DETERMINATION OF HYPERBOLICITY BY PARTIAL PROLONGATIONS

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In [3] it was shown that non-hyperbolic systems of partial differential equations may sometimes be altered by partial prolongations so they become hyperbolic. This paper solves two problems concerning this process for normal systems with two independent variables. First, if hyperbolicity is obtainable, it can be obtained after a bounded number of steps, the bound depending only on the algebraic structure of the given system and easily calculated. Second, an explicit procedure is described whereby any system which is absolutely equivalent to a hyperbolic system can be changed into a hyperbolic system. In addition much of the underlying algebraic structure of such systems and their partial prolongations is analyzed.

These problems are local in nature, so we generally use local coordinate expressions. Readers familiar with fibre bundle and jet terminology will easily see how to express many concepts in the language of modern differential geometry. *All functions are assumed infinitely differentiable.*

1. **Systems.** Let $\sum$ denote a system of quasi-linear partial differential equations in dependent variables $z^1, \cdots, z^m$ and two independent variables x^1, x^2 which can be solved for the partials $\partial z^\lambda / \partial x^2 = \partial_2 z^\lambda$:

$$\sum \begin{cases} \partial_2 z^\lambda = A^\lambda_\mu \partial_1 z^\mu + B^\lambda, \lambda = 1, \cdots, m , \\ \Theta^\alpha = C^\alpha_\lambda \partial_1 z^\lambda + D^\alpha = 0, \alpha = 1, \cdots, \alpha_1 , \\ f^\beta(x^i, z^\lambda) = 0, \beta = 1, \cdots, \beta_1 . \end{cases}$$

Here we use the summation convention, $\partial_1 z^\lambda = \partial z^\lambda / \partial x^1$, and $A^\lambda_\mu, B^\lambda, C^\alpha_\lambda, D^\alpha, f^\beta$ are (infinitely differentiable) functions of $x^1, x^2, z^1, \cdots, z^m$, on some open set $\bigcup$ in R^{m+2}.

The following notation will be useful. If $F = F(x^1, x^2, z^1, \cdots, z^m, \partial_1 z^1, \cdots, \partial_1 z^m)$ is a function on an open set V of R^{2m+2}, then for $(x^1, x^2, z^1, \cdots, z^m)$ in $\bigcup$, we define

$$\partial_1^* F = F_{x^1} + F_{z^\lambda} \partial_1 z^\lambda + F_{\partial_1 z^\lambda} \partial_{11} z^\lambda ,$$
$$\partial_2^* F = F_{x^2} + F_{z^\lambda} [A^\lambda_\mu \partial_1 z^\mu + B^\lambda]$$
$$+ F_{\partial_1 z^\lambda} \partial_1^* [A^\lambda_\mu \partial_1 z^\mu + B^\lambda] .$$

These are functions of $x^1, x^2, z^1, \cdots, z^m, \partial_1 z^1, \cdots, \partial_1 z^m, \partial_{11} z^1, \cdots, \partial_{11} z^m$ where $\partial_{11} z^\lambda = \partial^2 z^\lambda / \partial x^1 \partial x^1$.

DEFINITION 1. In this paper the collection Σ of equations will be called a *system* when there are equations

(1) $$\partial_2^* \Theta^\alpha = R_\gamma^\alpha \partial_1^* \Theta^\gamma + S_\gamma^\alpha \Theta^\gamma \;,$$

(2) $$\partial_2^* f^\beta = T_\alpha^\beta \Theta^\alpha \;, \qquad \partial_1^* f^\beta = U_\alpha^\beta \Theta^\alpha \;,$$

$\alpha = 1, \cdots, \alpha_1; \beta = 1, \cdots, \beta_1;$ where $R_\gamma^\alpha, S_\gamma^\alpha, T_\alpha^\beta, U_\alpha^\beta$ are functions of x^i, z^λ on U. Then U is the *domain* of Σ. If $z^\lambda = \varphi^\lambda(x^1)$, $x_0^1 \leqq x^1 \leqq x_1^1$ is any solution of $\Theta^\alpha = 0$ for fixed x_0^2, where $f^\beta(x_0^1, x_0^2, \varphi^\lambda(x_0^1)) = 0$, this 1-dimensional solution may be extended to a full solution by solving the initial value problem $\partial_2 z^\lambda = A_\mu^\lambda \partial_1 z^\mu + B^\lambda$, $\lambda = 1, \cdots, m$, $z^\lambda(x^1, x_0^2) = \varphi^\lambda(x')$. Such systems are called "involutive".

We now define for each point (x, z) in $\bigcup$ an *associated vector space* $V(x, z)$ to be that vector space over R with basis $\partial_1 z^1, \cdots, \partial_1 z^m$. We also define an *associated linear transformation* $A(x, z)$:

$$V(x, z) \rightarrow V(x, z) \quad \text{defined by} \quad A(x, z)(\partial_1 z^\lambda) = A_\mu^\lambda(x, z)\partial_1 z^\mu \;.$$

A *vector field* on an open subset W of $\bigcup$ is a smooth assignment of a vector in the vector space at each point of W, that is, $C_\lambda(x, z)\partial_1 z^\lambda$ in $V(x, z)$ where C_λ are infinitely differentiable on W.

PROPOSITION 1. *If Σ is a system then the vector fields $C_\lambda^\alpha(x, z)\partial_1 z^\lambda$, $\alpha = 1, \cdots, \alpha_1$, span a subspace of $V(x, z)$ which is invariant under $A(x, z)$.*

Proof. Comparing coefficients of $\partial_{11} z^\mu$ in equations (1) leads to

$$C_\lambda^\alpha A_\mu^\lambda = R_\gamma^\alpha C_\mu^\gamma \quad \text{all} \quad \alpha, \mu \;.$$

Hence

$$\begin{aligned} A(C_\lambda^\alpha \partial_1 z^\lambda) &= C_\lambda^\alpha A(\partial_1 z^\lambda) \\ &= C_\lambda^\alpha A_\mu^\lambda \partial_1 z^\mu = R_\gamma^\alpha C_\mu^\lambda \partial_1 z^\mu \;, \end{aligned}$$

at each point of $\bigcup$.

This subspace is *the invariant subspace* of Σ at (x, z) and denoted $C(x, z)$.

A change in the dependent variables of Σ results in a similarity transformation on $A(x, z)$, corresponding to a basis change of $V(x, z)$. The invariant subspace $C(x, z)$ is preserved. Thus, A, V and C are invariant under a change in dependent variables.

