

IDEALS IN ADMISSIBLE ALGEBRAS

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The notion of admissible algebra has been introduced by Koecher. They are commutative algebras whose enveloping Lie algebra (of multiplications) splits into the direct sum of an even and an odd part. It will be shown here that the class of admissible algebras cannot be defined by (nonassociative) polynomial identities. This is done by exhibiting an admissible algebra which possesses a homomorphic image which is not admissible. The main tool is the relationship between the admissibility of a homomorphic image of an admissible algebra A , a symmetry property of a certain ideal of the enveloping Lie algebra of A formed from the kernel of the homomorphism, and the ideal structure of an algebra constructed by Koecher from A .

Let A be a commutative algebra over a field F of characteristic not two. We do not assume that A is associative, nor that A has a unit element 1. For a in A , let $L(a)$ denote multiplication by a in A , i.e., $L(a)x = ax$ for x in A . Let $L(A)$ denote the vector space of linear mappings $L(a)$, a in A . Let $H(A)$ be the Lie algebra generated by $L(A)$. $H(A) = H^-(A) + H^+(A)$, where $H^-(A)$ is the space spanned by commutator products of an odd number of elements of $L(A)$, and $H^+(A)$ is the space spanned by commutator products of an even number of elements of $L(A)$. The commutator $[T, U]$ of T and U is $TU - UT$. If $1 \in A$, then the sum $L(A) + [L(A), L(A)]$ is direct, since commutators vanish on 1. We call A *admissible*, [1], if the identity mapping of $L(A)$ extends to an antiautomorphism $T \rightarrow T^*$ of $H(A)$. This means that $T^* = T$ for $T \in H^-(A)$, and $T^* = -T$ for $T \in H^+(A)$, so that $H^-(A) \cap H^+(A) = 0$. Conversely, if $H^-(A) \cap H^+(A) = 0$, then the map $(T + U)^* = T - U$ for $T \in H^-(A)$, $U \in H^+(A)$ shows that A is admissible.

Let A be admissible. We may then form the algebra $\mathfrak{L}(A) = H(A) \oplus A \oplus \bar{A}$, where \bar{A} is a vector space copy of A . The multiplication in $\mathfrak{L}(A)$ is given by $[(T_1, a_1, \bar{b}_1), (T_2, a_2, \bar{b}_2)] = (T, a, \bar{b})$, where $T = [T_1, T_2] + \alpha_1 \Delta \bar{b}_2 - \alpha_2 \Delta \bar{b}_1$, $a = T_1 a_2 - T_2 a_1$, $b = T_2^* b_1 - T_1^* b_2$, and the pairing Δ of A and \bar{A} into $H(A)$ is given by $a \Delta \bar{b} = L(ab) + [L(a), L(b)]$ (see [1]).

Now let M be an ideal of A . Let $j_1(M) = \{T \in H(A) \mid TA \subseteq M\}$. $j_1(M)$ is an ideal in $H(A)$. $j_1(M)$ contains the ideal $i(M)$ of $H(A)$ generated by the mappings $L(m)$, $m \in M$. If we let $I(M) = i(M) \oplus M \oplus \bar{M}$ and $J_1(M) = j_1(M) \oplus M \oplus \bar{M}$, and note that $i(M)^* \subseteq i(M)$, then we see that $I(M)$ is an ideal in $\mathfrak{L}(A)$. In this note, we shall give an example to show that $J_1(M)$ need not be an ideal in $\mathfrak{L}(A)$. We first

note some equivalent conditions for $J_1(M)$ to be an ideal of $\mathfrak{L}(A)$.

THEOREM. *Let A be a commutative admissible algebra, M an ideal of A . Then the following three conditions are equivalent:*

- (1) A/M is admissible
- (2) $J_1(M)$ is an ideal of $\mathfrak{L}(A)$
- (3) $j_1(M)^* \subseteq j_1(M)$.

Proof. (1) \Rightarrow (2). The natural homomorphism of A onto A/M induces a homomorphism of $\mathfrak{L}(A)$ onto $\mathfrak{L}(A/M)$, by replacing all variables from A which appear by their cosets modulo M . The kernel of this homomorphism has M and \bar{M} in the second and third components, respectively, and the first component is $j_1(M)$, i.e., the kernel is $J_1(M)$, an ideal of $\mathfrak{L}(A)$.

(2) \Rightarrow (3). Let $T \in j_1(M), a \in A$. Then $[0, 0, \bar{a}], (T, 0, \bar{0})] = (0, 0, \bar{b}) \in J_1(M)$, where $b = T^* a \in M$. Hence $T^* \in j_1(M)$.

