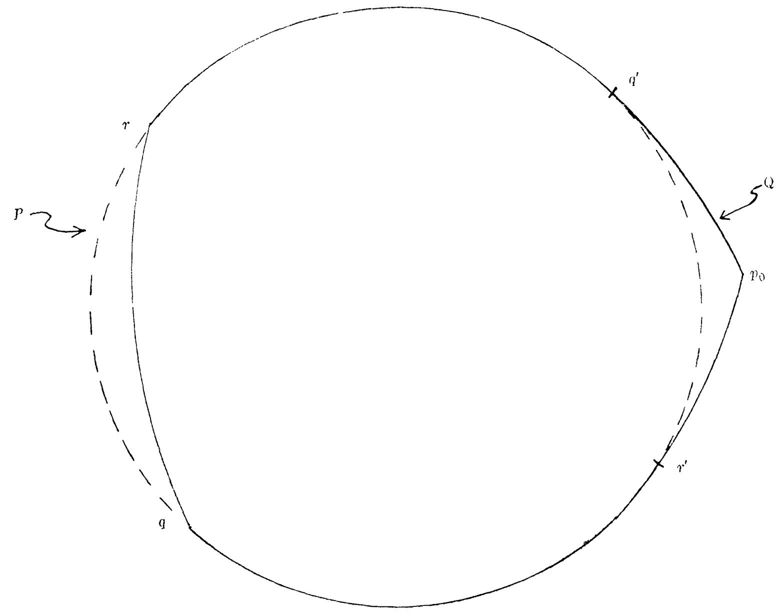

FIGURE 1.

This method gives a way of easily constructing sets of constant width which seems to be new. In particular, applying this method to the unit circle leads to a new class of sets of constant width. A similar construction may be carried out in d-dimensional space, and this process will be explored more fully in another paper [4].

The following lemma is more general than necessary, but may be useful for other problems of this nature.

We will say a family of sets in the plane is *locally finite* if every bounded set meets only a finite number of them.

LEMMA 1. *Let* $\{X_\alpha : \alpha \in A\}$ *be a locally finite family of convex sets in the plane and let* $X = \cup \{X_\alpha : \alpha \in A\}$. *If a set* P *of maximal constant width avoiding* X *exists, then it is a Reuleaux polygon with property* (*): *each of the open* (*curvilinear*) *edges of* P *contains at least one point of* X.

Proof. Suppose K is a set of maximal constant width λ which avoids X. We shall assume that it is not as described and show that there exists another set having a greater constant width which also avoids X. First we will show that for maximality K is a Reuleaux polygon and then that it has property (*).

Choose y_1 in bd K and y_2 in bd K counter-clockwise as far as possible from y_1 but so that the (y_1, y_2)-variant of K avoids X. Call this variant K_1. It is not hard to see that $K_1 = K$ if and only if K is a Reuleaux triangle or else y_1 and y_2 are opposite some common point and the set of points opposite y_2, $0(y_2)$, meets X. In a similar fashion choose y_3 in bd K_1 counterclockwise as far as possible from y_2 so that the (y_2, y_3)-variant of K_1 avoids X. Continue in this fashion.

After a finite number of steps this process will lead to a Reuleaux polygon avoiding X. For the y_i are determined either by one of the X_α or else by the fact that two adjacent y_j are a distance λ apart. Since the X_α are locally finite, each of these cases can occur only a finite number of times as the y_i get further around bd K from y_1. The assertion follows.

We have now constructed a Reuleaux polygon P of the same width as K which also avoids X. Note that if K itself were not a Reuleaux polygon satisfying (*), it is possible to modify the construction of P slightly (by not choosing the y_i to be at a maximal distance in some suitable step) so that P is a Reuleaux polygon, but does not satisfy (*). We now show that such a P does not have maximal width, contrary to our initial assumption.

