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A REPRESENTATION THEOREM FOR MEASURES ON INFINITE DIMENSIONAL SPACES

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If X is a locally compact, regular topological space, then the well known Riesz representation theorem sets up an isomorphism between the family of all bounded Radón outer measures on X and the set of continuous positive linear functionals on the family of continuous functions with compact support in X. In this isomorphism corresponding elements, l a linear functional and μ a measure, satisfy the relationship $l(f) = \int f d\mu$ for all continuous functions f with compact support in X.

Since an infinite product of locally compact, regular spaces is in general no longer locally compact with respect to the product topology, the Riesz representation theorem fails to hold for such spaces. In this paper, an analogue of the Riesz representation theorem is obtained for this case.

The main idea is to replace the various families mentioned above by the following:

(i) A family $\mathscr{C}$ of cylinders whose elements act like compact sets for a "pseudo-topology" $\mathscr{G}$, where $\mathscr{G}$ is closed under finite intersections and countable unions and is a subset of the product topology.

(ii) A family M of bounded outer measures, related to $\mathscr{C}$ and $\mathscr{G}$ in much the same way as bounded Radón outer measures are related to compact and open sets.

(iii) A family F of functions depending only on a finite number of coordinates, with respect to which they are continuous and have compact support.

(iv) A family L of positive linear functionals on the linear span of F.

Under the added hypothesis of σ-compactness of the coordinate spaces, we show that L and M are isomorphic in such a way that corresponding elements, l in L and μ in M, satisfy the relationship $l(f) = \int f d\mu$ for all f in F.

Moreover we show that the elements of M can be viewed as the projective limit measures of projective systems of bounded regular Borel measures.

From the integrability of the members of F, it follows that all bounded Borel functions which depend only on a finite number of coordinates are also integrable. Thus the simple functions used by Šilov [7] and the tame functions used by Segal [6] and Gross [2] in the development of an integration theory on Hilbert space are included among the integrable functions of the measures considered here. (For a good guide to the literature in this area see the bibliography in Gross [3].) Our results therefore not only characterize an important class of linear functionals in terms of projective limits of regular Borel measures, but also enable one to extend these functionals to a much wider class of functions through a standard integral with respect to a measure, thereby obviating the need to develop a special theory of integration in infinite dimensional spaces for this purpose.

1. General notation.

(1) $\varnothing$ is the empty set.

(2) ω is the set of natural numbers.

(3) $\boldsymbol{R}$ is the set of real numbers.

(4) $\mathscr{C}$ is a compact family if and only if for every subfamily $\mathscr{A}$ of $\mathscr{C}$, if the intersection of any finite number of members of $\mathscr{A}$ is nonvoid, then the intersection of all members of $\mathscr{A}$ is nonvoid.

(5) For f a function on X to $\boldsymbol{R}$ and $A \subset X$,
$f \mid A$ is the restriction of f to A,
1_A is the characteristic function of A,
$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$,
$f^+(x) = \max\{0, f(x)\}$ for $x \in X$,
support f = closure $\{x: f(x) > 0\}$ if X is a topological space.

(6) If for $n \in \omega$, α_n is a set, $a_n \in \boldsymbol{R}$, f_n is a function on X to R, then
$\alpha_n \uparrow \alpha$ if and only if $\alpha_n \subset \alpha_{n+1}$ and $\bigcup_{n \in \omega} \alpha_n = \alpha$,
$a_n \uparrow a$ if and only if $a_n \leqq a_{n+1}$ and $\lim_{n \in \omega} a_n = a$,
$f_n \uparrow f$ if and only if for all $x \in X, f_n(x) \leqq f_{n+1}(x)$ and $\lim_{n \in \omega} f_n(x) = f(x)$.

(7) For I an index set and X_i a set for each $i \in I$, $\prod_{i \in I} X_i = \{x: x$ is a function of I with $x_i \in X_i$ for each $i \in I\}$.

(8) μ is a Carathéodory measure on X if and only if μ is a function on the family of all subsets of X such that $\mu(\varnothing) = 0$ and $0 \leqq \mu(A) \leqq \sum_{n \in \omega} \mu(B_n) \leqq \infty$ whenever $A \subset \bigcup_{n \in \omega} B_n \subset X$.

(9) For μ a Carathéodory measure on X, A is μ-measurable if and only if $A \subset X$ and for every $B \subset X$, $\mu(B) = \mu(B \cap A) + \mu(B - A)$. $\mathscr{M}_\mu = \{A: A$ is μ-measurable$\}$.

(10) μ is a $\mathscr{G}$-outer measure on X if and only if μ is a Carathéodory measure on X, $\mathscr{G} \subset \mathscr{M}_\mu$, and for every $A \subset X$, $\mu(A) = \inf\{\mu(B): B \in \mathscr{G}$ and $A \subset B\}$.

(11) μ is the Carathéodory measure on X generated by τ and $\mathscr{G}$ if and only if $\mathscr{G}$ is a family of subsets of $X, \tau(A) \geqq 0$ for every $A \in \mathscr{G}$, and for $B \subset X$ $\mu(B) = \inf\{\sum_{A \in \mathscr{H}} \tau(A): \mathscr{H} \subset \mathscr{G}, \mathscr{H}$ is countable and $B \subset \bigcup_{A \in \mathscr{H}} A\}$.

(12) For X a topological space, μ is a Radón outer measure on X if and only if μ is a Carathéodory measure on X such that

(1) open sets are μ-measurable,

(2) If C is compact then $\mu(C) < \infty$,

(3) if A is open then $\mu(A) = \sup\{\mu(C): C \text{ is compact}, C \subset A\}$,

(4) if $B \subset X$ then $\mu(B) = \inf\{\mu(A): A \text{ is open}, B \subset A\}$.

