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CONCERNING SEMI-LOCAL-CONNECTEDNESS AND CUTTING IN NONLOCALLY CONNECTED CONTINUA

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In this paper relations between semi-local-connectedness and the sets of cut points are established in certain nonlocally connected continua.

If a compact metric continuum is not locally connected at any point of a dense G_δ subset and is semi-locally-connected at each point of a dense subset, then the set of weak cut points of the continuum is a first category subset. The presence of a dense G_δ set of weak cut points in a compact metric continuum which is not locally connected at any point of a dense G_δ subset implies that the continuum is not semi-locally-connected at any point of a dense open subset. If a compact metric continuum is not connected im kleinen at any point of a dense G_δ subset and is semi-locally-connected at each point of a dense subset, then the set of points which separate the continuum is nowhere dense. The presence of a dense set of separating points in a compact metric continuum which is not connected im kleinen at any point of a dense G_δ subset implies that the continuum is not semi-locally-connected at any point of a dense open set. If a semi-locally-connected compact metric continuum is not connected im kleinen at any of its points, then the set of weak cut points in the continuum is nowhere dense and countable.

Throughout this paper, M is a compact connected metric space. A point p of M is said to be a *weak cut point* of M provided that $M - \{p\}$ contains two distinct points x and y such that every subcontinuum of M containing $\{x, y\}$ contains p. Under these conditions p is said to cut M weakly between x and y. A point p of M is said to be a *separating point* of M provided that $M - \{p\}$ is not connected. For the definitions of unfamiliar terms and phrases see [5], [6], and [8].

THEOREM 1. *If D is an open subset of M and J is a G_δ subset of M which is dense in D and which has the property that M is not locally connected at any point of J, and M is semi-locally-connected at each point of a dense subset of D, then the set C of weak cut points of M which are contained in D is a first category subset of M. (Note that C may be void.)*

Proof. Assume that C is a second category set in M. For each positive integer n, define $C_n = \{x \in C \mid$ for some pair of points $\{y, z\}$ in $M - S(x, 1/n)$, the point x cuts M weakly between y and $z\}$. ($S(x, 1/n)$ is the circular region with its center at the point x and its radius $1/n$). Since $C = \bigcup_n C_n$, there exists a positive integer n such that $\bar{C}_n$ (the closure of C_n) contains an open set. It follows that there exists an open set G and a subset R of $C_n \cap G$ such that R is dense in G and for each point x in R there is a pair of points $\{y, z\}$ in $M - G$ such that x cuts M weakly between y and z. For each positive integer n, define $G_n = \{x \in \bar{G} \mid$ each open subset of $S(x, 1/n)$ which contains x is not connected$\}$. Note that $\overline{G_n} \subset G_{n+1}$ and therefore $J \cap \bar{G} = \bigcup_n G_n = \bigcup_n \overline{G_n}$. The set $J \cap \bar{G}$ is a second category subset of M. It follows that for some positive integer m, the set G_m contains an open set W. There exists an open set U in W such that for each point x in U every open subset of U containing x is not connected (i.e., U does not contain an open connected subset). Note that for each open subset V of U, none of the components of V are open in V.

There exists a point p in U such that M is semi-locally-connected at p. It follows that there exists a finite collection of mutually disjoint continua $H_1, H_2, \cdots, H_n$ in $M - \{p\}$ which cover $M - U$. Let $V = U - \bigcup_{i=1}^n H_i$. The set V is open and contained in U. Since no component of V is open in V, one can conclude that V has infinitely many components. Because $\bar{R} \supset G \supset V$, the set R must meet infinitely many of the components of V.

