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A MODULAR TOPOLOGICAL LATTICE

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The purpose of this paper is to present a construction of a compact connected topological lattice which is modular and not distributive. As a special case there will result the example which is a two dimensional subset of R^3, not embeddable in R^2.

The existence of such an example is related to structure questions in topological lattices considered by Dyer and Shields [3], Anderson [1], and others.

The first step is to present a general method for constructing a class of modular lattices. Let D denote a distributive lattice which is a chain, S a nonempty set, and L the S-fold product lattice of D. That is $L = \{f \mid f: S \to D\}$ and $f \leqq g$ if and only if $f(s) \leqq g(s)$ for every $s \in S$. It is known that $(L, \leqq)$ is a distributive lattice with its operations $\vee$ and $\wedge$ characterized by $[f \vee g](s) = f(s) \vee g(s)$ and

$$[f \wedge g](s) = f(s) \wedge g(s)$$

for every $s \in S$. Define

$$M = \{f \in L \mid \text{there exists } r \in S \text{ such that } s, t \in S - \{r\} \text{ implies}$$
$$f(s) \leqq f(r) \quad \text{and} \quad f(s) = f(t)\} .$$

For intuition about M and the arguments that follow, note that M simply consists of all of the constant functions of L and the functions of L which are essentially constant in the sense that they assume but two values — the larger value at exactly one point.

If the order of L, $\leqq$, is restricted to M, it will be established through a sequence of lemmas that $(M, \leqq)$ is a modular lattice. Recall a lattice $(M, \vee, \wedge)$ is modular if and only if for every $a, b, c \in M$, $b \leqq a$ implies that $a \wedge (b \vee c) = b \vee (a \wedge c)$.

LEMMA 1. *If $f \in M$ and f is not constant, there exists a unique $r \in S$ such that $s, t \in S - \{r\}$ implies $f(s) < f(r)$ and $f(s) = f(t)$.*

The proof of the lemma is immediate from the definition of M, and consequently for $f \in M$ and not constant, define *index* f to be the unique element described in Lemma 1.

LEMMA 2. *$(M, \leqq)$ is a sub $\wedge$-semilattice of $(L, \leqq)$.*

It suffices to show that if $f, g \in M$, then $f \wedge g \in M$. If f and g are constant then $f \wedge g$ is constant and therefore in M. If f is constant and g is not, let $b =$ index g. Then $s, t \in S - \{b\}$ implies

$$[f \wedge g](s) = f(s) \wedge g(s) \leqq f(s) \wedge g(b) = [f \wedge g](b)$$

and likewise $[f \wedge g](s) = [f \wedge g](t)$ and thus $f \wedge g \in M$. If f and g g are both not constant, let $a =$ index f and $b =$ index g. If $a = b$, then $s, t \in S - \{a\}$ implies $[f \wedge g](s) \leqq [f \wedge g](a)$ and $[f \wedge g](s) = [f \wedge g](t)$. If $a \neq b$, $[f \wedge g](a) = f(a) \wedge g(a)$, $[f \wedge g](b) = f(b) \wedge g(b)$, and for

$$x \in S - \{a, b\}\ [f \wedge g](x) = f(b) \wedge g(a) = [f \wedge g](a) \wedge [f \wedge g](b) .$$

Then since D is a chain, $[f \wedge g](x) = [f \wedge g](a)$ or $[f \wedge g](x) = [f \wedge g](b)$ depending upon which is minimal and therefore $f \wedge g \in M$.

LEMMA 3. *If a, b, c are distinct elements of S and $f \in M$, then $f(a) \wedge f(b) = f(b) \wedge f(c) = f(c) \wedge f(a)$.*

The facts of the lemma are an immediate consequence of the definition and is stated as a lemma for convenient reference.