In fact, we will construct a sequence of Reuleaux polygons $P_0, \cdots, P_m$ such that $P_0 = P$, P_{i+1} is a variant of P_i and P_{i+1} has fewer *closed* edges than P_i which contain a point of X. Since all the P_i will have the same number of edges, the process will produce a Reuleaux polygon P_m disjoint from X. Then a larger homothet of P_m will avoid X contrary to the assumption that K was maximal.

Suppose that P_0 has vertices $v_0, v_1, \cdots, v_{2n}$ and suppose that the open edge (v_{2n}, v_0) contains no point of X, but that v_0 or v_{2n} may belong to X. Let v'_n be a point on the arc between v_n and v_{n+1}, and let P_1 be the (v_{n-1}, v'_n) variant of P_0. The vertices of P_1 are

$$v_0, \cdots, v_{n-1}, v'_n, v_{n+1}, \cdots, v_{2n-1}, v'_{2n} .$$

If v'_n is close enough to v_n, P_1 will avoid X and in particular the half-open edge $[v'_{2n}, v]$ contains no points of X. Now choose v''_n on the arc of P_1 between v_{n-1} and v'_n and P_2 be the (v''_n, v_{n+1}) variant of P_1. The other new vertex of P_2 will be v'_0, near v_0. If v''_n is sufficiently close to v'_n, P_2 will also avoid X and the closed edge $[v'_{2n}, v'_0]$ will contain no point of X.

Note moreover that in the obvious correspondence between P_0 and P_2, every closed edge of P_2 containing a point of X corresponds to a closed edge of P_0 containing a point of X. In addition, we may repeat the above construction on the two open edges of P_2, (v_{n-1}, v''_n) and (v''_n, v_{n+1}) to produce Reuleaux polygons with at least two open edges and more closed edges avoiding X.

Continuing the process through at most $2n$ steps will lead to a Reuleaux polygon of width λ disjoint from X. By our earlier remarks this completes the proof.

3. Proof of theorem.

The following lemma is needed.

LEMMA 2. *Let L be a planar lattice and K a strictly convex set (its boundary contains no line segment) avoiding L. Then the boundary of K contains at most four points of L.*

Proof. Let $Z = K \cap L$. Since K is strictly convex, Z contains only two points in any one direction and these two points have no point of L between them.

Coordinatize the plane (not necessarily with perpendicular axes) so that L corresponds to the integer points of the coordinatization, so that every point of Z lies in the upper half plane, and so that the points $(0, 0)$ and $(1, 0)$ belong to Z. Now suppose $(k, n) \in Z$ for some $n \geqq 3$. Then taking a suitable convex combination of the three points $(0, 0)$, $(1, 0)$ and (k, n) which all lie on $\mathrm{bd}\, K$ shows that $(m, 1) \in \mathrm{int}\, K$, where $m = [k/n] + 1$ ($[x]$ being the greatest integer in x). Then K does not avoid L contrary to hypothesis. Hence every point of Z has y-coordinate 0 or 1. Since no more than two points of Z can be in either of the rows, the assertion is proved.

We can now prove the theorem. By the Blaschke Selection Theorem it is clear that a set of maximal constant width avoiding X exists. Since every set of constant width is strictly convex, and since every lattice is locally finite, the results of Lemmas 1 and 2 imply that the maximal width λ is attained by a Reuleaux triangle T. It only remains to establish the orientation of T.

By Lemma 1, each of the three edges of T contains a lattice point of L and it is clear that they must belong to a unit square of L. So suppose $a \equiv (0, 1)$, $b \equiv (1, 1)$ and $c \equiv (1, 0)$ belong to T. We wish to show $d \equiv (0, 0)$ also belongs to T. If $T \cap L$ consists of exactly three points, it follows from Lemma 1 that there is one vertex between each pair of lattice points. Let these vertices be a', b', and c' where a' is opposite a, etc.