(13) For X a topological space, μ is the topological measure cranked by τ if and only if τ is a function on the family of closed compact subsets of X, $\tau_*(A) = \sup\{\tau(C): C \text{ is closed compact and } C \subset A\}$ for A an open subset of X, and μ is the Carathéodory measure on X generated by τ_* and the family of open subsets of X.

(14) REMARKS. We mention here two well known facts about Carathéodory measures:

(1) The Carathéodory measure on X generated by τ and $\mathscr{G}$ is in fact a Carathéodory measure on X.

(2) If X is locally compact and regular and τ is a function on the family of closed compact subsets of X such that for A, B closed and compact we have $0 \leqq \tau(A) \leqq \tau(A \cup B) \leqq \tau(A) + \tau(B) < \infty$ and $\tau(A \cup B) = \tau(A) + \tau(B)$ if $A \cap B = \emptyset$, then the topological measure cranked by τ is a Radón outer measure on X. (See for example Sion [8].)

2. The family $\mathscr{G}$ of cylinders.

Throughout this paper we suppose that T is any index set and that for each $t \in T$, Y_t is a locally compact, σ-compact and regular topological space.

2.1. DEFINITIONS.

(1) $X = \prod_{t \in T} Y_t$.

(2) I is the set of nonvoid finite subsets of T, ordered by inclusion.

For $i, j \in I$ with $i \subset j$

(3) $X_i = \prod_{t \in i} Y_t$ is equipped with the product topology (which is locally compact, σ-compact and regular),

(4) $\mathscr{K}_i$ is the family of closed compact subsets of X_i,

(5) π_i (respectively π_{ij}) is the canonical projection of X (respectively X_j) onto X_i,

(6) For $A \subset X_i$, $\operatorname{cyl} A = \pi_i^{-1}[A]$.

If no confusion is possible we will for $t \in T$ identify t and $\{t\}$, Y_t and $X_{\{t\}}$. Thus $Y_t = X_{\{t\}} = X_t$ and $\mathscr{K}_{\{t\}} = \mathscr{K}_t$.

2.2. DEFINITIONS.

(1) $\mathscr{C} = \{\alpha$: there exists $i \in I$ and $\beta \in \mathscr{K}_i$ with $\alpha = \operatorname{cyl} \beta\}$. Thus $\mathscr{C}$ is the family of cylinder sets which for some $i \in I$ have a compact base in X_i.

(2) $\mathscr{G}_0$ is the closure under finite intersections of the family of complements of sets in $\mathscr{C}$.

(3) $\mathscr{G}$ is the closure of $\mathscr{G}_0$ under countable unions.

The essential properties of $\mathscr{C}$ are the following:

THEOREM 2.3. *$\mathscr{C}$ is a compact family.*

COROLLARY 2.4. *The closure of $\mathscr{C}$ under finite unions is a compact family.*

COROLLARY 2.5. *If $\alpha \in \mathscr{C}$ and, for each $n \in \omega, B_n \in \mathscr{G}_0$ with $\alpha \subset \bigcup_{n \in \omega} B_n$ then there exists $N \in \omega$ such that $\alpha \subset \bigcup_{n=0}^{N} B_n$.*

Proof of 2.3. Let $\mathscr{A}$ be any subfamily of $\mathscr{C}$ such that for any nonvoid finite $B \subset \mathscr{A}$ we have $\bigcap_{\alpha \in B} \alpha \neq \emptyset$. For each $\alpha \in \mathscr{A}$ let $i_\alpha \in I$ be such that $\alpha = \operatorname{cyl} \beta$ for some $\beta \in \mathscr{K}_{i_\alpha}$. Let $S = \bigcup_{\alpha \in \mathscr{A}} i_\alpha$ and for each $t \in S$ choose $\alpha_t \in \mathscr{A}$ with $t \in i_{\alpha_t}$ and let $C_t = \pi_t[\alpha_t]$. Then C_t is compact in X_t and $C_t \neq \emptyset$. Let z be a fixed point in X with $z_t \in C_t$ for each $t \in S$.

Then $C = \{x \in X: x_t \in C_t \text{ for } t \in S \text{ and } x_t = z_t \text{ for } t \in T\text{-}S\}$ is a compact subset of X with respect to the product topology. Now let $\mathscr{B}$ be the family of nonvoid finite subsets of $\mathscr{A}$. Then $\mathscr{B}$ is directed by inclusion. If for each $B \in \mathscr{B}$ we let $T_B = \bigcup_{\alpha \in B} i_\alpha$, then T_B is finite. For each $B \in \mathscr{B}$ choose $y^B \in \bigcap_{\beta \in B} \beta \cap \bigcap_{t \in T_B} \alpha_t$. Then $y_t^B \in C_t$ for each $t \in T_B$ and if x^B is defined by $x_t^B = y_t^B$ for $t \in T_B$ and $x_t^B = z_t$ for $t \in T - T_B$, then $x^B \in \bigcap_{\beta \in B} \beta$ and $x^B \in C$. Hence $\{x^B; B \in \mathscr{B}\}$ is a net in C, and since C is compact, this net has a cluster point x. If $\alpha \in \mathscr{A}$ then $x^B \in \alpha$ for any $B \in \mathscr{B}$ with $\{\alpha\} \subset B$. Therefore the net $\{x^B; B \in \mathscr{B}\}$ is eventually in α for each $\alpha \in \mathscr{A}$. Hence $x \in \alpha$ for each $\alpha \in \mathscr{A}$ and so $\bigcap_{\alpha \in \mathscr{A}} \alpha \neq \emptyset$.

Proof of 2.4. See Meyer [5] p. 33.

Proof of 2.5. Immediate from the definition of $\mathscr{G}_0$ and 2.4.

