

## A NOTE ON THE THEORY OF PRIMES

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In this paper we find those commutative rings for which the theory of primes is subsumed under classical ideal theory, that is, for which every finite prime is an ideal. The characterization is given in terms of domains with this property and they are shown to form a class of domains from number theory. In addition we give two characterizations of the primes of a subring of a global field. The first establishes them as the nontrivial preprimes whose complements are multiplicatively closed and the second relates the space of all primes to that of the quotient field.

The concept of a prime for commutative rings with identity was introduced by Harrison in 1966.

In what follows all rings are commutative and have a unity and all primes are finite.  $X(R)$  denotes the set of primes of a ring  $R$  and  $X'(R)$  denotes the set of valuation preprimes (preprimes  $T$  such that for each finite  $E \subset R$ ,  $T \cap E = \emptyset \Rightarrow$  there is  $P \in X(R)$  with  $T \subset P$  and  $P \cap E = \emptyset$ ). For a preprime  $T$  of  $R$  which is closed under subtraction, define the *idealizer*  $A(T)$  of  $T$  in  $R$  by  $A(T) = \{a \in R: aT \subset T\}$ .  $A(T)$  is a subring of  $R$  in which  $T$  is an ideal.

1. Call a ring a *C-ring* if every finite prime of it is an ideal. It is easy to check that the class of *C-rings* is closed under taking subrings and homomorphic images.

**THEOREM 1.** *The following are equivalent for a ring  $R$ :*

- (1)  $R$  is a *C-ring*;
- (2)  $X(R) = \{\text{maximal ideals of } R\}$ ;
- (3)  $R/P$  is a *C-ring*, for each minimal prime ideal  $P$  of  $R$ ;
- (4)  $X'(R) = \text{Spec}(R)$ .

*Proof.* That (4)  $\Rightarrow$  (2)  $\Rightarrow$  (1)  $\Rightarrow$  (3) is clear. In any case,  $\text{Spec}(R) \subset X'(R)$  [1, Lemma 2.6]. Let  $P \in X'(R)$ .  $P$  contains a minimal prime ideal  $Q$  of  $R$  and  $P/Q \in X'(R/Q)$ . Then  $P/Q$  is the intersection of the primes of  $R/Q$  which contain it; so, if  $R/Q$  is a *C-ring*, then  $P/Q \in \text{Spec}(R/Q)$  and  $P \in \text{Spec}(R)$ .

Because of condition (3), we turn to the classification of *C-domains*. If  $S$  denotes the ring of rational integers or a ring of polynomials in one variable over a finite field, then one checks that the polynomial

ring  $S[X]$  has primes which are not ideals. Thus, since a subring of a  $C$ -ring is also a  $C$ -ring, the transcendence degree of a  $C$ -domain is zero or one, the latter only if it has nonzero characteristic.

Let  $R$  denote a domain whose quotient field  $F$  is absolutely algebraic and has characteristic zero. Let  $R'$  denote the integral closure of  $R$  and  $S'$  the integral closure in  $F$  of its prime subring. Let  $S = R \cap S'$ .  $S'$  (and hence  $S$ ) is a  $C$ -domain and  $P \rightarrow P \cap S'$  gives a bijection from  $X(F)$  onto  $X(S')$ . In fact, the finite dimensional case follows from Proposition 3.4 of [1] and the general case follows immediately from it. Since for  $P \in X(F)$ ,  $P \cap R'$  is an ideal of  $R'$  if and only if  $R' \subset A(P)$ , we have that  $R'$  is a  $C$ -domain if and only if  $R' = S'$ . Now  $S' \subset R' \subset F$  yields that  $P \rightarrow P \cap S'$  gives a bijection from  $X(R')$  onto  $X(S')$  and  $P \rightarrow P \cap R'$  gives a bijection from  $X(F)$  onto  $X(R')$ . Since  $S \subset S'$  are both  $C$ -domains,  $P \cap S \in X(S)$  for any  $P \in X(R')$ .