Suppose $x(c')$ [the x-coordinate of c'] $> 1/2$. Rotate T a small distance counter-clockwise to T^* so that T^* still contains a and b on its boundary. If the rotation is small enough, $d \notin T^*$ and the distance between c and c' is increased (this latter statement is proved in [1] § 2 where it is shown that the curve $R(x; l; \lambda)$ defined there is strictly convex). Then it is clear that a larger homothet of T^* will avoid

L contrary to the choice of T. In a similar way we see that the y-coordinate of $a' \leqq 1/2$.

Now if $d \notin T$ either $c'd > \lambda$ or $a'd > \lambda$. If $c'd > c'c = \lambda$ then $x(c') > 1/2$ in contradiction to what was proved in the last paragraph. We arrive at a similar contradiction by assuming $a'd > \lambda$. Hence $d \in T$.

Hence two lattice points are opposite the same vertex of T and thus are equidistant from it. Without loss of generality, suppose c and d are both opposite c'. Then $x(c') = 1/2$ and T is as described in the theorem.

We may compute $\omega = \omega(T)$ as follows. If T is in the orientation just described, and we let

$$\alpha = y(c'),\ \beta = y(a') = y(b'),\ x(a') = \frac{1}{2} + \gamma,\ x(b') = \frac{1}{2} - \gamma\,,$$

we see:

$$(1) \qquad \gamma = \omega/2$$

$$(2) \qquad \omega^2 = 1/4 + \alpha^2$$

$$(3) \qquad \beta = \alpha - \sqrt{3}\,\omega/2$$

$$(4) \qquad \left(\frac{1}{2} + \frac{\omega}{2}\right)^2 + (1 - \beta)^2 = \omega^2\,.$$

Untangling (2), (3) and (4), we obtain:

$$(5) \quad 2\omega^4 + \omega^3(2\sqrt{3} - 1) + \omega^2(-2 - \sqrt{3}) + \omega(-1 - 3\sqrt{3}) - 2 = 0\,.$$

Solving (5) leads to the stated value for $\omega(T)$.

It is clear that the techniques used in proving this theorem can be extended to other similar problems. In particular, if L is any planar lattice the set of maximal constant width is again a Reuleaux triangle. In general, Lemma 1 ensures that the maximal figure is a Reuleaux polygon and makes it fairly easy to determine the number of sides, but it is more difficult to determine the exact orientation.

4. Remarks.

Let $\mathscr{M}$ be any 2-dimensional Minkowski space with unit ball S. We may define W to be a set of *constant width* λ *relative to* S if $\omega(W, u) = \lambda\omega(S, u)$ for any direction u. In analogy to the Euclidean case, we say R is a *relative Reuleaux polygon* if R is of constant relative width and is the intersection of a finite number of (properly chosen) translates of λS.

With only slight changes, the proof of Lemma 1 may be seen to be valid in $\mathscr{M}$ (where, of course, an "arc of radius λ" is an arc of λS, etc.). However, sets of constant width relative to S only satisfy the hypotheses of Lemma 2 if $\mathscr{M}$ is *rotund*—that is, if S is strictly convex.

So we have, in fact, proved the following:

LEMMA 3. *Let* $\{X_\alpha: \alpha \in A\}$ *be a locally finite family of convex sets in any 2-dimensional Minkowski space and let*

$$X = \cup \{X_\alpha: \alpha \in A\}.$$

Every set of maximal constant relative width avoiding X *is a relative Reuleaux polygon with property* (*).

THEOREM 2. *Let* L *be a planar lattice in a rotund, 2-dimensional Minkowski space. Every set of maximal constant width avoiding* L *is a relative Reuleaux triangle with property* (*).

The author wishes to thank G. D. Chakerian for calling this problem to his attention and for interesting discussions. It seems to have originally appeared as a problem in the American Math. Monthly [3]. The author also wishes to thank the referee for his suggestion strengthening the statement of Lemma 1.

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