The following well known elementary lemma will be needed later:

LEMMA 2.6. *If $i \in I$, A, B are open in X_i, $\gamma \in \mathscr{K}_i$ and $\gamma \subset A \cup B$, then there exist $\alpha, \beta \in \mathscr{K}_i$ with $\alpha \subset A, \beta \subset B$ and $\alpha \cup \beta = \gamma$.*

3. The family M of measures.

3.1. DEFINITION. $M = \{\mu: \mu$ is a bounded outer measure on X such that

(1) $\mathscr{G} \subset \mathscr{M}_\mu$,

(2) $\mu(A) = \sup\{\mu(\alpha): \alpha \in \mathscr{C}$ and $\alpha \subset A\}$ for $A \in \mathscr{G}$,

(3) $\mu(B) = \inf\{\mu(A): A \in \mathscr{G}$ and $B \subset A\}$ for $B \subset X\}$.

3.2. DEFINITIONS. For any set function τ on $\mathscr{C}$

(1) τ satisfies condition (a) if and only if τ is bounded, $\tau(\varnothing) = 0$ and for every $i \in I$ and $\alpha, \beta \in \mathscr{K}_i$ $0 \leqq \tau(\text{cyl}\,\alpha) \leqq \tau(\text{cyl}\,\alpha \cap \text{cyl}\,\beta) \leqq \tau(\text{cyl}\,\alpha) + \tau(\text{cyl}\,\beta)$ and $\tau(\text{cyl}\,\alpha \cup \text{cyl}\,\beta) = \tau(\text{cyl}\,\alpha) + \tau(\text{cyl}\,\beta)$ if $\alpha \cap \beta = \varnothing$.

(2) τ satisfies condition (b) if and only if for every $i \in I$, $\alpha \in \mathscr{K}_i$, $t \in T - i$ and sequence C in $\mathscr{K}_t$ with $C_n \subset \text{interior}\ C_{n+1}$ for $n \in \omega$ and $C_n \uparrow X_t$, if $j = i \cup \{t\}$ and $\beta_n = \{x \in X_j: x \mid i \in \alpha$ and $x_t \in C_n\}$ then $\tau(\text{cyl}\,\beta_n) \uparrow \tau(\text{cyl}\,\alpha)$. (Note that we certainly have $\text{cyl}\,\beta_n \uparrow \text{cyl}\,\alpha$.)

(3) $\tau_*(A) = \sup\{\tau(\alpha): \alpha \in \mathscr{C}$ and $\alpha \subset A\}$ for $A \in \mathscr{G}$.

The key results of this section are summed up in the following

THEOREM 3.3. *Let τ satisfy conditions* (a) *and* (b) *and μ be the Carathéodory measure on X generated by τ_* and $\mathscr{G}$. Then*

(1) *$\mu \in \boldsymbol{M}$ and μ agrees with τ_* on $\mathscr{G}$,*

(2) *if $i \in I$, $\mu_i(A) = \mu(\text{cyl}\,A)$ for $A \subset X_i$, $\tau_i(\alpha) = \tau(\text{cyl}\,\alpha)$ for $\alpha \in \mathscr{K}_i$ and ν_i is the topological outer measure on X_i cranked by τ_i, then ν_i is a bounded Radón outer measure and μ_i agrees with ν_i on $\mathscr{M}_{\nu_i}$.*

For the proof of this theorem two preliminary lemmas are needed.

LEMMA A. *Let τ satisfy condition* (b). *Then for $i, j \in I$ with $i \subset j$ and $\alpha \in \mathscr{K}_i$ we have $\tau(\text{cyl}\,\alpha) = \sup\{\tau(\text{cyl}\,\beta): \beta \in \mathscr{K}_j$ and $\text{cyl}\,\beta \subset \text{cyl}\,\alpha\}$.*

Proof. Follows easily from condition (b) and induction.

LEMMA B. *If τ satisfies conditions* (a) *and* (b) *then τ_* is countably subadditive on $\mathscr{G}$.*

Proof. Let $A_n \in \mathscr{G}$ for $n \in \omega$, $\varepsilon > 0$ and $\alpha \in \mathscr{C}$ with $\alpha \subset \bigcup_{n \in \omega} A_n$. For each $n \in \omega$, $A_n = \bigcup_{m \in \omega} B_{nm}$ where $B_{nm} \in \mathscr{G}_0$. So $\alpha \subset \bigcup_{n \in \omega} \bigcup_{m \in \omega} B_{nm}$ and hence by Corollary 2.5 there exist $N, M \in \omega$ such that $\alpha \subset \bigcup_{n=0}^{N} \bigcup_{m=0}^{M} B_{nm}$. Let, for $0 \leqq n \leqq N$, $E_n = \bigcup_{m=0}^{M} B_{nm}$ and i_n be such that $E_n = \text{cyl}\,A_n$ for some $A_n \subset X_{i_n}$. Let $i_\alpha \in I$ be such that $\alpha = \text{cyl}\,\gamma$ for some $\gamma \in \mathscr{K}_{i_\alpha}$. Let $i = i_\alpha, \bigcup_{n=0}^{N}, i_n$, then i is finite and $i_\alpha \subset i$. By Lemma A of this section choose $\beta \in \mathscr{K}_i$ with $\text{cyl}\,\beta \subset \alpha$ and $\tau(\alpha) \leqq \tau(\text{cyl}\,\beta) + \varepsilon$. Now for $0 \leqq n \leqq N$, $\pi_i[E_n]$ is open in X_i and $\beta \subset \pi_i[\alpha] \subset \bigcup_{n=0}^{N} \pi_i[E_n]$.