LEMMA 2. *If  $P \in X(F)$ , then  $P \cap R \in X(R)$ .*

*Proof.* Note that it suffices to consider the case where  $F$  is finite dimensional over its prime subfield. Moreover, since  $P \cap R' \in X(R')$  for any  $P \in X(F)$ , as already noted, it suffices to show that  $P \cap R \in X(R)$  for  $P \in X(R')$ . We may assume that  $R \neq R'$ , whence  $S \neq S'$ . Let  $c(S)$  (resp.  $c(R)$ ) denote the conductor of  $S$  in  $S'$  (resp.  $R$  in  $R'$ ). Recall that  $(0) \neq c(S) \subset c(R)$ . Now let  $P \in X(R')$  and just suppose that  $P \cap R$  is not in  $X(R)$ . Then  $P \cap R \subset T \in X(R)$ , since  $P \cap R$  is a preprime of  $R$ . But  $T$  is a preprime of  $R'$ , so  $T \subset Q \in X(R')$ . Then  $P \cap R \subset T = Q \cap R$ . Since  $P \cap R \neq T$ , we have  $P \neq Q$  and  $P \cap S' \neq Q \cap S'$ . Now  $P \cap S = P \cap R \cap S \subset Q \cap R \cap S = Q \cap S$ . But  $P \cap S' \in X(S')$  yields  $P \cap S = (P \cap S') \cap S \in X(S)$ . Thus  $P \cap S = Q \cap S$ . Hence, by [2, p. 91],  $c(S) \subset P \cap Q$ . By the approximation theorem, there is  $a \in F$  whose (normalized exponential) value is  $-1$  at  $P$ ,  $+1$  at  $Q$ , and nonnegative otherwise. Note that  $a$  is in  $R'$ . Let  $0 \neq b \in c(S) \subset P \cap Q$  and let  $n$  be its (positive) value at  $P$ . Then  $ba^n \in Q \cap S'$ . Also  $ba^n \in c(S)R' \subset c(R)R' \subset R$ . Thus  $ba^n \in Q \cap S' \cap R = Q \cap S = P \cap S$ . But this a contradiction, since  $ba^n$  was constructed to have value zero at  $P$ .

Now assume that  $R$  is a  $C$ -domain. We will show that then  $R'$  is a  $C$ -domain, so  $R' = S'$  and  $R$  is absolutely integral. Let  $P \in X(R')$ . We must show that  $A(P) = R'$ . If  $R = R'$  we are done, so we assume that  $R \neq R'$ . By the lemma,  $P \cap R \in X(R)$ . But then  $A(P) \cap R = A(P \cap R) = R$ , so  $R \subset A(P)$ . Since  $A(P)$  is integrally closed in  $R'$ , it is integrally closed in  $F$ . Then  $R \subset A(P)$  yields  $R' \subset A(P)$ , whence  $A(P) = R'$ .

We have shown that a domain of characteristic zero is a  $C$ -domain if and only if it is absolutely integral. An absolutely integral domain of nonzero characteristic is always a  $C$ -domain (such a domain has a unique prime and a unique proper ideal, namely  $(0)$ ). We are faced with determining which domains of nonzero characteristic and transcendence degree one are  $C$ -domains. This case is handled in a fashion similar to the above case with the usual care necessary to handle a finite set of primes (those arising from the "infinite" valuations of the quotient field). If  $R$  is such a domain,  $F$  its quotient field, and  $R'$  its integral closure, then  $P \rightarrow P \cap R'$  gives a bijection from  $X(F)$  onto  $X(R')$  or onto  $X(R') \cup \{(0)\}$ , depending on whether there is not or is a unique valuation ring of  $F$  which does *not* contain  $R$ . For  $P \in X(F)$ ,  $P \cap R \in X(R)$  unless there is a unique valuation ring of  $F$  which does not contain  $R$  and  $P$  is its maximal ideal, in which case  $P \cap R = (0)$ . Now if  $R$  is a  $C$ -domain, then so is  $R'$ . But then  $R'$  (or equivalently  $R$ ) is contained in all but one valuation ring of  $F$ . We omit the details. This completes the classification of  $C$ -domains.

