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SEMIVARIETIES AND SUBFUNCTORS OF THE IDENTITY FUNCTOR

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We study certain subcategories called semivarieties and obtain Kaplanasky's theorem on the decomposition of Abelian groups into a divisible group and a reduced group under the frame-work of category theory; We also investigate the connection of these epicoreflective subcategories with varieties.

Semivarieties are subcategories of a category with cartain axioms; such subcategories play an important role in Abelian categories and a description of these by means of coreflection, appears in the general context in the work of Mitchell [8, § 5, § 6, III]. Broader classes than these have also been studied by Amitsur [1], Carreau [2], under the title of $HI - RI$ properties of radicals and their classes. Our aim in this note is to give a categorical proof of Kaplanasky's Theorem 3 [6, § 5] and while so doing we generalize the concepts of varieties and variety functors of Fröhlich [4] under abstract frame work utilizing Maranda's [9] concept of a radical.

$\mathscr{C}$ is a category equipped with the following axioms:

I. $\mathscr{C}$ has a null object.

II. Every morphism α in $\mathscr{C}$, admits a factorization $\alpha = \nu\mu$, where ν is a normal epimorphism and μ is a monomorphism; we are writing the composition in the precise way $\begin{tikzcd}\cdot \arrow[r, "\alpha"] \arrow[rr, bend right, "\alpha\beta"'] & \cdot \arrow[r, "\beta"] & \cdot\end{tikzcd}$ where the dots are the unnamed objects.

III. Every family of objects has a direct and a free product.

IV. The subobjects and factor objects of any objects form a set.

It is immediate that $\mathscr{C}$ admits null morphisms. The image of a morphism in $\mathscr{C}$, defined in axiom II and some time written as (ν, L, μ) in the factorization $\begin{tikzcd}\cdot \arrow[r, "\nu"] \arrow[rr, bend right, "\alpha"'] & L \arrow[r, "\mu"] & \cdot\end{tikzcd}$, is uniquely determined to within equivalence. Every family of subobjects of an object A in $\mathscr{C}$ has a union and as such every morphism has a kernel. Dual consideration holds for factor objects and cokernel. A map admitting null subobject as the kernel is a monomorphism and a sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact if and only if α is a monomorphism, β is a normal epimorphism and the subobject (A, α) serves as the kernel of β. For details on the notation used and results mentioned in this section the reader is referred to [7], [10]. In view of axiom II, one can check that every retraction is a normal epimorphism as such a monomorphism which is a retraction is invertible, [7] and further the composition of two normal epimorphisms is again normal. Also if μ is a normal monomorphism admitting cokernel ν, then μ is the kernel of ν and dual result holds for the normal epimorphism ν, admitting kernel μ.

2. **Subfunctors of I.** We shall consider covariant functors from $\mathscr{C} \to \mathscr{C}$, and such subfunctors of the identity functor I.

DEFINITION. A subfunctor (F, μ) of I will be called a *normal subfunctor* if for each $A \in \mathscr{C}$, the natural map $\mu_A : F(A) \to A$ is a normal monomorphism; and (F, μ) is called *injective* if μ_A is a coretraction.

A subfunctor (F, μ) of $\mathscr{C}$ is said to be a *radical subfunctor*, (dually *coradical subfunctor*) in the sense of Maranda [9] if for any A, $(F(A), \mu_A)$ has cokernel (ε, C) then $1_{F(C)} \sim \omega[\omega = \omega_{AB}$ always stands for the null morphism of $H(A, B)]$, so that $F(\varepsilon) : F(A) \to F(C)$ is the null normal epimorphism $\omega_{F(A),0}$ (dually for the inclusion $\mu_A : F(A) \to A$; $F(\mu_A) : FF(A) \to F(A)$ is an equivalence).

3. **Semivarieties and cosemivarieties.** A *semivariety* $\mathscr{B}$ is a full subcategory of $\mathscr{C}$, satisfying the following axioms:

I. $f : A \to B$ is a monomorphism and $B \in B \Rightarrow A \in B$.

II. If $(A_i)_{i \in I}$ is an indexed set of objects of $\mathscr{B}$, then their direct product $\prod A_i \in \mathscr{B}$.

EXAMPLES. (i) All varieties [4].

(ii) Torsion free groups (respectively Abelian groups) form a semivariety of all groups (respectively of Abelian groups)

(iii) Reduced groups form a semivariety of all Abelian groups.

(iv) Nilpotent free rings (i.e., a ring in which $a^2 = 0 \Rightarrow a = 0$) in the category of rings.