By Lemma 2.6 and induction we can, for $0 \leqq n \leqq N$, find $\beta_n \in \mathscr{K}_i$ with $\beta_n \subset \pi_i[E_n]$ and $\beta = \bigcup_{n=0}^N \beta_n$. Then $\text{cyl}\,\beta = \bigcup_{n=0}^N \text{cyl}\,\beta_n \subset \bigcup_{n=0}^N E_n$. By condition (a) we have $\tau(\text{cyl}\,\beta) \leqq \sum_{n=0}^N \tau(\text{cyl}\,\beta_n)$. Hence $\tau(\alpha) \leqq \tau(\text{cyl}\,\beta) + \varepsilon \leqq \sum_{n=0}^N \tau(\text{cyl}\,\beta_n) + \varepsilon \leqq \sum_{n=0}^N \tau_*(E_n) + \varepsilon \leqq \sum_{n\in\omega} \tau_*(A_n) + \varepsilon$. It follows that $\tau_*(\bigcup_{n\in\omega} A_n) = \sup\{\tau(\alpha): \alpha \in \mathscr{C} \text{ and } \alpha \subset \bigcup_{n\in\omega} A_n\} \leqq \sum_{n\in\omega} \tau_*(A_n)$. We now proceed to prove Theorem 3.3.

Proof of 3.3(1) Since, by Lemma B, τ_* is countably subadditive on $\mathscr{G}$ and since $\mathscr{G}$ is closed under countable unions we have for $A \in \mathscr{G}$, $\mu(A) = \tau_*(A)$ and therefore for $B \subset X$, $\mu(B) = \inf\{\mu(A): A \in \mathscr{G}$ and $B \subset A\}$. Furthermore, since clearly $\mu(\alpha) \geqq \tau(\alpha)$ for each $\alpha \in \mathscr{C}$, it follows that for $A \in \mathscr{G}$, $\mu(A) = \sup\{\mu(\alpha): \alpha \in \mathscr{C}$ and $\alpha \subset A\}$. Now, since μ certainly is a bounded Carathéodory measure on X, all that remains is to show $\mathscr{G} \subset \mathscr{M}_\mu$. So let $A \in \mathscr{G}$, $B \subset X$ and $\varepsilon > 0$. Choose $B' \in \mathscr{G}$ with $B \subset B'$ and $\mu(B') \leqq \mu(B) + \varepsilon$. Let $\alpha \in \mathscr{C}$ with $\alpha \subset B' \cap A$ and $\mu(B' \cap A) \leqq \mu(\alpha) + \varepsilon$. Let $\beta \in \mathscr{C}$ with $\beta \subset B' - \alpha$ and $\mu(B' - \alpha) \leqq \mu(\beta) + \varepsilon$. By Lemma A we can suppose that $\alpha \cup \beta \in \mathscr{C}$ also. Then $\mu(B \cap A) + \mu(B - A) \leqq \mu(B' \cap A) + \mu(B' - \alpha) \leqq \mu(\alpha) + \mu(\beta) + 2\varepsilon = \mu(\alpha \cup \beta) + 2\varepsilon \leqq \mu(B') + 2\varepsilon \leqq \mu(B) + 3\varepsilon$.

Hence $\mu(B \cap A) + \mu(B - A) = \mu(B)$ for all $B \subset X$. It follows that $\mathscr{G} \subset \mathscr{M}_\mu$.

Proof of 3.3(2) Let ν_i be the topological outer measure on X_i cranked by τ_i. By condition (a) and Remark 1.14.2 we have that ν_i is a bounded Radón outer measure on X_i. Let A be open in X_i. Since $X_i - A$ is closed and X_i is σ-compact, we can for $n \in \omega$ choose $C_n \in \mathscr{K}_i$ such that $C_n \uparrow (X_i - A)$. Then $A = \bigcap_{n\in\omega}(X_i - C_n)$. Since $\text{cyl}\,(X_i - C_n) \in \mathscr{G}$ we have

$$\begin{aligned}
\mu_i(X_i - C_n) &= \mu\,(\text{cyl}\,(X_i - C_n)) = \tau_*\,(\text{cyl}\,(X_i - C_n)) \\
&= \sup\{\tau(\beta): \beta \in \mathscr{C} \text{ and } \beta \subset \text{cyl}\,(X_i - C_n)\} \\
&= \sup\{\tau\,(\text{cyl}\,\alpha): \alpha \in \mathscr{K}_i \text{ and } \alpha \subset (X_i - C_n)\} \\
&= \sup\{\tau_i(\alpha): \alpha \in \mathscr{K}_i \text{ and } \alpha \subset (X_i - C_n)\} \\
&= \nu_i(X_i - C_n)\,.
\end{aligned}$$

Furthermore since the $X_i - C_n$ are μ_i-measurable as well as ν_i-measurable we have $\mu_i(A) = \lim_{n\in\omega} \mu_i(X_i - C_n) = \lim_{n\in\omega} \nu_i(X_i - C_n) = \nu_i(A)$. Hence μ_i and ν_i agree on open sets. If $D \subset X_i$ then $\nu_i(D) = \inf\{\nu_i(A): A$ is open in X_i and $D \subset A\} = \inf\{\mu_i(A): A$ is open in X_i and $D \subset A\} \geqq \mu_i(D)$. Hence $\mu_i \leqq \nu_i$ always.