(3) \Rightarrow (1). We wish to show $H^-(A/M) \cap H^+(A/M) = 0$. Let $a \rightarrow a'$ be the natural homomorphism of A onto A/M . Let $T' = L(a') + \sum [[L(b'_i), L(c'_i)], L(d'_i)] + \dots = \sum [L(e'_j), L(f'_j)] + \dots$ be in $H^-(A/M) \cap H^+(A/M)$. Then $U' = L(a') - \sum [L(e'_j), L(f'_j)] + \sum [[L(b'_i), L(c'_i)], L(d'_i)] - \dots = 0$, so $U = L(a) - \sum [L(e_j), L(f_j)] + \sum [[L(b_i), L(c_i)], L(d_i)] - \dots$ is in $j_1(M)$. If we write $U = U^- + U^+, U^- \in H^-(A), U^+ \in U^+(A)$, then $U^* = U^- - U^+ \in j_1(M)$, so $U^-, U^+ \in j_1(M)$. But $U^- = L(a) + \sum [[L(b_i), L(c_i)], L(d_i)] + \dots$, so $T' = 0$. This proves the theorem. We remark that (1) \Rightarrow (2) follows from (2.8) of [1].

If A is a commutative associative algebra, then $H(A) = L(A)$, so A is admissible. Hence the conditions of the theorem hold for any ideal in a commutative associative algebra. If A is a commutative Jordan algebra, then $L(A)$ is a Lie triple system, and $H(A) = L(A) + [L(A), L(A)]$. If, in addition, A has a unit element 1 , then $H(A) = L(A) \oplus [L(A), L(A)]$, so A is admissible. Hence the conditions of the theorem hold for any ideal in a commutative Jordan algebra with unit element.

We now present an example of a commutative Jordan algebra without unit element, which is admissible, but which has an ideal M which does not satisfy the conditions of the theorem. We consider three-by-three matrices over F , with e_{ij} denoting the usual matrix units. Consider the four-dimensional Jordan algebra A with basis $e = e_{11} + e_{33}, x = e_{12}, y = e_{23}, z = e_{13}$, and multiplication $ab + ba$ for $a, b \in A$. $L(A)$ is four-dimensional, with basis $L(e), L(x), L(y), L(z)$. Let $T_1 = [L(e), L(x)], T_2 = [L(e), L(y)]$. T_1 and T_2 form a basis for $[L(A), L(A)]$. Let $\alpha L(e) + \beta L(x) + \gamma L(y) + \delta L(z) = \rho T_1 + \tau T_2$. Applying to e yields $\alpha = \delta = 0, \rho = -\beta, \tau = -\gamma$. Applying to x yields $\gamma = 0$, and applying to y yields $\beta = 0$. Hence $L(A) \cap [L(A), L(A)] = 0$. Since

A is a Jordan algebra, $H(A) = L(A) \oplus [L(A), L(A)]$, and A is admissible.

Let $M = Fz$, an ideal of A . Set $T = L(x) + T_1$ in $H(A)$. $T \in j_1(M)$ but $T^* = L(x) - T_1$ satisfies $T^*(e) = 2x \notin M$, so $j_1(M)^* \not\subseteq j_1(M)$. Noting the other conditions of the theorem, $A' = A/M$ is not admissible as $L(x') = [L(x'), L(e')] \neq 0$, and $J_1(M)$ is not an ideal in $\mathfrak{L}(A)$, as $[(0, 0, \bar{e}), (T, 0, 0)] = (0, 0, 2\bar{x}) \notin J_1(M)$.

COROLLARY. *The class of admissible algebras is not definable by polynomial identities.*

This follows since our example shows that the class of admissible algebras is not closed under homomorphic images.

The algebra A/M of the example is a commutative Jordan algebra which is not admissible. Another example would be the algebra with basis $e = e_{11} + e_{22}$, $x = e_{21}$, $y = e_{31}$, $z = e_{32}$, and multiplication $ab + ba$. Then $L(y) = [L(y)L(e)] \neq 0$.

In [1], an ideal $j(M)$ of $H(A)$ is introduced, $j(M) = \{T \in H(A) \mid TA \subseteq M, T^*A \subseteq M\}$. Then $j(M)^* \subseteq j(M)$ and $J(M) = j(M) \oplus M \oplus \bar{M}$ will be an ideal in $\mathfrak{L}(A)$. For commutative Jordan algebras with unit element, $j(M) = j_1(M)$, but our example shows that in general $j(M) \not\subseteq j_1(M)$, even for admissible Jordan algebras A . For an admissible commutative Jordan algebra A , $\mathfrak{L}(A)$ is a Lie algebra. This is proved in [1] for Jordan algebras with unit element. If A does not have a unit element, adjoin 1 to get the Jordan algebra A_1 . The Lie algebra $\mathfrak{L}(A_1)$ has a subalgebra $H_{A_1}(A) \oplus A \oplus \bar{A}$, where $H_{A_1}(A)$ is the enveloping Lie algebra of the multiplications in A_1 by elements of A . This subalgebra has a homomorphic image obtained by restricting the elements of $H_{A_1}(A)$ to A . But this image is $\mathfrak{L}(A)$, so $\mathfrak{L}(A)$ is a Lie algebra.

In conclusion, it might be interesting to find an example of a commutative admissible algebra A with unit element with an ideal M which fails to satisfy the conditions of the theorem. Such an A could not be a Jordan algebra or an associative algebra.

REFERENCE

1. M. Koecher, *Imbedding of Jordan algebras into lie algebras*, I, Amer. J. Math. (1967), 787-816.

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