PROPOSITION 3.1. (1) *If (V, μ) is a subfunctor of I, then the objects A for which $(V(A), \mu_A) \sim (0, \omega)$ form a semivariety $\mathscr{B}_V$.*

(2) *If $\mathscr{B}$ is a semivariety then every object A has a minimal normal subobject $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ whose cokernel lies in $\mathscr{B}$. These $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ determine a normal radical subfunctor $(V_{\mathscr{B}}, \mu_{\mathscr{B}})$ of I.*

(3) *If $V \sim V_{\mathscr{B}}$ then $\mathscr{B} = \mathscr{B}_V$. Conversely $\mathscr{B} = \mathscr{B}_V$ implies V is a subfunctor of $V_{\mathscr{B}}$; but if (V, μ) is normal and a radical subfunctor then $\mathscr{B} = \mathscr{B}_V \Rightarrow V \sim V_{\mathscr{B}}$.*

Proof. (1) Let $\mathscr{B}_V = \{A \mid (V(A), \mu_A) \sim (0, \omega)\}$ and all their morphisms. Since (V, μ) is a subfunctor of $I, 0 \in \mathscr{B}_V$. Thus $\mathscr{B}_V$ is a full subcategory.

Next if $A \xrightarrow{f} B$ is a monomorphism, then $V(f): V(A) \to V(B)$ is a monomorphism. Thus $B \in \mathscr{B}_V \Rightarrow V(B) \sim 0$ and as such $1_{V(A)} = \omega$ which implies $A \in \mathscr{B}_V$.

Next if $(A_i)_{i \in I}$ is a family of objects of $\mathscr{B}_V$, then we have a commutative diagram

$$\begin{tikzcd} \prod A_i \arrow[r, "\pi_i"] & A_i \\ V(\prod A_i) \arrow[u, "\mu_{\Pi A_i}"] \arrow[r, "V(\mu_i)"'] & V(A_i) \arrow[u, "\mu_{A_i}"'] \end{tikzcd}$$

where the top horizontal is the natural projection for each i which is a normal epimorphism. Now $A_i \in \mathscr{B}_V \Rightarrow (V(A_i), \mu_{A_i}) \sim (0, \omega)$ that is $\mu_{\Pi A_i}\pi_i = \omega$, i.e., $\mu_{\Pi A_i} = \omega$ so $1_V(_{\Pi A_i}) = \omega$ giving $\prod A_i \in \mathscr{B}_V$.

(2) Suppose $\mathscr{B}$ is a semivariety. We consider all normal subobjects $(V_i(A), \mu_i(A))$ of A whose cokernel (ε_i, C_i) lies in $\mathscr{B}$. Then $\varepsilon_i : A \to C_i$ determine a unique map $\varepsilon : A \to \prod C_i$ such that $\varepsilon\pi_i = \varepsilon_i$ (where $\prod C_i \xrightarrow{\pi i} C_i$, stands for the direct product). Now $\prod C_i \in \mathscr{B}$. We denote by $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ the kernel of ε; then this determines a normal subfunctor $(V_{\mathscr{B}}, \mu_{\mathscr{B}})$ of I. We note $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ is also the kernel of the epimorphism ν, in the image (ν, C, μ) of ε. Then $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ is minimal normal subobject (in the sense of partial ordering of subobjects, see [7]) whose cokernel lies in $\mathscr{B}$ is obvious from the construction. Evidently $V_{\mathscr{B}}$ has the radical property.

(3) Suppose $\mathscr{B}$ is a semivariety and $V \sim V_{\mathscr{B}}$, then $\mathscr{B}_V$ consists of all objects A for which $(V(A), \mu_A) \sim (0, \omega)$. Now $\mathscr{B}$ consists of all objects A whose cokernel is $(1_A, A)$; thus $\mathscr{B} = \mathscr{B}_V$.

Now suppose $\mathscr{B} = \mathscr{B}_V$. If for any object A, $V(A) \xrightarrow{\mu_A} A$ and $V_{\mathscr{B}}(A) \xrightarrow{\mu_{\mathscr{B}}} A$ admit the cokernels (ε^*, D^*) and (ε, C) respectively then $C \in \mathscr{B} = \mathscr{B}_V$, i.e., $(V(C), \mu_C) \sim (0, \omega)$ so $\mu_A\varepsilon = \omega$, therefore there exists a monomorphism $\lambda_A : V(A) \to V_{\mathscr{B}}(A)$ such that $\lambda_A\mu_{\mathscr{B}} = \mu_A$. That this λ is a functor monomorphism is easy to check. If further V is normal and a radical subfunctor, then for the cokernel (ε^*, D^*), $1_{V(D^*)} \sim \omega$, i.e., $D^* \in \mathscr{B}_V = \mathscr{B}$. Thus $(\varepsilon^*, D^*) \leqq (\varepsilon, C)$, i.e., $\varepsilon^* \sim \varepsilon$, giving $(V, \mu) \sim (V_{\mathscr{B}}, \mu_{\mathscr{B}})$.