DEFINITION 2. A system Σ_1 in variables $x^1, x^2, z^1, \cdots, z^m u^1, \cdots, u^p$ is called a *partial prolongation* (which we abbreviate *pp*) of Σ if the projection map $\pi(x, z, u) = (x, z)$ carries its domain $\bigcup_1$ onto the domain

∪ of Σ and if Σ_1 contains the equations of Σ together with equations for each u^π, namely,

$$(3) \qquad u^\pi = E_\lambda^\pi \partial_1 z^\lambda + F^\pi, \pi = 1, \cdots, p,$$

where E_λ^π and F^π depend only on (x, z). Thus Σ_1 contains in addition to the equations of Σ equations (3) and

$$(4) \qquad \partial_2 u^\pi = F_\rho^\pi \partial_1 u^\rho + G_\lambda^\pi \partial_1 z^\lambda + H^\pi, \pi = 1, \cdots, p.$$

Finally, in order to satisfy the Definition 1 of a system (see equations (2)), if $L(x, z, u) = 0$ is any equation which can be deduced from the equations of Σ_1, then $\partial_1^* L = 0$ and $\partial_2^* L = 0$ must be in Σ_1. The pp is *normal* if it can be defined so the vector fields $E_\lambda^\pi \partial_1 z^\lambda$ and $C_\lambda^\alpha \partial_1 z^\lambda$ are linearly independent at each point of ∪. The invariant space associated with Σ_1 at (x, z, u) is spanned by $C_\lambda^\alpha(x, z) \partial_1 z^\lambda, \alpha = 1, \cdots, \alpha_1$, and $E_\lambda^\pi(x, z) \partial_1 z^\lambda, \pi = 1, \cdots, p$.

Partial prolongations are the basic tool we shall use in modifying given systems. It was shown in [3] that there is a natural correspondence between the solutions of Σ and Σ_1. This correspondence is what was behind Élie Cartan's use of the term 'absolute equivalence' for systems which could be related by a sequence of partial prolongations [1]. See [2] for a discussion of these general questions. We first explore some consequences of Definition 2.

PROPOSITION 2. *If at each point of* ∪ *the vectors* $C_\lambda^\alpha \partial_1 z^\lambda$, *and* $E_\lambda^\pi \partial_1 z^\lambda$ *are linearly independent then* F_ρ^π *and* S_α^π *depend only on* (x, z), *not* $u^1, \cdots, u^p$.

Proof. Applying condition (1) of Definition 1 to the equations $\Psi^\pi = u^\pi - E_\lambda^\pi \partial_1 z^\lambda - F^\pi = 0$ in Σ_1 gives $\partial_2^* \Psi^\pi \equiv 0$ modulo $(\partial_1^* \Psi^\rho, \partial_1^* \Theta^\alpha, \Psi^\rho, \Theta^\alpha)$, i.e.,

$$(5) \qquad \begin{aligned} \partial_2^* [u^\pi - E_\lambda^\pi \partial_1 z^\lambda - F^\pi] = {}&R_\rho^\pi \partial_1^* [u^\rho - E_\lambda^\rho \partial_1 z^\lambda - F^\rho] \\ &+ S_\alpha^\pi \partial_1^* [C_\lambda^\alpha \partial_1 z^\lambda + D^\alpha] \\ &+ T_\rho^\pi [u^\rho - E_\lambda^\rho \partial_1 z^\lambda - F^\rho] \\ &+ \mathsf{U}_\alpha^\pi [C_\lambda^\alpha \partial_1 z^\lambda + D^\alpha], \end{aligned}$$

for some $R_\rho^\pi, S_\alpha^\pi, T_\rho^\pi$ and U_α^π. Comparing coefficients of $\partial_1 u^\rho$ and $\partial_{11} z^\lambda$ we have $F_\rho^\pi = R_\rho^\pi$, and

$$(6) \qquad \begin{aligned} -E_\mu^\pi A_\lambda^\mu &= -R_\rho^\pi E_\lambda^\rho + S_\alpha^\pi C_\lambda^\alpha \\ &= -F_\rho^\pi E_\lambda^\rho + S_\alpha^\pi C_\lambda^\alpha. \end{aligned}$$

Using the fact that $A_\lambda^\mu, E_\lambda^\rho$ and C_λ^α are functions of (x, z) alone, if we take the partial derivative of (6) with respect to u^σ we obtain $0 =$

$-F^{\pi}_{\rho},\,_{u^{\sigma}}E^{\rho}_{\lambda} + S^{\pi}_{\alpha},\,_{u^{\sigma}}C^{\alpha}_{\lambda}$.   The linear independence hypothesis now implies the result.

PROPOSITION 3.   *If Σ_1 is a normal pp the invariant subspace $C_1(x, z, u)$ of Σ_1 at (x, z, u) is independent of $u^1, \cdots, u^p$ and as a subspace of $V(x, z)$ is invariant under $A(x, z)$.*

Proof.   Independence of $u^1, \cdots, u^p$ follows from Proposition 2. Invariance follows from equations (6).

Thus, the possible pp's of Σ depend on the subspaces of $V(x, z)$ which are invariant under $A(x, z)$.

DEFINITION 3.   A *change in dependent variables* will mean the obvious system in new variables

$$\bar{z}^{\lambda} = h^{\lambda}(x^1, x^2, z^1, \cdots, z^m), \lambda = 1, \cdots, m$$

defined on $\bigcup$ which must have inverse functions $z^{\lambda} = g^{\lambda}(x^1, x^2, \bar{z}^1, \cdots, \bar{z}^m)$. For example, in Σ_1 of Definition 2 we may replace u^{π} by $\bar{u}^{\pi} = u^{\pi} - F^{\pi}$. Then Σ_1 is a pp on $\bar{u}^{\pi} = E^{\pi}_{\lambda}\partial_1 z^{\lambda}$.

DEFINITION 4.   If Σ contains equations of the form

$$z^{r+k} = f^k(x^1, x^2, z^1, \cdots, z^r), k = 1, \cdots, m - r ,$$

then by Definition 1 it contains

$$\partial_2 z^{r+k} = \sum_{\lambda=1}^{r} f^k,\,_{z^{\lambda}}[A^{\lambda}_{\mu}\partial_1 z^{\mu} + B^{\lambda}] + f^k,\,_{x^2} ,$$

$$\partial_1 z^{r+k} = \sum_{\lambda=1}^{r} f^k,\,_{z^{\lambda}}\partial_1 z^{\lambda} + f^k,\,_{x^1} .$$

The system $\bar{\Sigma}$ obtained from Σ by replacing $z^{r+k}, \partial_1 z^{r+k}, \partial_2 z^{r+k}$ by their expressions in terms of $x^1, x^2, z^1, \cdots, z^r, \partial_1 z^1, \cdots, \partial_1 z^r$ is called an *admissible restriction* of Σ.

PROPOSITION 4.   *If Σ_1 is a pp of Σ and Σ_0 is an admissible restriction of Σ, then there is an admissible restriction of Σ_1, Σ_{10}, which is a pp of Σ_0.*

This follows from the definitions.