Now let $B \in \mathscr{M}_{\nu_i}$. Given $\varepsilon > 0$ choose A open in X_i with $B \subset A$ and $\nu_i(A) < \nu_i(B) + \varepsilon$. Since $\nu_i(A) = \nu_i(B) + \nu_i(A - B)$ we have $\nu_i(A - B) < \varepsilon$ and consequently $\mu_i(A - B) < \varepsilon$. But $\mu_i(A) \leqq \mu_i(A-B) +$

$\mu_i(B) < \mu_i(B) + \varepsilon$. Hence

$$\begin{aligned}\mu_i(B) &= \inf\{\mu_i(A): A \text{ is open in } X_i \text{ and } B \subset A\}\\ &= \inf\{\nu_i(A): A \text{ is open in } X_i \text{ and } B \subset A\}\\ &= \nu_i(B)\ .\end{aligned}$$

3.4. *$\boldsymbol{M}$ as related to projective limit measures.* Suppose that for each $i \in I$, $\mathscr{B}_i$ is the σ-ring generated by $\mathscr{K}_i$ and ν_i is a measure on $\mathscr{B}_i$. We call $\{\nu_i: i \in I\}$ a projective system of measures if whenever $i, j \in I$ with $i \subset j$ we have for $A \in \mathscr{B}_i$

$$\nu_i(A) = \nu_j(\pi_{ij}^{-1}[A])\ .$$

We say that the projective system $\{\nu_i: i \in I\}$ admits a projective limit measure ν if ν is a measure on the σ-ring $\mathscr{B}$ of subsets of X generated by $\{\text{cyl}\, B: B \in \mathscr{B}_i \text{ for some } i \in I\}$ such that for each $i \in I$ and $A \in \mathscr{B}_i$, $\nu(\text{cyl}\, A) = \nu_i(A)$. Such a measure ν, if it exists, is unique and can thus be called the projective limit measure of the system $\{\nu_i: i \in I\}$.

For more general definitions of projective or inverse systems of measures see Choksi [1], Mallory [4] or Meyer [5].

Now, if for $i \in I$ we call ν_i a bounded regular Borel measure whenever ν_i is a bounded measure on $\mathscr{B}_i$ such that for every $A \in \mathscr{B}_i$

$$\begin{aligned}\nu_i(A) &= \inf\{\nu_i(B): B \text{ is open and } A \subset B\}\\ &= \sup\{\nu_i(C): C \in \mathscr{K}_i \text{ and } C \subset A\}\end{aligned}$$

we then have

THEOREM 3.4.1. *$\mu \in \boldsymbol{M}$ if and only if μ is a $\mathscr{S}$-outer measure on X and $\mu \mid \mathscr{B}$ is the projective limit measure of a projective system $\{\mu_i: i \in I\}$ of bounded regular Borel measures μ_i on $\mathscr{B}_i$.*

Proof. Suppose $\mu \in \boldsymbol{M}$. Then μ is a $\mathscr{S}$-outer measure on X. If $\mu_i(A) = \mu(\text{cyl}\, A)$ for $A \in \mathscr{B}_i$ then clearly $\{\mu_i: i \in I\}$ forms a projective system of measures and $\mu \mid \mathscr{B}$ is clearly the projective limit measure of this system. Using 3.3(2) one can easily check that each μ_i is in fact a bounded regular Borel measure on $\mathscr{B}_i$.

Conversely let μ be a $\mathscr{S}$-outer measure on X and $\mu \mid \mathscr{B}$ be the projective limit measure of a projective system $\{\mu_i: i \in I\}$ of bounded regular Borel measures μ_i on $\mathscr{B}_i$. Let for each $i \in I$ and $\alpha \in \mathscr{K}_i$, $\tau(\text{cyl}\, \alpha) = \mu_i(\alpha)$. Then τ is a set function on $\mathscr{C}$ satisfying conditions (a) and (b). Let ν be the Carathéodory measure on X generated by τ_* and $\mathscr{S}$. Then by 3.3(1), $\nu \in \boldsymbol{M}$. Clearly $\nu \mid \mathscr{B}$ is the projective limit measure of the system $\{\nu_i: i \in I\}$ where $\nu_i(A) = \nu(\text{cyl}\, A)$ for $A \in \mathscr{B}_i$. From 3.3(2), we see that $\mu_i = \nu_i$ for each $i \in I$. Hence $\mu \mid \mathscr{B} = \nu \mid \mathscr{B}$.

Since $\mathscr{G} \subset \mathscr{B}$, $\mu \mid \mathscr{G} = \nu \mid \mathscr{G}$ and therefore, since both μ and ν are $\mathscr{G}$-outer measures on X, we have $\mu = \nu$. Hence $\mu \in \boldsymbol{M}$.

4. The representation theorem.

4.1. DEFINITIONS. (1) For $i \in I$, $C_0(X_i)$ is the set of continuous real valued functions on X_i with compact support.

(2) For $i \in I$ and $h \in C_0(X_i)$, $\operatorname{cyl} h$ is the function on X given by $(\operatorname{cyl} h)(x) = h(x \mid i)$ for every $x \in X$.

(3) $\boldsymbol{F} = \{f$: there exists $i \in I$ and $h \in C_0(X_i)$ with $f = \operatorname{cyl} h\}$.

4.2. DEFINITIONS. (1) $\boldsymbol{L} = (l$: l is a positive linear functional on the linear span of $\boldsymbol{F}$ such that

(1) there exists $K > 0$ with $|l(f)| \leqq K \|f\|_\infty$ for all $f \in \boldsymbol{F}$,

(2) if $i, j \in I$ with $i \subset j$, $f \in C_0(X_i)$ and, for $n \in \omega$, $f_n \in C_0(X_j)$ with $\operatorname{cyl} f_n \uparrow \operatorname{cyl} f$ then $l(\operatorname{cyl} f_n) \uparrow l(\operatorname{cyl} f)$.$\}$

(Note that in the definition of $\boldsymbol{L}$ above, condition (1) does not necessarily imply condition (2).)

(2) For $l \in \boldsymbol{L}$, τ^l is the set function on $\mathscr{G}$ given by $\tau^l(\alpha) = \inf\{l(f): 1_\alpha \leqq f \in \boldsymbol{F}\}$ for $\alpha \in \mathscr{G}$.