Dual to the concept of a semivariety is that of a cosemivariety.

DEFINITION. A full subcategory $\mathscr{F}$ of C is called a *cosemivariety* if

I. $f : A \to Q$ a normal[1] epimorphism $A \in \mathscr{F}, \Rightarrow Q \in \mathscr{F}$.

[1] This is required since axiom II, is not self dual.

II. $(A_i)_{i \in I}$ is a family of objects of $\mathscr{F}$, then their free product $\prod^* A_i \in \mathscr{F}$.

EXAMPLES. (i) All covarieties. In particular in the category of Abelian groups, torsion groups from a covariety.

(ii) Divisible groups form a cosemivariety of all Abelian groups. Mirror image of Proposition 3.1. will be

PROPOSITION 3.1*. (1) *If F is subfunctor of the identity functor I of $\mathscr{C}$, then the collection of all objects A for whjch $(F(A), \mu_A) \sim (A, 1_A)$ from a cosemivariety $\mathscr{F}^F$.*

(2) *If $\mathscr{F}$ is a cosemivariety, then every object A has a maximal subobject $(F^{\mathscr{F}}(A), \mu^{\mathscr{F}}(A) \in \mathscr{F}$. This $(F^{\mathscr{F}}, \mu^{\mathscr{F}})$ is a coradical subfunctor of I.*

(3) *$F \sim F^{\mathscr{F}} \Rightarrow \mathscr{F} = \mathscr{F}^F$. Conversely $\mathscr{F} = \mathscr{F}^F$ implies $F^{\mathscr{F}}$ is a subfunctor of F; further if F is a coradical subfunctor then $\mathscr{F} = \mathscr{F}^F \Rightarrow F \sim F^{\mathscr{F}}$.*

We exhibit the proof for (1), leaving the second and third part for the reader to dualize their counter parts in Proposition 3.1 in the same way.

Let $\mathscr{F}^F = \{A \mid (F(A), \mu_A) \sim (A, 1_A)\}$ and all their morphisms; the null object is obviously $\in \mathscr{F}^F$.

Next if $A \xrightarrow{\alpha} B$ is a normal epimorphism and $A \in \mathscr{F}^F$, then we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \\ F(A) \arrow[u, "\mu_A"] \arrow[r, "F(\alpha)"'] & F(B) \arrow[u, "\mu_B"'] \end{tikzcd}$$

in which μ_A is an equivalence. Now $\mu_A \alpha$ is a normal epimorphism as observed in § 1. Let δ be its kernel. Then $\delta F(\alpha) = \omega$, i.e., $F(\alpha) = \mu_A \alpha\, \acute{\mu}(\mu_A \alpha$ is normal and therefore cokernal of $\delta)$. Now $\mu_A \alpha \acute{\mu} \mu_B = F(\alpha)\mu_B = \mu_A \alpha$. Thus $\acute{\mu}\mu_B = 1$, i.e., μ_B is a retraction and hence invertible, i.e., $B \in \mathscr{F}^F$.

Next suppose $(A_i)_{i \in I}$ be a family of objects of $\mathscr{C}$ belonging to $\mathscr{F}$, then we have for each i, a commutative diagram

$$\begin{tikzcd} A_i \arrow[r, "\tau i"] & \prod^* A_i \\ F(A_i) \arrow[u, "\mu_{A_i}"] \arrow[r, "F(\tau i)"'] & F(\prod^* A_i) \arrow[u, "\mu_{\prod^* A_i}"'] \end{tikzcd}$$

where τ_i are the natural monomorphisms associated with the free product. Now μ_{A_i} is invertible, let $\mu_{A_i}^{-1}$ be its inverse. Then $\mu_{A_i}^{-1}F(\tau_i) : A_i \rightarrow F(\prod^* A_i)$ determines a unique $\lambda : \prod^* A_i \rightarrow F(\prod^* A_i)$ such that $\tau_i \lambda = \mu_{A_i}^{-1}F(\tau_i)$.

Now $\tau_i(\lambda \mu_{\Pi^* A_i}) = \mu_{A_i}^{-1}F(\tau_i)\mu_{\Pi^* A_i} = \mu_{A_i}^{-1}\mu_{A_i}\tau_i = \tau_i$.

Thus $\lambda \mu_{\Pi^* A_i} = 1$ showing that $\mu_{\Pi^* A_i}$ is a retraction and hence invertibe, i.e., $\prod^* A_i \in \mathscr{F}^F$.