PROPOSITION 5.   *If Σ_1 is a normal pp of Σ then the equations of Σ_1 are uniquely determined by the vector fields $E^{\pi}_{\lambda}\partial_1 z^{\lambda}$ up to a change of dependent variables.   That is, G^{z}_{λ} and H^{π} are uniquely determined up to arbitrary linear combinations of the equations $C^{\alpha}_{\lambda}\partial_1 z^{\lambda} + D^{\alpha} = 0$*

and $E_{\tilde{\lambda}}^{\pi}\partial_1 z^{\lambda} + F^{\pi} - u^{\pi} = 0$.   (*We say then that Σ_1 is the pp on $E_{\tilde{\lambda}}^{\pi}\partial_1 z^{\lambda}$.*)

Proof. By changing variables make $F^{\pi} = 0$. Since $E_{\tilde{\lambda}}^{\pi}\partial_1 z^{\lambda}$ and $C_{\tilde{\lambda}}^{\alpha}\partial_1 z^{\lambda}$ are linearly independent, F_{ρ}^{π} and S_{α}^{π} are uniquely determined by equation (6). Comparing coefficients of $\partial_1 z^{\lambda}$ in (5) using (4), we find

$$\begin{aligned}
G_{\tilde{\lambda}}^{\pi} - E_{\tilde{\lambda}}^{\pi},_{x^2} &- E_{\mu}^{\pi}(A_{\lambda}^{\mu},_{x^2} + B^{\mu},_{z\lambda}) \\
&= -R_{\rho}^{\pi}E_{\tilde{\lambda}}^{\rho},_{x^1} + S_{\alpha}^{\pi}(C_{\tilde{\lambda}}^{\alpha},_{x^1} + D^{\alpha},_{z\lambda}) - T_{\rho}^{\pi}E_{\tilde{\lambda}}^{\rho} \\
&\quad + \bigcup_{\alpha}^{\pi}C_{\tilde{\lambda}}^{\alpha} \ .
\end{aligned}$$

Thus, $G_{\tilde{\lambda}}^{\pi}$ is uniquely determined up to a linear combination of $E_{\tilde{\lambda}}^{\rho}$ and $C_{\tilde{\lambda}}^{\alpha}$. Comparing the terms (5) which are independent of $\partial_1 z^{\lambda}$ leads to the same conclusion about H^{π}.

PROPOSITION 6.   *Let Σ_1 be a normal pp of Σ. Let $V_1(x, z, u)$ and $V(x, z)$ be the vector spaces associated with Σ_1 and Σ, respectively. Let $C_1(x, z, u) = C_1(x, z)$ and $C(x, z)$ be the invariant subspaces of Σ_1 and Σ, respectively, and $A_1(x, z, u)$, $A(x, z)$ their linear transformations. Then there exist natural horizontal exact sequences such that the following diagram is commutative:*

$$\begin{array}{ccccccc}
V(x, z)/C(x, z) & \xrightarrow{\ \alpha\ } & V_1(x, z, u)/C_1(x, z) & \xrightarrow{\ \beta\ } & C_1(x, z)/C(x, z) & \longrightarrow & 0 \\
\downarrow{\scriptstyle A(x, z)} & & \downarrow{\scriptstyle A_1(x, z, u)} & & \downarrow{\scriptstyle B(x, z)} & & \\
V(x, z)/C(x, z) & \xrightarrow{\ \alpha\ } & V_1(x, z, u)/C_1(x, z) & \xrightarrow{\ \beta\ } & C_1(x, z)/C(x, z) & \longrightarrow & 0
\end{array}$$

where

$$\begin{aligned}
\alpha(\partial_1 z^{\lambda} + C(x, z)) &= \partial_1 z^{\lambda} + C_1(x, z) \ , \\
\beta(\partial_1 z^{\lambda} + C_1(x, z)) &= 0 + C(x, z) \ , \\
\beta(\partial_1 u^{\pi} + c_1(x, z)) &= E_{\tilde{\lambda}}^{\pi}\partial_1 z^{\lambda} + C(x, z) \ ,
\end{aligned}$$

and $B(x, z)$ is the natural linear transformation induced by $A(x, z)$.

Proof. Let us omit the dependence on (x, z) or (x, z, u). We have $A(\partial_1 z^{\lambda}) = A_{\mu}^{\lambda}\partial_1 z^{\mu} = A_1(\partial_1 z^{\lambda})$, $A_1(\partial_1 u^{\pi}) = F_{\rho}^{\pi}\partial_1 u^{\rho} + G_{\tilde{\lambda}}^{\pi}\partial_1 z^{\lambda}$. Then C_1/C is spanned by $E_{\tilde{\lambda}}^{\pi}\partial_1 z^{\lambda} + C$, where $A(E_{\tilde{\lambda}}^{\pi}\partial_1 z^{\lambda}) = F_{\rho}^{\pi}E_{\tilde{\lambda}}^{\rho}\partial_1 z^{\lambda} + C$. By definition $\beta\alpha = 0$ and the kernel of β is V/C_1 since the $E_{\tilde{\lambda}}^{\pi}\partial_1 z^{\lambda}$ are independent modulo C. This kernel of β is the image of α. Hence exactness. Commutativity is routinely checked.

DEFINITION 5.   The *eigenvalues* of Σ at (x, z) are the eigenvalues of $A(x, z)$ on $V(x, z)/C(x, z)$.

PROPOSITION 7.   *If Σ_1 is a normal pp of Σ then the eigenvalues*

of Σ_1 at (x, z, u) are those of Σ at (x, z).

Proof. Let $w^\gamma = W_\lambda^\gamma \partial_1 z^\lambda$ fill out a basis of $V(x, z)$ together with $v^\alpha = C_\lambda^\alpha \partial_1 z^\lambda$ and $u^\pi = E_\lambda^\pi \partial_1 z^\lambda$. Suppose

$$A(u^\pi) = F_\rho^\pi u^\rho + \tilde{G}_\alpha^\pi v^\alpha, \; A(w^\gamma) = M_\varepsilon^\gamma w^\varepsilon + N_\pi^\gamma u^\pi + T_\alpha^\gamma v^\alpha .$$

The eigenvalues of Σ at (x, z) are those of

$$\begin{pmatrix} F & N \\ O & M \end{pmatrix},$$

where $F = (F_\rho^\pi)$, $N = (N_\pi^\gamma)$, $M = (M_\varepsilon^\gamma)$.

If $y^\pi = \partial_1 u^\pi$, then $A_1(y^\pi) = F_\rho^\pi y^\rho + P_\gamma^\pi w^\gamma + Q_\rho^\pi u^\rho + L_\alpha^\pi v^\alpha$, so the eigenvalues of Σ_1 on V_1/C_1 are those of the

$$\begin{pmatrix} M & P \\ O & F \end{pmatrix}$$

where $P = (P_\gamma^\pi)$.

We have found that a normal pp of Σ is completely determined by vector fields $E_\lambda^\pi \partial_1 z^\lambda$ which together with $C_\lambda^\alpha \partial_1 z^\lambda$ span a subspace of $V(x, z)$ which is invariant under $A(x, z)$. However, there need not be pp's on any given invariant vector fields. This is because of the equations which must be satisfied in (5) for the coefficients of $\partial_1 z^\lambda \partial_1 z^\mu$:

$$-E_\lambda^\pi,_{z^\nu} A_\mu^\nu - E_\mu^\pi,_{z^\nu} A_\lambda^\nu = -R_\rho^\pi (E_\lambda^\rho,_{z^\mu} + E_\mu^\rho,_{z^\lambda})$$
$$-S_\alpha^\pi (C_\lambda^\alpha,_{z^\mu} + C_\mu^\alpha,_{z^\lambda}) .$$

One situation when these equations are always satisfied is when the entire space $V(x, z)$ is used for the pp.