**THEOREM 3.**  *$R$  is a  $C$ -domain if and only if  $R$  is absolutely integral or  $R$  has nonzero characteristic, has transcendence degree one, and is contained in all but one valuation ring of its quotient field.*

**REMARKS.** The hypothesis on the transcendence degree of  $R$  is superfluous in Theorem 3, but we admit it to emphasize that  $C$ -domains lie in the realm of number theory. Since the Krull dimension of a  $C$ -domain is zero or one, the same is true for  $C$ -rings. Using Proposition 2.11 of [1], it is easy to check that  $X(R) = \text{Spec}(R)$  if and only if  $R/\sqrt{0}$  is a generalized Boolean ring. Theorem 1 answers a question arising in the theory of valuations of commutative rings introduced by Manis [3], namely, every valuation of a commutative ring is trivial if and only if it is a  $C$ -ring.

2. Call a domain a *global ring* if its quotient field is a global field, that is, a finitely generated field of adjusted transcendence degree one (equals transcendence degree plus one or zero depending on whether the characteristic is zero or not). We seek to characterize the primes of a global ring. Since the infinite primes are easily seen to arise in the same way and correspond exactly to the infinite primes of its quotient field [1, Propositions 3.5 and 3.6], we will consider only finite primes.

Let  $R$  denote any domain of adjusted transcendence degree one

and  $F$  its quotient field. Let  $T$  denote a nonzero preprime of  $R$  whose complement in  $R$  is multiplicatively closed. Then  $T$  is closed under subtraction,  $A = A(T)$  also has adjusted transcendence degree one, and the complement of  $T$  in  $A$  is multiplicatively closed. Thus  $T$  is a maximal ideal of  $A$  and a (finite) prime of  $A$ . We seek to show that  $T$  is a prime of  $R$ . We may assume that  $A \neq R$ , i.e., that  $T$  is not an ideal of  $R$ . Let  $S$  and  $B$  denote the integral closures in  $F$  of  $R$  and  $A$  respectively. Then  $B \subset S$ ,  $R \cap B = A$ , and  $B \neq S$ .

Let  $E(R)$  denote the set of valuation rings of  $F$  which do *not* contain  $R$ . Since  $B \subset S$  and  $B \neq S$ ,  $E(R)$  is not empty. It suffices to consider the case where  $R$  is a global ring and  $E(R)$  is finite. In fact, assume that  $T$  is properly contained in a prime  $P$  of  $R$ . Let  $a \in P$ ,  $a \notin T$ , and  $0 \neq b \in T$ . Let  $R_1$  denote the subring of  $R$  generated by  $a$  and  $b$ . Then  $R_1$  is a global ring (otherwise  $a$  is a root of unity and  $1 \in P$ ).  $E(R_1)$  is the set of valuation rings of the quotient field of  $R_1$  which exclude  $a$  or  $b$ , so  $E(R_1)$  is finite.  $T \cap R_1$  is a nonzero preprime of  $R_1$  whose complement in  $R_1$  is multiplicatively closed, and  $T \cap R_1$  is properly contained in the preprime  $P \cap R_1$ .

We omit the proof of the following lemma since it is a straightforward application of the approximation theorem and the fact that the conductor of a global ring in its integral closure is not zero.

LEMMA 4. *Let  $L$  denote a global ring and  $V_1, \dots, V_n$  a set of valuation rings of the quotient field of  $L$  which do not contain  $L$ . Then  $L \cap V_1 \cap \dots \cap V_n$  is an irredundant intersection.*

Let  $M = M(T, R) = \{V \in E(R) : A \subset V\}$ . Since  $B \neq S$ ,  $M$  is not empty. Let  $M = \{V_1, \dots, V_n\}$ . Then  $B = S \cap V_1 \cap \dots \cap V_n$  and  $A = R \cap V_1 \cap \dots \cap V_n$ . Let  $P_i$  denote the maximal ideal of  $V_i \in M$  and let  $N = \{P_1, \dots, P_n\}$ . Recall that  $N \subset X(F)$ .