DEFINITION. For $f, g \in M$ define $f \bigtriangledown g: S \rightarrow D$ by the following

(i) if f is constant or g is constant, or if f and g are both not constant and index $f =$ index g, then $f \bigtriangledown g = f \vee g$,

(ii) if f and g are both not constant and index $f \neq$ index g, let $a =$ index f and $b =$ index g, then

$$[f \bigtriangledown g](x) = f(x) \vee g(x) \quad \text{for} \quad x \in \{a, b\}$$
$$[f \bigtriangledown g](x) = [f(a) \vee g(a)] \wedge [f(b) \vee g(b)] \quad \text{for} \quad x \in S - \{a, b\} .$$

LEMMA 4. *If $f, g \in M$, then*

(1) $f \bigtriangledown g \in M$ and $f \vee g \leqq f \bigtriangledown g$, and

(2) $h \in M$, $f \leqq h$, $g \leqq h$ implies $f \bigtriangledown g \leqq h$.

In case (i) of the definition of $f \bigtriangledown g$, easily $f \bigtriangledown g \in M$ and the other results are immediate from $f \bigtriangledown g = f \vee g$. In case (ii) let $a =$ index f and $b =$ index g, then since D is a chain $[f \bigtriangledown g](x) = [f \bigtriangledown g](a)$ or $[f \bigtriangledown g](x) = [f \bigtriangledown g](b)$ for $x \in S - \{a, b\}$. So in this case also $f \bigtriangledown g \in M$ and $f \vee g \bigtriangledown g$. Also relative to this case, if $h \in M$, $f \leqq h$, and $g \leqq h$, then $f(x) \vee g(x) = [f \bigtriangledown g](x)$ for $x = a$ or $x = b$. But from Lemma 3, $x \in S - \{a, b\}$ implies $h(a) \wedge h(b) \leqq h(x)$ and thus for

$$x \in S - \{a, b\}\ [f \bigtriangledown g](x) = [f \bigtriangledown g](a) \wedge [f \bigtriangledown](b) \leqq h(a) \wedge h(b) \leqq h(x) .$$

Therefore $f \bigtriangledown g \leqq h$.

LEMMA 5. *If $f, g, h \in M$, $a, b \in S$, $a \neq b$, $[f \vee g](x) = h(x)$ for $x \in \{a, b\}$*

and $[f \vee g](x) \leqq h(a) \wedge h(b) = h(x)$ *for* $x \in S - \{a, b\}$, *then* $h = f \,\bar{\vee}\, g$.

From the hypothesis $f \vee g \leqq h$ and therefore from Lemma 4 $f \,\bar{\vee}\, g \leqq h$. But $h(a) = [f \vee g](a) \leqq [f \,\bar{\vee}\, g](a)$ and $h(b) \leqq [f \,\bar{\vee}\, g](b)$. Then from Lemma 3, for $x \neq a$ and $x \neq b$

$$h(x) = h(a) \wedge h(b) \leqq [f \,\bar{\vee}\, g](a) \wedge [f \,\bar{\vee}\, g](b) \leqq [f \,\bar{\vee}\, g](x)$$

and $h \leqq f \,\bar{\vee}\, g$.

THEOREM 1. *$(M, \leqq)$ is a modular lattice with operations $\bar{\vee}$ and $\wedge$.*

Lemmas 2 and 4 establish that $(M, \leqq)$ is a lattice with operations $\bar{\vee}$ and $\wedge$, it remains to establish that it is modular. Let $f, g, h \in M$ and $f \leqq g$. It suffices to establish $g \wedge (f \,\bar{\vee}\, h) \leqq f \,\bar{\vee}\, (g \wedge h)$ since in any lattice $f \,\bar{\vee}\, (g \wedge h) \leqq g \wedge (f \,\bar{\vee})$. The argument will be a case argument.

If $f \,\bar{\vee}\, h = f \vee h$, then

$$g \wedge (f \,\bar{\vee}\, h) = g \wedge (f \vee h) = f \vee (g \wedge h) \leqq f \,\bar{\vee}\, (g \wedge h)$$

since L is itself modular and $g \wedge h \in M$ allows Lemma 4 to apply.