DEFINITION 6. The *total prolongation* (abbreviated *tp*) $P\Sigma$ of Σ is obtained as the pp on vector fields which span all of $V(x, z)$ at each point.

PROPOSITION 8. *The tp of Σ always exists.*

Proof. The system $P\Sigma$ is variables $x^i, z^\lambda, p^1, \cdots, p^m$ is obtained by adding to the equations of Σ the equations

$$(7) \quad \begin{aligned} & p^\lambda = \partial_1 z^\lambda , \\ & \partial_2 p^\lambda = A_\mu^\lambda,_{z^\nu} \partial_1 p^\mu + [A_\mu^\lambda,_{z^\nu}(A_\omega^\nu p^\omega + B^\nu) + A_\mu^\lambda,_{x^2}] p^\mu \\ & \qquad + B^\lambda,_{z^\mu} p^\mu + B^\lambda,_{x^2}, \lambda = 1, \cdots, m, \\ & C_\lambda^\alpha \partial_1 p^\lambda + (\partial_1^* C_\lambda^\alpha) p^\lambda + \partial_1^* D^\alpha = 0, \alpha = 1, \cdots, \alpha_1 . \end{aligned}$$

The last set of equations is added because $p\Sigma$ contains $C_\lambda^\alpha p^\lambda + D^\alpha = 0$ in order to satisfy conditions (2) in Definition 1 for functional equations in systems. It is a calculation to check that equations (7) together with those in Σ constitute a system.

The tp is not normal, but using the equations $C_\lambda^\alpha p^\lambda + D^\alpha = 0$ it is possible to find an admissible restriction of $P\Sigma$ which is a normal pp of Σ. The invariant subspace associated with $P\Sigma$ is spanned by $\partial_1 z^1, \cdots, \partial_1 z^m$ and $C_\lambda^\alpha \partial_1 p^\lambda, \alpha = 1, \cdots, \alpha_1$.

PROPOSITION 9. *Let V, C and A denote the vector space, invariant subspace and linear transformation associated with Σ at a point (x, z). Let PV, PC and PA denote the same entities for $P\Sigma$ at (x, z, p). There is a natural isomorphism $\pi: V/C \to PV/PC$ satisfying $\pi A = (PA)\pi$. In particular subspaces of PV/PC invariant under PA correspond in a one-to-one way to those of V/C invariant under A.*

Proof. Let $\pi(\partial_1 z^\lambda + C) = \partial_1 p^\lambda + PC$. The rest is a calculation.

PROPOSITION 10. *If Σ_0 is an admissible restriction of Σ, then $P\Sigma_0$ is an admissible restriction of $P\Sigma$.*

This follows from a calculation. The next result shows an important use of tp's.

PROPOSITION 11. *Let $E_\lambda^\pi \partial_1 z^\lambda, \pi = 1, \cdots, p$, be vector fields on the domain of Σ which, together with $C_\lambda^\alpha \partial_1 z^\lambda$, span a subspace of $V(x, z)$ at each point which is invariant under $A(x, z)$. Then $E_\lambda^\pi \partial_1 p^\lambda = \pi(E_\lambda^\pi \partial_1 z^\lambda)$ satisfy a similar condition in $PV(x, z, p)$. Moreover, there always exists a pp of $P\Sigma$ on the vector fields $E_\lambda^\pi \partial_1 p^\lambda$.*

Proof. If $A(E_\lambda^\pi \partial_1 z^\lambda) = F_\rho^\pi E_\lambda^\rho \partial_1 z^\lambda + S_\alpha^\pi C_\lambda^\alpha \partial_1 z^\lambda$, let

$$u^\pi = E_\lambda^\pi \partial_1 p^\lambda \ ,$$
$$\partial_2 u^\pi = F_\rho^\pi \partial_1 u^\rho + (\partial_2^* E_\lambda^\pi)\partial_1 \rho^\lambda$$
$$+ E_\lambda^\pi[(\partial_1^* A_\mu^\lambda)\partial_1 p^\mu + p^\mu \partial_1^*(\partial_2^* A_\mu^\lambda) + (\partial_2^* A_\mu^\lambda)\partial_1 p^\mu + \partial_1^*(\partial_2^* B^\lambda)]$$
$$- F_\rho^\pi(\partial_1^* E_\lambda^\rho)\partial_1 p^\lambda \ .$$

Observe that $\partial_1^* A_\mu^\lambda, \partial_2^* A_\mu^\lambda, \partial_2^* B^\lambda, \partial_1^* E_\lambda^\rho$ can be expressed in terms of x, z, p using $\partial_1 z^\lambda = p^\lambda$, while $\partial_1^*(\partial_2^* A_\mu^\lambda)$ and $\partial_1^*(\partial_2^* B^\lambda)$ are linear in $\partial_1 p^\nu$. Consequently the above equation is linear in $\partial_1 p^\nu$.

PROPOSITION 12. *Let Σ_1 be a normal pp of Σ on $E_\lambda^\pi \partial_1 z^\lambda, \pi = 1, \cdots, p$. Let $(P\Sigma)_1$ be the pp of $P\Sigma$ on the $E_\lambda^\pi \partial_1 p^\lambda$. Then $(P\Sigma)_1$ is an admissible restriction of $P\Sigma_1$ and is a normal pp of $P\Sigma$.*

This is merely a calculation.

COROLLARY. *If $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ is a sequence of systems where each Σ_i is a normal pp of Σ_{i-1}, $i = 1, \cdots, h$, then there exists a sequence $\Sigma_0' = P\Sigma_0 \subset \Sigma_1' \subset \cdots \subset \Sigma_h'$ where Σ_i' is a normal pp of Σ_{i-1}' and $P\Sigma_i$ has Σ_i' as an admissible restriction, $i = 1, \cdots, h$.*

PROPOSITION 13. *If Σ_1 and Σ_2 are pp's of Σ, then $\Sigma_1 \cup \Sigma_2$ determines a pp of Σ, Σ_1 and Σ_2.*

Proof. All that needs to be added to $\Sigma_1 \cup \Sigma_2$ are equations $\partial_1^* L = 0, \partial_2^* L = 0$ for any equation $L(x, z) = 0$ which can be deduced from the equations in Σ_1 and Σ_2.

PROPOSITION 14. *If Σ_1 is a pp of Σ using vector fields which lie in the invariant subspace $C(x, z)$ of Σ at each point, then Σ_1 has Σ as an admissible restriction.*

Proof. Suppose Σ_1 contains

$$u^\pi = E_\lambda^\pi \partial_1 z^\lambda = K_\alpha^\pi C_\lambda^\alpha \partial_1 z^\lambda \ .$$

Then Σ_1 contains the equations

$$u^\pi = -K_\alpha^\pi D^\alpha$$

so the u^π may be eliminated by an admissible restriction (see Definition 4).