Our basic theorem now is

THEOREM 4.3. *For each $l \in \boldsymbol{L}$ there exists a unique $\mu^l \in \boldsymbol{M}$ such that the relationship $l(f) = \int f d\mu^l$ holds for all $f \in \boldsymbol{F}$. Moreover the mapping $l \to \mu^l$ is an isomorphism between $\boldsymbol{L}$ and $\boldsymbol{M}$.*

For the proof of this theorem we will need three preliminary lemmas.

LEMMA C. *For $l \in \boldsymbol{L}$, $i \in I$ and $\alpha \in \mathscr{K}_i$,*

$$\tau^l(\operatorname{cyl} \alpha) = \inf\{l(\operatorname{cyl} f): 1_\alpha \leqq f \in C_0(X_i)\}\,.$$

Proof. Suppose $h \in \boldsymbol{F}$ and $1_{\operatorname{cyl}\alpha} \leqq h$. We want to find $f \in C_0(X_i)$ with $1_\alpha \leqq f$ and $\operatorname{cyl} f \leqq h$. By definition there exists $j \in I$ and $g \in C_0(X_j)$ such that $h = \operatorname{cyl} g$. Let $k = i \cup j$. For $z \in X_k$ let $h_k(z) = h(y)$ for some $y \in X$ with $y \mid k = z$. (Note that $h_k(z)$ is independent of y provided $y \mid k = z$, and that $\operatorname{cyl} h_k = h$.) Since $g \in C_0(X_j)$ and $h_k(z) = g(z \mid j)$ we have that h_k is uniformly continuous on X_k. Hence if for $x \in X_i$

$$\begin{aligned} f^*(x) &= \inf\{h_k(z): z \in X_k \text{ and } z \mid i = x\} \\ &= \inf\{h(y): y \in X \text{ and } y \mid i = x\} \end{aligned}$$

then f^* is continuous on X_i. Moreover it is clear that $\text{cyl}\, f^* \leqq h$ and $1_\alpha \leqq f^*$. Since X_i is locally compact and regular there exists $f \in C_0(X_i)$ with $1_\alpha \leqq f \leqq f^*$. Hence $1_\alpha \leqq f$ and $\text{cyl}\, f \leqq h$. It follows that

$$\begin{aligned}\tau^l(\text{cyl}\,\alpha) &= \inf\{l(h): 1_{\text{cyl}\,\alpha} \leqq h \in \boldsymbol{F}\}\\ &= \inf\{l(\text{cyl}\,f): 1_\alpha \leqq f \in C_0(X_i)\}\ .\end{aligned}$$

LEMMA D. *Let $l \in \boldsymbol{L}$, $i \in I$, $\alpha \in \mathscr{K}_i$. Then for every $\varepsilon > 0$ there exists A open in X_i with $\alpha \subset A$ such that for any $j \in I$ with $i \subset j$ and $f \in C_0(X_j)$ with $\|f\|_\infty \leqq 1$ and $\{x: f(x) > 0\} \subset \pi_{ij}^{-1}[A - \alpha]$ we have $l(\text{cyl}\, f) \leqq \varepsilon$.*

Proof. By Lemma C choose $h \in C_0(X_i)$ with $1_\alpha \leqq h$ and $l(\text{cyl}\, h) \leqq \tau^l(\text{cyl}\,\alpha) + \varepsilon/2$. Let

$$A = \{x: (1 + \varepsilon/1 + 2\, l(\text{cyl}\, h))h(x) > 1\}\ .$$

Then A is open and $\alpha \subset A$. Now let $j \in I$ with $i \subset j$. Suppose first that $g \in C_0(X_j)$ with $0 \leqq g \leqq 1$ and support $g \subset \pi_{ij}^{-1}[A - \alpha]$. Let $\beta = \pi_{ij}[\text{support } g]$. Then α, β are disjoint compact subsets of A and so let V, W be disjoint neighborhoods of α and β respectively with $V \cup W \subset A$. Let $v, w \in C_0(X_i)$ with $1_\alpha \leqq v \leqq 1_U$ and $1_\beta \leqq w \leqq 1_W$. Then $v + w \leqq (1 + \varepsilon/1 + 2l(\text{cyl}\, h))h$ and therefore

$$\begin{aligned}l(\text{cyl}\, v) + l(\text{cyl}\, w) &\leqq l(\text{cyl}\, h) + \varepsilon/2\\ &\leqq \tau^l(\text{cyl}\,\alpha) + \varepsilon \leqq l(\text{cyl}\, v) + \varepsilon\ .\end{aligned}$$

Hence $l(\text{cyl}\, w) \leqq \varepsilon$ and since $\text{cyl}\, g \leqq \text{cyl}\, w$ we have by condition (2) of 4.2(1), that $l(\text{cyl}\, g) \leqq l(\text{cyl}\, w)$. Thus $l(\text{cyl}\, g) \leqq \varepsilon$.

Now let $f \in C_0(X_j)$ with $\|f\|_\infty \leqq 1$ and $\{x: f(x) > 0\} \subset \pi_{ij}^{-1}[A - \alpha]$. For $n \in \omega$ let $\beta_n = \{x: f(x) \geqq 1/n\}$. Then $\beta_n \in \mathscr{K}_j$ and $\beta_n \subset$ interior β_{n+1}. Let $g_n \in C_0(X_j)$ with $1_{\beta_n} \leqq g_n \leqq 1_{\beta_{n+1}}$ and let $f_n = f \cdot g_n$. Then support $f_n \subset \beta_{n+1} \subset \pi_{ij}^{-1}[A - \alpha]$, $0 \leqq f_n \leqq 1$, and hence by the above argument, $l(\text{cyl}\, f_n) \leqq \varepsilon$. Since $f_n \uparrow f^+$ we have by condition (2) of 4.2(1), that $l(\text{cyl}\, f_n) \uparrow l(\text{cyl}\, f^+)$. It follows that $l(\text{cyl}\, f^+) \leqq \varepsilon$ and therefore $l(\text{cyl}\, f) \leqq \varepsilon$.