If $h \leqq g$, then $f \,\bar{\vee}\, h \leqq g$ and $g \wedge (f \,\bar{\vee}\, h) = f \,\bar{\vee}\, h = f \,\bar{\vee}\, (g \wedge h)$. If $f \leqq h$, then $f \leqq g \wedge h$ and $g \wedge (f \,\bar{\vee}\, h) = g \wedge h = f \,\bar{\vee}\, (g \wedge h)$.

If f and h are not constant, $a =$ index f, $b =$ index g, $a \neq b$, $h \neq g$, and $f \not\leqq h$. Then $f(b) < f(a)$ and $h(a) < h(b)$. Further, $f(a) \leqq h(a)$ implies $f \leqq h$ and therefore $h(a) < f(a)$. Also $h(b) \leqq g(b)$ and $h(a) < f(a) \leqq g(a)$ implies $h \leqq g$ and therefore $h \not\leqq g$ implies $g(b) < h(b)$. Therefore in this case $h(a) < f(a) \leqq g(a)$ and $f(b) \leqq g(b) < h(b)$. Hence

$$\begin{aligned}[g \wedge (f \,\bar{\vee}\, h)](a) &= g(a) \wedge [f(a) \vee h(a)] = f(a) \\ &= f(a) \vee [g(a) \wedge h(a)] = [f \vee (g \wedge h)](a) .\end{aligned}$$

Likewise

$$[g \wedge (f \,\bar{\vee}\, h)](b) = g(b) = [f \vee (g \wedge h)](b) .$$

If $x \in S - \{a, b\}$, then

$$\begin{aligned}[g \wedge (f \,\bar{\vee}\, h)](x) &= g(x) \wedge [f(a) \vee h(a)] \wedge [f(b) \vee h(b)] \\ &= g(x) \wedge f(a) \wedge h(b) = g(x) \wedge g(a) \wedge f(a) \wedge h(b) \\ &= g(b) \wedge g(a) \wedge f(a) \wedge h(b) = f(a) \wedge g(b) \\ &= [g \wedge (f \,\bar{\vee}\, h)](a) \wedge [g \wedge (f \,\bar{\vee}\, h)](b) .\end{aligned}$$

But $[f \vee (g \wedge h)](x) \leqq [g \wedge (f \,\bar{\vee}\, h)](x)$ and $g \wedge (f \,\bar{\vee}\, h) \in M$, therefore by Lemma 5 $g \wedge (f \,\bar{\vee}\, h) = f \,\bar{\vee}\, (g \wedge h)$.

COROLLARY. *If* card $S<3$, *M is a distributive lattice. If* $3\leqq$card *S and* $2\leqq$ card *D, then M is a modular nondistributive lattice.*

If card $S<3$, then $M=L$ and M is distributive. If $3\leqq$ card S and $2\leqq$ card D, let s_1, s_2, s_3 be three distinct elements of S and $c<d$ be two elements of D. Define f_1, f_2, f_3 by $f_i(x)=d$ if $x=s_i$ and $f(x)=c$ for $x\in S-\{s_i\}$. Also define g and k by $g(s)=d$ for every $s\in S$ and $k(s)=c$ for every $s\in S$. Then $f_1\wedge f_2=f_2\wedge f_3\wedge f_1=k$ and $f_1\bigtriangledown f_2=f_2\bigtriangledown f_3=f_3\bigtriangledown f_1=g$ and $\{f_1, f_2, f_3, g, k\}$ is a modular five sublattice of M. Therefore M is not distributive [2].

At this stage the algebraic nature of M has been established, in the section that follows the topological nature of M will be studied. It will be assumed in the following that D is topological chain, that is D is a Hausdorff topological space with the operations $\vee$ and $\wedge$ continuous [3]. If L is considered with the product topology, it is as usual a topological lattice and M may be considered as a topological space in the relative topology that it inherits from L. In this context, the following theorem results.