2. Hyperbolicity.

DEFINITION 7. Suppose that in a neighborhood of each point in the domain of Σ there exist linearly independent vector fields $v^\pi = v_\lambda^\pi(x, z)\partial_1 z^\lambda$, $\pi = 1, \cdots, m - \alpha_1$, which together with $C_\lambda^\alpha \partial_1 z^\lambda$ span $V(x, z)$ at each point. Suppose further that

$$A_\mu^\lambda(x, z)v_\lambda^\pi(x, z) \equiv K^\pi v_\mu^\pi(x, z)$$

modulo $C(x, z)$. That is, $v^\pi + C$ is an eigenvector of A on V/C at each point with eigenvalue K^π. Then Σ is called *hyperbolic*.

DEFINITION 8. Let there exist a sequence $\Sigma \subset \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_h$ of systems in which each system is a *pp* of the preceding system and in which Σ_h is hyperbolic. Then Σ is *absolutely equivalent* to a hyperbolic system.
In this event Σ has the same behavior as a hyperbolic system [3].

PROPOSITION 15. *Hyperbolicity is preserved under admissible restrictions and total prolongations.*

Proof. If $z^{r+\lambda} = f^{r+\lambda}(x^1, x^2, z^1, \cdots, z^r)$, $\lambda = 1, \cdots, m - r$, define an admissible restriction, then $C(x, z)$ contains $\partial_i z^{r+\lambda} - \partial_1^* f^{r+\lambda} = 0$. Hence v^π must be linearly independent when $z^{r+\lambda}$ and $\partial_i z^{r+\lambda}$ are replaced by $f^{r+\lambda}$ and $\partial_1^* f^{r+\lambda}$, respectively. The second assertion follows from Proposition 9.

PROPOSITION. 16. *If Σ is absolutely equivalent to a hyperbolic system the eigenvalues of Σ are real at each point.*

Proof. Proposition 7.

DEFINITION 9. A vector $v \in V(x, z)$ is *essential* if for some eigenvalue λ of Σ at (x, z) there is a vector $u \in V(x, z)$ such that
 (1) $[A(x, z) - \lambda I]u \equiv v$,
 (2) $[A(x, z) - \lambda I]^s v \equiv 0, s \geqq 1$,
 (3) $v \not\equiv 0$ modulo $C(x, z)$.
(Here I is the identity.) The essential vectors belonging to an eigenvalue λ generate a vector space $V_\lambda(x, z) \supset C(x, z)$. Let

$$\rho(\Sigma)(x, z) = \sum_\lambda \dim V_\lambda(x, z)/C(x, z)$$

be called the *degeneracy* of Σ at (x, z).
 When Σ is hyperbolic, $\rho(\Sigma) = 0$ at all points.

THEOREM 1. *Let $\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ be a sequence of normal pp's leading to a hyperbolic system Σ_h. Let*

$$(x, z), (x, z, u_1), \cdots, (x, z, u_1, \cdots, u_h)$$

be points in their domains, let $V_j(x, z, u_1, \cdots, u_j) \supset C_j(x, z, u_1, \cdots, u_j)$ be the associated and invariant spaces of Σ_j, where

$$V(x, z) \subset V_1(x, z, u_1) \subset \cdots \subset V_h(x, z, u_1, \cdots, u_h) \ .$$

Then if $v \in V(x, z)$ is an essential vector of Σ, $v \in C_h(x, z, u_1, \cdots, u_h)$.

Proof. By Proposition 6, if $A_h(x, z, u_1, \cdots, u_h)$ is the linear transformation of Σ_h, then $A_h(x, z, u_1, \cdots, u_h) \mid V(x, z) \equiv A(x, z)$ modulo $C(x, z)$. Hence v satisfies (1) and (2) of Definition 9 as an element in V_h. Being hyperbolic Σ_h can have no essential vectors, so $v \in C_h$.

PROPOSITION 17. *Let $\Sigma \subset \Sigma_1 \cdots \subset \Sigma_h$ be a sequence of normal pp's where Σ_h is hyperbolic. Let $v^\pi = v_\lambda^\pi \partial_i z^\lambda$ be linearly independent vector fields of Σ modulo $C(x, z)$ which are essential and which generate a subspace of $V(x, z) / C(x, z)$ invariant under $A(x, z)$. Let $\bar{\Sigma}$ be the pp of Σ on the vector fields v^π. Then there exists a sequence of pp's $\Sigma \subset \bar{\Sigma} \subset \bar{\Sigma}_1 \cdots \subset \bar{\Sigma}_h$ where $\bar{\Sigma}_1$ is a pp of Σ and $\bar{\Sigma}_h$ has Σ_h as an admissible restriction, hence is hyperbolic.*

Proof. Set $\bar{\Sigma}_j = \Sigma_j \cup \bar{\Sigma}$ and use Proposition 13 repeatedly. The hyperbolicity of $\bar{\Sigma}_h$ follows from Theorem 1 and Proposition 14.

As a result of this proposition it can be supposed that any sequence of pp's leading to a hyperbolic system begins with a pp on essential vectors. We shall show that this sort of pp is the optimal way of obtaining hyperbolicity. We next examine more closely those essential vectors.

DEFINITION 10. A system Σ has a *Jordan basis* if there exist on its domain vector fields

$$U_k^s = U_{k\lambda}^s(x, z)\partial_1 z^\lambda , \qquad 1 \leqq s \leqq m_k , \qquad k = 1, \cdots, r$$

and eigenvalues $\lambda_k(x, z)$ of $A(x, z)$ modulo $C(x, z)$ such that

$$[A(x, z) - \lambda_k(x, z)I]u_k^1(x, z) \equiv 0 \text{ modulo } C(x, z) ,$$

$$[A(x, z) - \lambda_k(x, z)I]u_k^s(x, z) \equiv u_k^{s-1}(x, z) \text{ modulo } C(x, z) ,$$

for $2 \leqq s \leqq m_k, 1 \leqq k \leqq r$, and these vector fields determine a basis of $V(x, z)/C(x, z)$. We also assume the functions $\lambda_k(x, z)$ are infinitely differentiable.

PROPOSITION 18. *Suppose Σ has a Jordan basis u_k^s. Then a vector in $V(x, z)$ is essential with respect to an eigenvalue μ if and only if it is a linear combination modulo $C(x, z)$ of those $u_k^s(x, z)$ where $\lambda_k(x, z) = \mu, m_k > 1$ and $1 \leqq s \leqq m_k - 1$. Moreover, $\rho(\Sigma)(x, z) = \sum_{k=1}^r (m_k - 1)$.*

Proof. "If" is immediate. Suppose

$$u \equiv \sum_{k=1}^r \sum_{s=1}^{m_k} a_s^k u_k^s \text{ modulo } C$$

is essential at (x, z). Now, observe that

$$(A - \mu I)^h u_k^s \equiv (\lambda_k - \mu)^h u_k^s + \binom{h}{1}(\lambda_k - \mu)^{h-1}u_k^{s-1} + \cdots$$

$$+ \binom{h}{j}(\lambda_k - \mu)^{h-j}u_k^{s-j} + \cdots + u_k^{s-h} ,$$

so

$$(A - \mu I)^h u \equiv \sum_{k=1}^r a_{m_k}^k(\lambda_k - \mu)^h u_k^{m_k} + L$$

where L involves only u_k^s for $s \leqq m_k - 1$. Hence, if $(A - \mu I)^h u \equiv 0$, $a_{m_k}^k = 0$ whenever $\lambda_k \neq \mu$. Then

$$(A - \mu I)^h u \equiv \Sigma_k a_{m_k-1}^k(\lambda_k - \mu)^h u_k^{m_k-1} + L'$$

where L' involves $u_k^s, s \leqq m_k - 2$. Again these coefficients $a_{m_k-1}^k = 0$ unless $\lambda_k = \mu$. Continuing we find u depends only on $u_k^s, 1 \leqq s \leqq m_k,$ $\lambda_k = \mu$. But $u \equiv (A - \mu I)v$, so this same result must hold for v. This implies that u depends only on $u_k^s, 1 \leqq s \leqq m_k - 1$.