LEMMA E. *For $l \in \boldsymbol{L}$, τ^l satisfies conditions* (a) *and* (b).

Proof. Condition (a) follows easily from Lemma C using well known standard arguments. To prove condition (b), let $i \in I$, $\alpha \in \mathscr{K}_i$, $t \in T - i$ and C be a sequence in $\mathscr{K}_t$ with $C_n \subset$ interior C_{n+1} for $n \in \omega$, and $C_n \uparrow X_t$. Let $j = i \cup \{t\}$ and $\beta_n = \{x \in X_j: x \mid i \in \alpha \text{ and } x_t \in C_n\}$. Then $\text{cyl}\,\beta_n \uparrow \text{cyl}\,\alpha$. Given $\varepsilon > 0$, by Lemma D, there exists A open in

X_i with $\alpha \subset A$ such that for any $g \in C_0(X_j)$ with $\|g\|_\infty \leqq 1$ and $\{x: g(x) > 0\} \subset \pi_{ij}^{-1}[A - \alpha]$ we have $l(\operatorname{cyl} g) \leqq \varepsilon$. Choose $f \in C_0(X_i)$ with $1_\alpha \leqq f \leqq 1_A$ and let $k_n \in C_0(X_t)$ with $1_{C_n} \leqq k_n \leqq 1_{C_{n+1}}$.

For $x \in X_j$ let $f_n(x) = f(x \mid i) \cdot k_n(x_t)$. Then $f_n \in C_0(X_j)$, $1_{\beta_n} \leqq f_n$ and $\operatorname{cyl} f_n \uparrow \operatorname{cyl} f$. By Lemma C choose $h_n \in C_0(X_j)$ with $1_{\beta_n} \leqq h_n \leqq f_n$ and $l(\operatorname{cyl} h_n) \leqq \tau^l(\operatorname{cyl} \beta_n) + \varepsilon$. We note that $f_n - h_{n+1} \in C_0(X_j)$, $\|f_n - h_{n+1}\|_\infty \leqq 1$ and $\{x: (f_n - h_{n+1})(x) > 0\} \subset \pi_{ij}^{-1}[A - \alpha]$. Hence by Lemma D,

$$l(\operatorname{cyl}(f_n - h_{n+1})) < \varepsilon .$$

Since $f_n = f_n - h_{n+1} + h_{n+1}$ we have $l(\operatorname{cyl} f_n) = l(\operatorname{cyl}(f_n - h_{n+1})) + l(\operatorname{cyl} h_{n+1})$

$$\leqq l(\operatorname{cyl} h_{n+1}) + \varepsilon \leqq \tau^l(\operatorname{cyl} \beta_{n+1}) + 2\varepsilon .$$

Hence

$$\begin{aligned} \tau^l(\operatorname{cyl} \alpha) &\leqq l(\operatorname{cyl} f) = \lim_{n \in \omega} l(\operatorname{cyl} f_n) \\ &\leqq \lim_{n \in \omega} l(\operatorname{cyl} h_{n+1}) + \varepsilon \leqq \lim_{n \in \omega} \tau^l(\operatorname{cyl} \beta_{n+1}) + 2\varepsilon . \end{aligned}$$

Thus $\tau^l(\operatorname{cyl} \alpha) \leqq \lim_{n \in \omega} \tau^l(\operatorname{cyl} \beta_n)$ and since certainly the reverse inequality holds, we have $\tau^l(\operatorname{cyl} \alpha) = \lim_{n \in \omega} \tau^l(\operatorname{cyl} \beta_n)$.

Proof of 4.3. Let $l \in \boldsymbol{L}$. By Lemma E, τ^l satisfies conditions (a) and (b) and hence by 3.3.1 the Carathéodory outer measure μ^l on X generated by τ^l_* and $\mathscr{G}$ is in $\boldsymbol{M}$.

Now suppose $f \in \boldsymbol{F}$. By definition there exists $i \in I$ and $h \in C_0(X_i)$ such that $f = \operatorname{cyl} h$. If for every $A \subset X_i$ we let $\mu_i^l(A) = \mu^l(\operatorname{cyl} A)$ then $\int \operatorname{cyl} h d\mu^l = \int h d\mu_i^l$. If for $\alpha \in \mathscr{K}_i$ we let $\tau_i^l(\alpha) = \tau^l(\operatorname{cyl} \alpha)$ and let ν_i^l be the topological outer measure on X_i cranked by τ_i^l, then by 3.3(2), ν_i^l is a Radón outer measure on X_i and μ_i^l agrees with ν_i^l on all ν_i^l-measurable sets. Hence since $h \in C_0(X_i)$ we have

$$\int h d\nu_i^l = \int h d\mu_i^l .$$

Furthermore if $l_i(g) = l(\operatorname{cyl} g)$ for $g \in C_0(X_i)$ then l_i is a positive continuous linear functional on $C_0(X_i)$ and by Lemma C

$$\tau_i^l(\alpha) = \inf \{l_i(g): 1_\alpha \leqq g \in C_0(X_i)\} \text{ for } \alpha \in \mathscr{K}_i .$$