LEMMA 5.  *$T = A \cap P$ , for  $P \in N$ .*

*Proof.* Just suppose that  $T \neq A \cap P$ , for all  $P \in N$ . Let  $Q \in X(F)$  with  $T \subset Q$ . Since  $T = A \cap Q$  and  $T$  is an ideal and a prime of  $A$ ,  $A \subset A(Q)$ .  $Q \notin N$ , so  $A(Q) \notin E(R)$ . Thus  $R \subset A(Q)$  and  $R \cap Q$  is a maximal ideal of  $R$ . Since  $T$  is not an ideal of  $R$ ,  $R \cap Q \neq T = A \cap Q$ . Let  $b \in R \cap Q$  with  $b \notin A \cap Q$ . Now, for  $P_i \in N$ ,  $A \cap P_i$  is a maximal ideal of  $A$  and since  $A \cap P_i \neq T = A \cap Q$ , there is  $a_i \in A \cap P_i$  with  $a_i \notin A \cap Q$ . Let  $a = a_1 a_2 \dots a_n$ . Let  $w$  (resp.  $v_i$ ) denote the normalized exponential valuation of  $F$  associated with  $Q$  (resp.  $P_i$ ). Then  $w(b) > 0$ ,  $w(a) = 0$ , and  $v_i(a) > 0$ . We fix  $m > 0$  so that  $v_i(a^m b) > 0$ , for  $1 \leq i \leq n$ . Then  $a^m b \in R \cap V_1 \cap \dots \cap V_n \cap Q = A \cap Q = T$ . But

$a, b \in R$ , so  $a^mb \in T$  implies that  $a \in T$  or  $b \in T$ . By this contradiction  $T = A \cap P$ , for some  $P \in N$ .

Now reindex  $N$  if necessary so that  $T = A \cap P_i$  for  $1 \leq i \leq s$ , and  $T \neq A \cap P_i$  for  $s < i \leq n$ . Just suppose  $s < n$ . Let  $R_1 = R \cap V_1 \cap \dots \cap V_s$ .  $T$  is a preprime of  $R_1$  with multiplicatively closed complement in  $R_1$ . By Lemma 4,  $A \neq R_1$  and  $M(T, R_1) = \{V_{s+1}, \dots, V_n\}$ . Applying the first part of this proof to  $T$  and  $R_1$ , we get  $T = R_1 \cap P_i$ , for some  $s < i \leq n$ . But then  $T = T \cap R = R \cap P_i$ , a contradiction.

**LEMMA 6.** *Let  $L$  denote a global ring,  $T$  a nonzero preprime of  $L$  with a multiplicatively closed complement in  $L$ , and  $A$  the idealizer of  $T$  in  $L$ . Assume that  $E(L)$  is finite and that  $M(T, L) = \{V\}$ . Then  $T \in X(L)$ .*

*Proof.* Let  $P$  denote the maximal ideal of  $V$ . Then Lemma 5 yields  $T = A \cap P$ . But  $M(T, L) = \{V\}$  yields  $A = L \cap V$ . Thus  $T = L \cap V \cap P = L \cap P$ . But if  $L$  has characteristic zero, Lemma 2 yields  $T = L \cap P \in X(L)$ . If  $L$  has nonzero characteristic, we consider  $E(L)$ . At least  $V \in E(L)$ . If  $E(L) = \{V\}$ , then  $A = L \cap V$  is a finite field and  $T = (0)$ . Hence  $E(L)$  is not a singleton and  $T = L \cap P \in X(L)$ .

We can now give the first characterization.

**PROPOSITION 7.** *Let  $R$  be a domain of adjusted transcendence degree one. A necessary and sufficient condition that a nonzero preprime of  $R$  be a prime is that its complement be multiplicatively closed.*

*Proof.* Necessity holds for arbitrary commutative rings. [1, Proposition 2.1]. To prove the sufficiency, we have already noted that we may assume that  $R$  is a global ring, that  $E(R)$  is finite, and that the idealizer  $A$  of  $T$  in  $R$  is not  $R$ . In the notation established above,  $M(T, R) = \{V_1 \dots, V_n\}$ . By Lemma 6, we need only show that  $n = 1$ . Just suppose  $n \geq 2$ . Let  $L = R \cap V_s \cap \dots \cap V_n$ . Then  $A = L \cap V_1 \cap V_2$ . Let  $R_1 = L \cap V_1$ , so  $A = R_1 \cap V_2$ . By Lemma 6 applied to  $R_1$  and  $T$ , we have that  $T \in X(R_1)$ ; and by Lemma 5,  $T = A \cap P_1 \subset R_1 \cap P_1$ , so  $T = P_1 \cap R_1$ . Then, since  $A$  is the idealizer of  $T$  in  $R_1$ ,  $A = V_1 \cap R_1 = R_1$ , in contradiction to Lemma 4.