PROPOSITION 19. *If Σ has real eigenvalues, then there exists a dense open set V in its domain such that every point of V has a neighborhood on which Σ has a Jordan basis and the invariant subspace C is spanned by linearly independent vector fields.*

Proof. This is a matter of examining various systems of linear and polynomial equations with coefficients which are functions of (x, z).

DEFINITION 11. If each point in the domain of Σ has a neighborhood on which the $C_\lambda^\alpha \partial_1 z^\lambda$ are linearly independent and on which a Jordan basis can be found, let $E\Sigma$ be the pp of Σ on the essential vector fields. This is called the *essential pp* of Σ. (Note that $E\Sigma$ may be defined locally. By Proposition 5 two pp's on local essential vector fields must coincide on the overlapping neighborhoods.)

We recall that $E\Sigma$ need not exist due to quadratic conditions, but by Proposition 9 the essential vector fields of Σ and $P\Sigma$ correspond. By Proposition 11, there always exists a pp on these vector fields which come from Σ. Thus we prove

PROPOSITION 20. *Any system Σ may be restricted to a dense open set in its domain on which $EP\Sigma$ exists as a normal pp of $P\Sigma$. Similarly for $EP^k\Sigma, k \geqq 1$.*

PROPOSITION 21. *Let $\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ be a sequence of normal pp's where Σ_h is hyperbolic. Then on an open dense subset of the domain of Σ there is a sequence of pp's*

$$EP\Sigma \subset \Sigma_1' \subset \cdots \subset \Sigma_h'$$

where Σ_h' is hyperbolic.

Proof. Use corollary to Proposition 12, Proposition 20, and Proposition 17.

PROPOSITION 22. *Let $\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ be a sequence of pp's. Suppose the vector space $V(x, z)$ associated with Σ is contained in the invariant space $C_h(x, z, u_1, \cdots, u_h)$ associated with Σ_h. Then there exists a sequence*

$$\Sigma \subset P\Sigma \subset \Sigma_2' \subset \cdots \subset \Sigma_h'$$

where Σ_h' has Σ_h as an admissible restriction.

Proof. Let $\Sigma_j'' = \Sigma_j \cup P\Sigma = \Sigma_j \cup (\Sigma_{j-1} \cup P\Sigma)$. Clearly $\Sigma_1'' = \Sigma_1 \cup P\Sigma$ has $P\Sigma$ as an admissible restriction. The result follows from Proposition 13 and 14.

PROPOSITION 23. *Let Σ_1 be a normal pp of $P^k\Sigma$, $k \geq 1$, on vector fields with coefficients which are functions of (x, z). Then the essential vectors of Σ_1 can be spanned by vector fields having coefficients which are functions of the variables of $P\Sigma$.*

Proof. This is a tautology for $k = 1$. For $k > 1$, assuming the result for $k - 1$ to be true, by Proposition 12 Σ_1 is an admissible restriction of a pp $P\Sigma'$ where Σ' is the normal pp of $P^{k-1}\Sigma$ on the corresponding vector fields in $P^{k-1}\Sigma$. By induction the essential vectors of Σ', hence of $P\Sigma'$, hence Σ_1, are spanned by vector fields with coefficients functions of the variables of $P\Sigma$.

DEFINITION 12. *A pp Σ_1 of Σ is complete if Σ_1 has no essential vectors which involve only the variables of Σ, i.e., of the form $a_\lambda \partial_1 z^\lambda$.*
For example, $P\Sigma$ is a complete pp of Σ.

THEOREM 2. *Let $\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ be a sequence of normal pp's where Σ_h is hyperbolic. For any $k \geq m - \alpha_1 - \rho(\Sigma) + 1$, there exists a sequence of pp's on an open dense subset of the domain of Σ,*

$$P^k\Sigma \subset E'P^k\Sigma \subset \Sigma_1' \subset \cdots \subset \Sigma_h'$$

where Σ_h' is hyperbolic, Σ_1' is a pp of $P^k\Sigma$, $E'P^k\Sigma$ is normal complete pp of $P^k\Sigma$ and $\rho(\Sigma)$ is the maximum value of the degeneracy of Σ. (Recall $m =$ no of dependent variables z^λ, $\alpha_1 = \dim C(x, z)$. Thus, $m - \alpha_1 = \dim V(x, z)/C(x, z)$.)

Proof. Let $u_\lambda^\tau(x, z)\partial_1 z^\lambda$ be vector fields which span the essential vectors of Σ on an open set in the domain of Σ. By Proposition 9, $u_\lambda^\tau \partial_1 p_k^\lambda$ span the essential vectors of $P^k\Sigma$, where p_k^λ are the variables in $P^k\Sigma$ which are analogous to the p^λ in Proposition 9. Since $k \geq 1$, $EP^k\Sigma$ exists and is a normal pp of $P^k\Sigma$ for a suitable restriction by Proposition 20.

Now, by the corollary to Proposition 12, there is a sequence of normal pp's $P\Sigma \subset \tilde{\Sigma}_1 \subset \cdots \subset \tilde{\Sigma}_h$ where $\tilde{\Sigma}_h$ is hyperbolic. Repeating, we get a similar sequence $P^k\Sigma \subset \Sigma_1^* \subset \cdots \subset \Sigma_h^*$ where Σ_h^* is hyperbolic. By Proposition 21 there is a sequence of pp's $P^k\Sigma \subset EP^k\Sigma \subset \tilde{\Sigma}_1' \subset \cdots \subset \tilde{\Sigma}_h'$ where $\tilde{\Sigma}_h'$ is hyperbolic and $\tilde{\Sigma}_1'$ is a pp of $P^k\Sigma$.

If $EP^k\Sigma$ is a complete pp of $P^k\Sigma$, the theorem is proved. Otherwise there exist essential vector fields of $EP^k\Sigma$ of the form $v_\lambda^\sigma\partial_1 p_k^\lambda$. By Proposition 23 the coefficients v_λ^σ can be supposed to be functions of the variables (x, z, p_1) of $P\Sigma$. By further restricting the open dense domain if necessary we can assume the $v_\lambda^\sigma\partial_1 p_k^\lambda$ and $u_\lambda^\tau\partial_1 p_k^\lambda$ are linearly independent. They determine an invariant subspace of $P^k\Sigma$ modulo C_k (the invariant space associated with $P^k\Sigma$), because the $v_\lambda^\sigma\partial_1 p_k^\lambda$ are determined by the intersection of the space of essential vectors in $EP^k\Sigma$ with the image of V_k (the total space associated with $P^k\Sigma$) in $EP^k\Sigma$, at each point. That is, they are merely those essential vectors in $EP^k\Sigma$ which depend only on $\partial_1 p_k^\lambda$, the variables of $P^k\Sigma$.