Consider the continuum H_1 and the closed set $\cup_{i=2}^n H_i$. There exists a component K_1 of V such that $\bar{K}_1$ meets both H_1 and $\cup_{i=2}^n H_i$. Define L_1 to be the subcollection of $H_1, H_2, \cdots, H_n$ consisting of continua which meet $\bar{K}_1$. Let H_1' be the continuum $\bar{K}_1 \cup L_1^*$ ($L_1^* = \bigcup_{H_i \in L_1} H_i$). There exists a component K_2 of V such that $\bar{K}_2$ meets both H_1' and the closed set $F_1 = (\cup_{i=1}^n H_i) - L_1^*$. Define L_2 to be the subcollection of $H_1, H_2, \cdots, H_n$ consisting of continua which meet $\bar{K}_2$. Let H_2' be the continuum $H_1' \cup \bar{K}_2 \cup L_2^*$ and continue the process. Clearly after a finite number of steps the elements of $H_1, H_2, \cdots, H_n$ will be connected by a finite collection K of closures of components of V. For each point x of $V - K^*$ and each pair of points $\{y, z\}$ in $M - G$, the continuum $(\cup_{i=1}^n H_i) \cup K^*$ contains $\{y, z\}$ and misses x. It follows that no point of $V - K^*$ cuts M weakly between two points in $M - G$. But R must meet $V - K^*$. This contradicts the choice of R. Therefore C is a first category subset of M.

COROLLARY. *If M is not locally connected at any point of a dense G_δ subset of M and is semi-locally-connected at each point of a dense subset of M, then the set of weak cut points of M is a first category subset of M.*

THEOREM 2. *If M is not locally connected at any point of a dense G_δ subset of M and M contains a dense G_δ set of weak cut points, then M is not semi-locally-connected at any point of a dense open set.*

Proof. For each open set U in M, there exists an open set V contained in U such that M is not semi-locally-connected at any point of V (Theorem 1). It is therefore possible to define a dense open subset of M which has the desired property.

Grace has asked the following question. "Does each compact metric continuum which is totally nonsemi-locally-connected on a dense open set contain a dense G_δ set of weak cut points?" See [3]. If this is the case, then Theorem 2 provides a characterization of those "totally" nonlocally connected, compact metric continua which are not semi-locally-connected at each point of a dense open subset.

One might hope to put further restrictions on the set of weak cut points or the set of separating points in nonlocally connected continua of the type mentioned in Theorem 1. Grace has described a compact plane continuum which is not locally connected at any point and is semi-locally-connected at each point of a dense subset. This continuum contains a countable dense set of separating points [2]. Therefore the result stated in Theorem 1 is sharp for nonlocally connected continua. However it is possible to put further limitations on these sets in certain nonconnected im kleinen continua.

THEOREM 3. *If D is an open subset of M and J is a G_δ subset of M dense in D which has the property that M is not connected im kleinen at any point of J, and M is semi-locally-connected at each point of a dense subset of D, then the set of points in D which separate M is nowhere dense in D.*

Proof. Let C denote the set of points in D which separate M. Assume that C is somewhere dense in D. It follows that $\bar{C} \cap D$ contains an open set G such that each component of G is nowhere dense [2, Lemma 2]. Since M is compact, the boundary of G is compact. Let p be a point of G such that M is semi-locally-connected at p. It follows that there is a finite collection of continua H in $M - \{p\}$ whose interiors cover the boundary of G. Let $U = (M - H^*) \cap G$. The set U is open in M and contains p. Since the components of G are nowhere dense and $U \subset G \subset \bar{C}$, there exists an infinite sequence $K_1, K_2, \cdots$ of distinct components of G and an infinite sequence $x_1, x_2 \cdots$ of points such that for each positive integer j, the point x_j is in $K_j \cap C \cap U$. For each j, let $M - \{x_j\} = A_j \cup B_j$, where A_j and B_j are separated sets. The set $A_j \cup \{x_j\}$ is a continuum which contains a point of G in its interior and therefore meets the boundary of G. The x_j-component of $G \cap (A_j \cup \{x_j\})$, which is a subset of K_j, meets an element H_j of H. Apply the same argument to $B_j \cup \{x_j\}$. Let H'_j denote an element of H which meets the x_j-component of $G \cap (B_j \cup \{x_j\})$. Since A_j and B_j are separated, H_j and H'_j are distinct elements of H each of which meet K_j. Note that K_j is the only component in $K_1, K_2, \cdots$ which meets both H_j and H_j, for if this were not the case, then x_j would not separate M between H_j and H'_j. But for each j, the component K_j must meet two distinct elements of H. This contradicts the fact that H is a finite collection. Therefore C is nowhere dense.