THEOREM 2. *If D is a topological chain, then*
(1) *M is a closed subset of L,*
(2) *M is compact if D is compact, and*
(3) *M is connected if D is connected.*

Since with card $S\leqq 2$, $M=L$, it suffices to consider $3\leqq$ card S and to establish (1) and (3).

(1) $L-M$ is open for if $f\notin M$, then f is not constant and there exist distinct $a, b, c\in S$ such that $f(b)<f(a)$ and $f(b)<f(c)$. Then since D is a chain $f(b)<f(a)\wedge f(c)$. If there exists $z\in D$ such that $f(b)<z<f(a)\wedge f(c)$, define $W=\{g\in L\mid z<g(a),\ z<g(c), \text{ and } g(b)<z\}$ and define $W=\{g\in L\mid f(b)<g(a),\ f(b)<g(c), \text{ and } g(b)<f(a)\wedge f(c)\}$ if no such z exists. In either case, $f\in W$, W is open, and $W\cap M=\varnothing$.

(3) If D is connected, consider the map $T\colon D\to M$ where for each $d\in D$ $T(d)=k_d$ and k_d is the constant function generated by d. Clearly T is continuous and K the set of all constant functions is a connected subset of M. If $f\in M-K$, let $a=$ index f, $m=\max f$, and $r=\min f$ define the map H from $[r, m]=\{x\in D\mid r\leqq x\leqq m\}$ into M by $H(x)=f_x$ where $f_x(a)=x$ and $f_x(s)=r$ for $s\in S-\{a\}$. Again H is continuous and since $[r, m]$ is connected then the range of H is a connected subset of M containing f and intersecting K. Therefore M is connected.

Note. It is clear that $\wedge$ will be continuous as an operation on

M since it is continuous on L. Thus when D is a topological chain M is a closed topological sub- $\wedge$ -semilattice of L. In order to study *the operation* $\bigtriangledown$ *relative to continuity, it is necessary to restrict* S to being finite, in view of the following lemma.

LEMMA 6. *If D is a topological chain and* $2 \leqq \text{card}\, D$ *and S is infinite, then* $\bigtriangledown$ *is not continuous.*

Let $c < d$ in D and define $k: S \to D$ by $k(s) = c$ for every $s \in S$. Then $k \bigtriangledown k = k$. Let $r \in S$ and define $W_r = \{f \in M \mid f(r) < d\}$; then W_r is an open subset of M containing k. Let U be any open set of M containing k, then there exist $s_1, s_2, \cdots, s_n$ distinct elements of S and $U_1, U_2, \cdots, U_n$ open sets of D such that if $W = \{f \in M \mid f(s_i) \in U_i$ for $i = 1, 2, \cdots, n\}$, $k \in W \subset D$. Now $k \in W$ implies

$$c \in \cap \{U_i \mid i = 1, 2, \cdots, n\} .$$

Since S is infinite there exist $a, b \in S - \{s_1, s_2, \cdots, s_n\}$ such that $a \neq b$. Define h and g by $h(a) = d$ and $h(x) = c$ if $x \in S - \{a\}$, and $g(b) = d$ and $g(x) = c$ if $x \in S - \{b\}$. Therefore $h, g \in W$ and $h \bigtriangledown g \notin W_r$ since $h \bigtriangledown g$ is the constant function defined by d. Therefore $U \bigtriangledown U \not\subset W_r$ and $\bigtriangledown$ is not continuous.