By Proposition 11 we can form the normal pp of $P^k\Sigma$ on the vector fields $v_\lambda^\sigma\partial_1 p_k^\lambda$ and $u_\lambda^\tau\partial_1 p_k^\lambda$ provided that $k \geq 2$. But if $k = 1$, that means $m - \alpha_1 - \rho(\Sigma) = 0$, so every vector in $V(x, z)$ would be essential, which is impossible by the very Definition 11 of essential.

Letting $\mathscr{E}P^k\Sigma$ denote this pp, we see that $\mathscr{E}P^k\Sigma$ is a pp of $EP^k\Sigma$ on some of that system's essential vector fields. Applying Proposition 17 there exists a sequence

$$P^k\Sigma \subset EP^k\Sigma \subset \mathscr{E}P^k\Sigma \subset \tilde{\Sigma}_1'' \subset \cdots \subset \tilde{\Sigma}_h''$$

where $\tilde{\Sigma}_1''$ is a pp of $P^k\Sigma$ and $\tilde{\Sigma}_h''$ is hyperbolic.

If $\rho(\Sigma) = m - \alpha_1 - 1$, all vectors of $P^k\Sigma$ are essential except one, which must have been added in forming $\mathscr{E}P^k\Sigma$. Hence $\mathscr{E}P^k\Sigma$ is an admissible restriction of $P^{k+1}\Sigma$ and is a normal complete pp of $P^k\Sigma$. Otherwise $m - \alpha_1 - \rho(\Sigma) + 1 \geq 3$ and so the process above may be repeated for any further essential vectors in $\mathscr{E}P^k\Sigma$ of the form $w_\lambda^\tau\partial_1 p_k^\lambda$ in the variables of $P^k\Sigma$.

This process must end since there are at least $\rho(\Sigma) + h$ linearly independent vector fields added at the hth stage of the form $a_\lambda\partial_1 p_k^\lambda$, so we must arrive at $P^{k+1}\Sigma$ for $h \leq m - \alpha_1 - \rho(\Sigma) \leq k - 1$.

REMARK. Observe that $E'P^k\Sigma$ is explicitely *constructed.*

PROPOSITION 24. *If* $k \geq m - \alpha_1 - \rho(\Sigma) + 1, j \geq 0$, *then* $P^j E'P^k\Sigma$ *has* $E'P^{k+j}\Sigma$ *as an admissible restriction, where* $E'P^k\Sigma$ *and* $E'P^{k+j}\Sigma$ *are constructed as in Theorem 2.*

Proof. This follows by applying Proposition 12 at each stage in the process described in Theorem 2 and using Proposition 9 repeatedly.

PROPOSITION 25. *Let* $\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ *be a sequence of pp's. Then there exists a sequence* $\Sigma \subset \Sigma_1' \subset \cdots \subset \Sigma_h'$ *of normal pp's on an open dense set of the domain of* Σ *where* Σ_j' *is an admissible restriction of* Σ_j.

Proof. By restricting the domain if necessary, Σ_1 is obtained by vector fields linearly independent of Σ and possibly some in $C(x, z)$. Use Proposition 14 to obtain Σ'_1 and make the same admissible restriction in all succeeding systems. Repeat this reasoning starting with Σ_1, etc.

THEOREM 3. *Let* $\Sigma \subset E'\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ *be a sequence of normal pp's leading to the hyperbolic system* Σ_h. *Let* $E'\Sigma$ *be a complete pp of* Σ *and suppose* Σ_1 *is a pp of* Σ. *Then on an open dense set, either* $\rho(E'\Sigma) < \rho(\Sigma)$ *or else there exists a sequence of normal pp's*

$$P\Sigma \subset \Sigma'_2 \subset \cdots \subset \Sigma'_h$$

of one shorter length, leading to a hyperbolic system Σ'_h.

Proof. Let $u_k^s = u_{k\lambda}^s(x, z)\partial_i z^\lambda$ be a Jordan basis on an open set $\bigcup$,

$$1 \leqq s \leqq m_k, k = 1, \cdots, r\colon (A - \lambda_k I)u_k^s \equiv u_k^{s-1}, (A - \lambda_k I)u_k' \equiv 0$$

modulo $C(x, z)$. Let $W(x, z)$ be the subspace of $V(x, z)$ used in the pp $E'\Sigma$, i.e., $W(x, z)$ is the invariant subspace of $E'\Sigma$ by Proposition 3. Since $\{u_k^s \mid m_k > 1, 1 \leqq s \leqq m_k - 1\}$ are all essential, they are contained in $W(x, z)$.

Let $W_\mu(x, z)$ be the subspace of $W(x, z)$ associated with the eigenvalue $\mu(x, z)$ of $A(x, z)$, modulo $C(x, z)$. Then $W_\mu(x, z) \supset \{u_k^s \mid \lambda_k = \mu, m_k > 1, 1 \leqq s \leqq m_k - 1\}$, where λ_k is the eigenvalue associated with u_k^s.

If $W_\mu(x, z)$ contains vectors independent of these vectors, then on an open dense set we can find a factor field of the form

$$\sum_{\lambda_k = \mu} C_k(x, z)u_k^{m_k} \in W_\mu(x, z) .$$

Let $m_j = \sup \{m_k \mid \lambda_k = \mu, C_k(x, z) \neq 0]$. Denoting

$$w_j^{m_j} = \sum C_k u_k^{m_k} ,$$
$$(A - \mu I)^r w_j^{m_j} \equiv \sum_{m_k \geqq m_j - r} C_k u_{v_j}^{m_j - r} = w_j^{m_j - r} ,$$

$r = 1, \cdots, m_j - 1$, we can replace u_j^s by w_j^s and obtain a new Jordan basis. Repeating this process we find that in a neighborhood of any point in an open dense set one can find a Jordan basis for Σ of the form $w_k^s, 1 \leqq s \leqq m_k, 1 \leqq k \leqq r$ where for $1 \leqq s \leqq m_k, 1 \leqq k \leqq r'$ and $1 \leqq s \leqq m_k - 1, r' < k \leqq r''$, these elements span $W(x, z)$. Moreover, $m_k = 1$ if $k > r''$.

There are two cases.