Thus if (F, μ) is any subfunctor of I, determining a semivariety $\mathscr{B}_F$ and a cosemivariety $\mathscr{F}^F$, then we have a functor transformation

$$F^{\mathscr{F}^F} \xrightarrow{\ \delta\ } F \xrightarrow{\ \lambda\ } F_{\mathscr{B}_F}, \quad \text{and}$$

for any object A, we have a commutative diagram

$$\begin{tikzcd}
 & A & \\
F^{\mathscr{F}^F}(A) \arrow[ur, "\mu^{\mathscr{F}^F}"] \arrow[dr, "\delta_A"'] & & F_{\mathscr{B}_F}(A) \arrow[ul, "\mu_{\mathscr{B}_F}"'] \\
 & F(A) \arrow[uu, "\mu"] \arrow[ur, "\lambda_A"'] &
\end{tikzcd}$$

where $\mu_{\mathscr{B}_F}$ is a normal monomorphism.

It is clear from our construction that if $\mathscr{F}$ is a cosemivariety determining a normal radical subfunctor $(F^{\mathscr{F}}, \mu^{\mathscr{F}})$ of I, then for any $A \in \mathscr{C}$ the cokernel of $\mu_A : F^{\mathscr{F}}(A) \rightarrow A \in \mathscr{B}_{F^{\mathscr{F}}}$, and by Proposition 3.1 $F^{\mathscr{F}} \sim F_{\mathscr{B}_{F^{\mathscr{F}}}}$, and we collect these informations in the following.

PROPOSITION 3.2. *If $\mathscr{F}$ is a cosemivariety, such that the functor $(F^{\mathscr{F}}, \mu^{\mathscr{F}})$ determined by it is a normal radical subfunctor, then for any object A, we have the exact sequence*

$$0 \longrightarrow F^{\mathscr{F}}(A) \xrightarrow{\mu_A} A \xrightarrow{\varepsilon} C \longrightarrow 0$$

where $F^{\mathscr{F}}(A) \in \mathscr{F}$ and $C \in \mathscr{B}_{F^{\mathscr{F}}}$.

We are content to leave the reader to mirror the images of the above observations.

If $\mathscr{C}$ is the category of Abelian groups, and $\mathscr{F}$ is the cosemivariety of torsion groups (or divisible groups) then if to each group A, we assign the torsion subgroup $F^{\mathscr{F}}(A)$ (or the miximal divisible subgroup $F^{\mathscr{F}}(A)$), then $A/F^{\mathscr{F}}(A)$ lies in the semivariety of torsion free groups (or reduced groups).

PROPOSITION 3.3. *If $\mathscr{A}$ is an Abelian category in which a cosemivariety $\mathscr{F}$ determines a radical which is injective, then every object A is the direct sum of an object in $\mathscr{F}$ and $\mathscr{B}_{F^{\mathscr{F}}}$, and as* *such the identity functor splits having components in the cosemivariety and the semivariety.*

Proof. We have an exact sequence $0 \longrightarrow F^{\mathscr{F}}(A) \xrightarrow{\mu} A \longrightarrow A/_{F^{\mathscr{F}}(A)} \longrightarrow 0$ in which the natural injection is a coretraction. Thus there exists a $\delta : A \longrightarrow F^{\mathscr{F}}(A)$ such that $\mu\delta = 1$. So the theorem is obvious from Theorem 2.68 [5]. Thus for the category of abelian groups, if $F^{\mathscr{F}}$ is the functor determined by the cosemivariety of divisible groups, then we have Theorem 3. [6, § 5].

It is easy to see that the variety functors (covariety functors or torsion radicals) [[4], [9]] are indeed radical functors (coradical functors) and a coradical subfunctor (F, μ) determines a cotriple $[(F\mu)^{-1}, \mu, F]$ in the sense of Eilenberg and Moore [3] where $F\mu$ is the equivalence $F^2 \longrightarrow F$ which appears in the definition of coradical subfunctor given by $F\mu_A = F(\mu_A)$ for any object A of $\mathscr{C}$, and this is cogenerated by the adjoint F of the inclusion of the associated cosemivariety $\mathscr{F} = \mathscr{F}^F$ into $\mathscr{C}$, where F can be considered as well a functor from $\mathscr{C} \longrightarrow \mathscr{C}$ dropping the inclusion completely.

We notice if $V_{\mathscr{B}}$ the normal radical functor associated with the semivariety $\mathscr{B}$ is a variety functor, then $\mathscr{B}$ is a variety; converse association holds, i.e, the normal radical functor associated with a variety will be a variety functor (11) in categories with the additional axiom.

(v) *If α is a monomorphism and β is a normal epimorphisms such that $\alpha\beta$ admits image $\nu\mu$, then α normal $\Rightarrow \mu$ is normal.*

Added in proof. While this work was in press, the author was given to understand by P. Lecouturier that certain generalization of Fröhlich's work (a weaker version of Proposition 3.1) has also been obtained by him in more *restricted class of categories* (in which every epimorphism is normal, etc). However he does not obtain the characterization of semivarieties by normal radical functors.

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