DEFINITION. For S finite and $2 \leqq \text{card}\, S$, denote max

$$f = \max \{f(s) \mid s \in S\} , \qquad I(f) = \{s \in S \mid f(s) = \max f\} .$$

Then define $f^-: S \to D$ by

(1) if $I(f)$ is not a unit set, $f^-(s) = \max f$ for every $s \in S$, and
(2) if $I(f)$ is a unit set, $f^-(s) = \max f$ for $s \in I(f)$, and

$$f^-(s) = \max \{f(t) \mid t \in S - I(f)\} \quad \text{for} \quad s \in S - I(f) .$$

LEMMA 7. *If S is finite and* $2 \leqq \text{card}\, S$, *then*
(1) $f \in L$ *implies* $f^- M$ *and* $f \in M$ if and only if $f = f^-$,
(2) $f \leqq f^-$, $f^{--} = f^-$, *and* $f \leqq g$ *implies* $f^- \leqq g^-$,
(3) $f, g \in M$ *implies* $f \bigtriangledown g = (f \vee g)^-$.

The lemma is a straight forward catalog of the properties following from the definition directly.

LEMMA 8. *If S is finite,* $2 \leqq \text{card}\, S$ *and D is a topological chain, then the function* $J: L \to M$ *defined by* $J(f) = f^-$ *is a retraction of L onto M.*

From Lemma 7 it suffices to show that J is continuous. This is done by letting U be an open set in D and $r \in S$, defining $W = \{f \in M \mid f(r) \in U\}$ and showing that $J^{-1}(W)$ is open. It is shown to be open by case argument. Let $g \in L$ and $g^- \in W$. If $I(g)$ is not a unit set, then g^- is constant and max $g \in U$. Define $V_1 = \{f \in L \mid f(s) \in U$ for every $s \in S\}$. Since S is finite, V_1 is open and contains g. Further $h \in V_1$ implies $h^-(r) \in U$. If $I(g) = \{r\}$, let b be an element of $S - \{r\}$. Define $V_2 = \{f \in L \mid f(r) \in U_1$, and $f(s) \in U_2$ for $s \in S - \{r\}\}$ where $U_1 = U \cap \{x \in D \mid z < x\}$ and $U_2 = \{x \in D \mid x < z\}$ if there exist $z \in D$ such that $g^-(b) < z < g^-(r)$, and if there does not exist such an element z, $U_1 = U \cap \{x \in D \mid g^-(b) < x\}$ and $U_2 = \{x \in D \mid x < g^-(r)\}$. Then $g \in V_2$, V_2 is open and $f \in V_2$ implies $f^-(r) \in U$. The other case is handled in a similar fashion.

THEOREM 3. *If D is a topological chain and S is finite, then $(M, \leqq)$ is a modular topological lattice which is nondistributive if* card $S > 2$ *and* card $D > 1$.

If card $S = 1$, $M = L$ and therefore $(M, \leqq)$ is a topological distributive lattice. If card $S \geqq 2$, then Lemma 7 and 8 establish that $\bar{\vee}$ is continuous since it is the composition of continuous maps. Therefore $(M, \leqq)$ is a topological lattice since $\wedge$ is continuous for every S. Theorem 1 establishes the modularity of M while its corollary the nondistributive nature of M when $3 \leqq$ card S and $2 \leqq$ card D.

DEFINITION. Let n be a positive integer and $3 \leqq n$, then let M_n denote the lattice constructed as the M above in the case where $S = \{1, 2, \cdots, n\}$ and $D = \{x \in R \mid 0 \leqq x \leqq 1\}$ with its usual order and the operations of D being $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. When S is the set of positive integers and D as previously described let M_∞ denote the lattice M constructed. Then the following results are immediate.

THEOREM 4. *For each positive integer $n \geqq 3$, M_n is a compact connected topological lattice which is modular and not distributive.*

COROLLARY 1. *M_3 is a compact connected topological lattice, modular and not distributive, which is a two dimensional subset of R^3 that cannot be embedded in R^2.*

COROLLARY 2. *M_∞ is a compact connected topological semilattice, which is a modular lattice, and not a topological lattice.*

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