Case 1. For some $m_k > 1, w_k^{m_k} \notin W(x, z)$. Then $E'\Sigma$ is obtained

locally by a pp of the form $z_k^s = w_{k\lambda}^s \partial_1 z^\lambda; 1 \leq s \leq m_k$ for $1 \leq k < r'$; $1 \leq s \leq m_k - 1$ for $r' < k \leq r''$ where some $m_k > 1$ for $r' < k \leq r''$.

$$\partial_2 z_k^s = \lambda_k \partial_1 z_k^s + \partial_1 z_k^{s-1} + \sum_{h=r'+1}^{r} R_{kh}^s w_h^{m_h} + T_k^s ,$$

for $s > 1$,

$$\partial_2 z_k^1 = \lambda_k \partial_1 z_k^1 + \sum_{h=r'+1}^{r} R_{kh}^1 w_h^{m_h} + T_k^s .$$

We shall show this system has lower degeneracy than Σ. The space $V'(x, z^\lambda, z_k^s)$ associated with $E'\Sigma$ is spanned by $\partial_1 z^\lambda, \partial_1 z_k^s$. Its invariant subspace $C'(x, z^\lambda, z_k^s)$ is spanned by $C_i^\alpha \partial_1 z^\lambda$ together with $w_k^s, 1 \leq s \leq m_k$ for $1 \leq k \leq r'; 1 \leq s \leq m_k - 1$ for $r' < k \leq r''$. Thus V'/C' is spanned by $\partial_1 z_k^s, w_h^{m_h}, h > r'$. Set $t_k^s = \partial_1 z_k^s + \sum_{h>r'} f_{kr}^s w_h^{m_h}, 1 \leq s \leq m_k$ for $1 \leq k \leq r'$ and $1 \leq s \leq m_k - 1$ for $r' < k \leq r''$, where for $\lambda_h \neq \lambda_k$,

$$f_{kh}^1 = \frac{R_{kh}^1}{\lambda_k - \lambda_h} , \qquad f_{kh}^s = \frac{f_{kh}^{s-1} - R_{kh}^s}{\lambda_k - \lambda_h}$$

and for $\lambda_k = \lambda_h, f_{kh}^s = R_{kh}^{s+1}$. Then

$$(A' - \lambda_k I) t_k^s \equiv t_k^{s-1} \quad \text{if} \quad s > 1 ,$$
$$(A' - \lambda_k I) t_k^1 \equiv \sum_{\lambda_h = \lambda_k, h > r'} R_{kh}^1 w_h^{m_h} = t_k^0 .$$

Then these t_k^0 must be essential vectors. But they depend only on $\partial_1 z^\lambda$ and by hypothesis $E'\Sigma$ is complete. Hence $R_{kh}^1 = 0, \lambda_h = \lambda_k, h > r'$. Hence a Jordan basis of $E'\Sigma$ is $t_k^s, 1 \leq s \leq m_k$ for $1 \leq k \leq r'; t_k^s, 1 \leq s \leq m_k - 1$ for $r' < k \leq r''; w_k^{m_k}$ for $r'' < k \leq r$. Since $(A' - \lambda_k I) t_k^s \equiv t_k^{s-1}, (A' - \lambda_k I) t_k^1 \equiv 0$ and $(A' - \lambda_k I) w_k^{m_k} \equiv 0$, it follows that $\rho(E'\Sigma) = \Sigma_{k=1}^{r'}(m_k - 1) + \Sigma_{k=r'+1}^{r''}(m_k - 2)$. Since some $m_k > 1$ for $r' < k \leq r''$, this number is smaller than $\rho(\Sigma) = \Sigma_{k=1}^{r''}(m_k - 1)$. (Recall $m_k = 1$ for $k > r''$.)

Case 2. Every $m_k = 1$ if $w_k^{m_k} \notin W(x, z)$. Thus, $E'\Sigma$ is obtained by prolonging on the vector fields $w_k^s, 1 \leq s \leq m_k, k = 1, \cdots, r'$, where $m_k = 1$ for $k > r'$. In Σ the vectors $w_k^{m_k}, k > r'$ satisfy $(A - \lambda_k I) w_k^{m_k} \equiv 0$. If we prolong $E'\Sigma$ on these vector fields $w_k^{m_k}, k > r'$, we shall add

(8) $$z_k = w_k^{m_k}, \partial_2 z_k = \lambda_k \partial_1 z_k + T_k, k > r' .$$

This in no way alters the degeneracy of $E'\Sigma$, whose essential vectors are $\partial_1 z_k^s, 1 \leq s \leq m_k - 1$, and whose degeneracy is $\rho(\Sigma)$. Moreover, if we add these equations to each of the systems in the sequence

$$E'\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$$

observe that $E'\Sigma$ becomes $P\Sigma$. Then since Σ_1 is a pp of Σ which contains $E'\Sigma$, when we add these equations Σ_1 must have $P\Sigma$ as an admissible restriction. Taking this admissible restriction in each succeeding systems leads to a sequence of pp's

$$P\Sigma \subset \Sigma_2' \subset \cdots \subset \Sigma_h' \ .$$

Now Σ_j' is an admissible restriction of the pp of Σ_j obtained by adding (8) to Σ_j. The essential vectors of Σ_j do not depend on $\partial_1 z^\lambda$ since $E'\Sigma$ is a complete pp of Σ, hence they do not depend on z_k. The vectors $\partial_1 z_k$ are not essential, so adding (8) in no way effects the degeneracy of Σ_j, since degeneracy measures the number of essential vectors. Since admissible restrictions do not effect degeneracy, $\rho(\Sigma_j') = \rho(\Sigma_j)$. In particular Σ_h' is hyperbolic. Finally, use Proposition 25.

THEOREM 4. *Let* $\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ *be a sequence of pp's leading to a hyperbolic system* Σ_h. *Then on an open dense set for some k there is a complete normal pp $E'P^k\Sigma$ of $P^k \Sigma$ having lower degeneracy than* Σ, *unless* Σ *is itself hyperbolic.*

Proof. By Proposition 25 we can suppose the sequence is normal. By Theorem 2, if $k \geq m - \alpha_1 - \rho(\Sigma) + 1$ there exists a sequence of pp's on an open dense set, $P^k\Sigma \subset E'P^k\Sigma \subset \Sigma_1' \subset \cdots \subset \Sigma_h'$ where $E'P^k\Sigma$ is a normal complete pp of $P^k\Sigma$, Σ_1' is a pp of $P^k\Sigma$ and Σ_h' is hyperbolic. By Theorem 3 either $\rho(E'P^k\Sigma) < \rho(\Sigma)$, in which case the theorem is proved, or else there exists a sequence $P^{k+1}\Sigma \subset \Sigma_2'' \subset \cdots \subset \Sigma_h''$ of normal pp's where Σ_h'' is hyperbolic. Repeating this, there must be some $E'P^j\Sigma$ of lower degeneracy than Σ unless Σ is itself hyperbolic.

THEOREM 5. *If* $\Sigma \subset \Sigma_1 \subset \cdots \subset \Sigma_h$ *is a sequence of normal pp's leading to the hyperbolic system* Σ_h, *then on an open dense set, for all* $k \geq m - \alpha_1 - \rho(\Sigma) + 1$, *there exists a complete pp $E'P^k\Sigma$ such that* $\rho(E'P^k\Sigma) < \rho(\Sigma)$.

Proof. By Theorem 4 this is true for some $k + j$. By Proposition 24, $\rho(E'P^k\Sigma) = \rho(P^jE'P^k\Sigma) = \rho(E'P^{k+j}\Sigma) < \rho(\Sigma)$.

COROLLARY. *If* Σ *is absolutely equivalent to a hyperbolic system, then the proceedure in Theorem 2 for successively constructing $E'P^k\Sigma$ will result in a hyperbolic system.*

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