For a commutative ring  $R$ ,  $X(R)$  is topologized by taking the sets  $U(a) = \{P \in X(R) : a \notin P\}$ , for all  $a \in R$ , as subbasic open sets. Let  $R$  denote a global ring and  $F$  its quotient field. We fix an element  $x \in R$  so that  $x$  is not absolutely algebraic, if  $\text{char}(R) \neq 0$ , and  $x = 1$ , otherwise. Let  $F_0$  (resp.  $R_0$ ) denote the subfield (resp. subring) of  $F$  generated by  $x$ . Let  $V = U(1/x) \subset X(F)$ . Let  $S$  (resp.  $S_0$ ) denote the

integral closure of  $R$  (resp.  $R_0$ ) in  $F$ . We fix  $c \in R_0$  with  $cS \subset R$  and  $c \neq 0$ . Note that for  $P \in V$ ,  $x \in A(P)$ ; so  $c \in R_0 \subset A(P)$ .

For commutative rings  $K \subset L$ , let  $\pi(L/K)$  denote the map from  $X(L)$  to the power set of  $K$  given by  $\pi(L/K)(P) = P \cap K$ . We have noted that  $\pi(F/S_0)$  restricted to  $V$  is a bijection onto the set of maximal ideals of  $S_0$ , and  $\pi(F_0/R_0)$  restricted to  $\{P \in X(F_0) : 1/x \notin P\}$  is a bijection onto  $X(R_0)$ [cf. 1, Proposition 3.4 ff.]. Thus  $S_0 \subset S \subset F$  implies that  $\pi(F/S)$  restricted to  $V$  is injective and that for  $P \in V$ ,  $\pi(F/S)(P) \neq (0)$ . Then by proposition 7,  $\pi(F/S)(V) \subset X(S)$ . Note that, for  $P, Q \in X(F)$  with  $P \cap R = Q \cap R$ , either both  $P$  and  $Q$  are in  $V$  or both are not. In fact,  $P \in V$  if and only if  $P \cap R_0 \neq (0)$ .

LEMMA 8. *Let  $P, Q \in V$ . If  $P \neq Q$  and  $P \cap R = Q \cap R$ , then  $c \in P \cap Q$ .*

*Proof.* Let  $T = (P \cap S) + (Q \cap S) + (P \cap S)(Q \cap S)$ .  $T$  is closed under addition and multiplication, but  $T$  properly contains  $P \cap S \in X(S)$ . Thus  $T$  is not a preprime of  $S$  and hence  $1 \in T$ . Then  $c \in cT \subset P \cap Q$ .

Since  $X(F)$  is cofinite space,  $C = \{P \in X(F) : 1/x \in P \text{ or } c \in P\}$  is finite. By Lemma 8,  $\pi(F/R)$  restricted to the complement  $C'$  of  $C$  is injective.  $X(R)$  lies in the range of  $\pi(F/R)$ , and we may choose a subset  $D$  of  $C$  on which  $\pi(F/R)$  is injective and so that  $\pi(F/R)(D) = \pi(F/R)(C)$ . Then  $D \cup C'$  is a cofinite set and an open set, and  $\pi(F/R)$  restricted to  $D \cup C'$  is a bijection onto  $X(R)$ . Since  $X(F)$  and  $X(R)$  are cofinite spaces, this map is a homeomorphism. We have proven.

PROPOSITION 9. *Let  $R$  denote a global ring and  $F$  its quotient field. Then  $X(R)$  is homeomorphic to an open (cofinite) subset of  $X(F)$  and the homeomorphism is induced by  $\pi(F/R)$ .*

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