Hence by the Riesz Representation Theorem l_i and ν_i^l satisfy the relationship

$$l_i(g) = \int g d\nu_i^l \text{ for all } g \in C_0(X_i) .$$

$$\begin{aligned} \text{Hence } l(f) = l(\operatorname{cyl} h) = l_i(h) &= \int h d\nu_i^l = \int h d\mu_i^l \\ &= \int \operatorname{cyl} h d\mu^l = \int f d\mu^l . \end{aligned}$$

To show uniqueness, suppose $\mu \in \boldsymbol{M}$ and $l(f) = \int f d\mu$ for all $f \in \boldsymbol{F}$. For each $i \in I$ let $\mu_i(A) = \mu(\text{cyl}\, A)$ for $A \subset X_i$, $\tau_i(\alpha) = \mu(\text{cyl}\, \alpha)$ for $\alpha \in \mathscr{K}_i$ and ν_i be the topological outer measure on X_i cranked by τ_i. By 3.3(2), ν_i is a Radón outer measure on X_i and μ_i agrees with ν_i on $\mathscr{M}_{\nu_i}$, hence also on $\mathscr{B}_i$. Furthermore for all $f \in C_0(X_i)$

$$\int f d\nu_i = \int f d\mu_i = l(\text{cyl}\, f) = \int f d\mu_i^l = \int f d\nu_i^l$$

and therefore by the Riesz representation theorem $\nu_i = \nu_i^l$. It follows that μ_i and μ_i^l agree on $\mathscr{B}_i$. Hence the projective systems $\{\mu_i \mid \mathscr{B}_i : i \in I\}$ and $\{\mu_i^l \mid \mathscr{B}_i : i \in I\}$ are equal and so their respective projective limit measures, which by 3.4.1 are $\mu \mid \mathscr{B}$ and $\mu^l \mid \mathscr{B}$, are also equal. Since $\mathscr{G} \subset \mathscr{B}$ we have that μ and μ^l agree on $\mathscr{G}$ and so $\mu = \mu^l$. The mapping $l \to \mu^l$ is now clearly an isomorphism between $\boldsymbol{L}$ and $\boldsymbol{M}$.

5. Example to show that σ-compactness of the coordinate spaces is needed. Let $\boldsymbol{R}$ have the discrete topology (which is not σ-compact) and consider $\boldsymbol{R}^2$ with the product topology. For $h \in C_0(\boldsymbol{R})$ and $x \in \boldsymbol{R}^2$, let $(\text{cyl}_1\, h)(x) = h(x_1)$ and $(\text{cyl}_2\, h)(x) = h(x_2)$.

Let $\boldsymbol{F}_0 = C_0(\boldsymbol{R}^2)$
$\boldsymbol{F}_1 = \{f : f = \text{cyl}_1\, h \text{ for some } h \in C_0(\boldsymbol{R})\}$
$\boldsymbol{F}_2 = \{f : f = \text{cyl}_2\, h \text{ for some } h \in C_0(\boldsymbol{R})\}$.

Using the notations of this paper, we let $T = \{1, 2\}$, $Y_1 = Y_2 = \boldsymbol{R}$ with the discrete topology and define X, $\mathscr{C}$, $\mathscr{G}$, $\boldsymbol{M}$, $\boldsymbol{F}$ and $\boldsymbol{L}$ as before.

First we note that $\boldsymbol{F} = \boldsymbol{F}_0 \cup \boldsymbol{F}_1 \cup \boldsymbol{F}_2$ and that since pairwise intersections of $\boldsymbol{F}_0$, $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ consist of the zero element only, every f in the linear span of $\boldsymbol{F}$ has a unique representation as $f = f_0 + f_1 + f_2$ where $f_n \in \boldsymbol{F}_n$ for $n = 0, 1, 2$. For fixed $z \in \boldsymbol{R}^2$ (which equals X) define l by $l(f) = f_0(z) + 2f_1(z) + 2f_2(z)$ for f in the linear span of $\boldsymbol{F}$. Then $l \in \boldsymbol{L}$ but we shall show that there is no $\mu \in \boldsymbol{M}$ such that $l(f) = \int f d\mu$ for all $f \in \boldsymbol{F}$. Suppose we did find such a $\mu \in \boldsymbol{M}$. Then if $A = \{x \in \boldsymbol{R}^2 : x_1 = z_1\}$ we have $1_A \in \boldsymbol{F}_1 \subset \boldsymbol{F}$ and hence

$$\mu(A) = \int 1_A d\mu = l(1_A) = 2 \cdot 1_A(z) = 2 .$$

We next note that $1_{\{z\}} \in \boldsymbol{F}_0 \subset \boldsymbol{F}$ and so

$$\mu(\{z\}) = \int 1_{\{z\}} d\mu = l(1_{\{z\}}) = 1_{\{z\}}(z) = 1 .$$

Furthermore since A, $\{z\}$ and $A - \{z\}$ are all μ-measurable we have

$$\mu(A - \{z\}) = \mu(A) - \mu(\{z\}) = 2 - 1 = 1$$

On the other hand $A - \{z\} \subset \boldsymbol{R}^2 - \{z\}$ which is in $\mathscr{G}$.
Hence

$$\begin{aligned}\mu(A - \{z\}) &\leqq \mu(\boldsymbol{R}^2 - \{z\}) \\ &= \sup\{\mu(\alpha)\colon \alpha \in \mathscr{C} \text{ and } \alpha \subset \boldsymbol{R}^2 - \{z\}\} \\ &= \sup\{l(1_\alpha)\colon \alpha \in \mathscr{C} \text{ and } \alpha \subset \boldsymbol{R}^2 = \{z\}\} = 0\end{aligned}$$

since $l(1_\alpha) = 0$ for any $\alpha \in \mathscr{C}$ with $z \notin \alpha$. Hence $A - \{z\}$ would have to have measure zero and one simultaneously, which is impossible.

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