COROLLARY. *If M is not connected im kleinen at any point of a dense G_δ subset of M and is semi-locally-connected at each point of a dense subset of M, then the set of points which separate M is nowhere dense in M.*

From Theorem 1 one can conclude that the set of weak cut points in the continuum M in the preceding corollary is a first category subset. The following example indicates that this result is sharp.

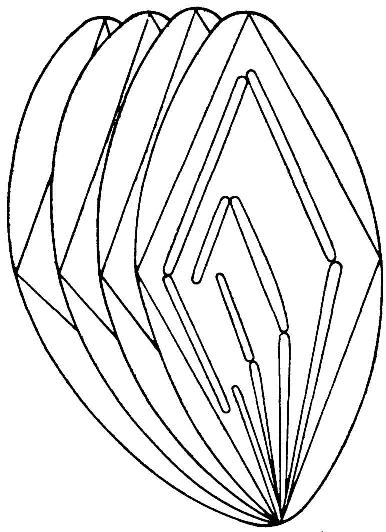
EXAMPLE 1. A compact continuum M in Euclidean 3-space which is totally nonaposyndetic, hence nowhere connected im kleinen, locally remotely connected on a dense subset, hence semi-locally-connected on a dense subset, and which contains a dense set of weak cut points.[1]

In [2] a compact plane continuum H is described which has the property that each connected open subset of H is dense in H. This continuum is the common part of a sequence $H_1, H_2, \cdots$ of nested continua contained in the unit disk. The continuum H is connected im kleinen at each point of the dense G_δ subset R of H consisting of the points that for each i, are interior points of H_i. Note that H is also locally remotely connected at each point of R. Let T denote the dense set of points which separate H. Let p be the point of H_1 which separates H_1 into infinitely many components. Let S be the topological product of H and the cantor set C. Define the continuum M to be the decomposition of S in which the set of points V in S with first coordinate p is shrunk to a point v. See the figure.

Since H is nonaposyndetic at p, M is nonaposyndetic at the point v. If q is a point of $M - \{v\}$, clearly M is nonaposyndetic at q with respect to v. Hence M is totally nonaposyndetic.

Define D to be the set of all points (x, y) in M such that x is in R. Obviously, D is a dense subset of M. Furthermore, M is locally remotely connected at each point of D. To see this let (x, y) be a

[1] A continuum M is said to be *locally remotely connected* at a point p if for each open set U in M containing p there exists an open set V such that $p \in V \subset U$ and $M - V$ is connected (F. B. Jones)



point of D and let G be an open set in M which contains (x, y). Since H is locally remotely connected at x, there exists an open set Q in $H - \{p\}$ such that $x \in Q \subset$ (the projection of G into H) $- \{v\}$ and $H - Q$ is connected. There exists a positive real number d such that the neighborhood N of (x, y), which consists of all points (p, q) in M with p in Q and $y - d < q < y + d$, is contained in G. Since v is not contained in N, it is clear that $M - N$ is connected.

Let E denote the set of points (x, y) in M such that x is in T. The set E is dense in M. Each point of E weakly cuts M. Note that v is the only point which separates M.

THEOREM 4. *If M is not connected im kleinen at any point of a dense G_δ subset of M and M contains a dense set of separating points, then M is not semi-locally-connected at any point of a dense open set.*

Proof. For each open set U in M, there exists an open set V contained in U such that M is not semi-locally-connected at any point of V (Theorem 3). Therefore one can define a dense open subset of M on which M is not semi-locally-connected at any point.

Consider the class of totally nonaposyndetic continua which are semi-locally-connected at each point of a dense open set. The continuum in [4, Example 3] is in this class and, the set of weak cut points in this continuum is a nowhere dense set. The following theorem indicates that this will always be the case for continua of this type.

THEOREM 5. *If M is not connected im kleinen at any point of a dense G_δ subset of M and is semi-locally-connected at each point* *of a dense open subset of M, then the set of weak cut points is nowhere dense in M.*

Proof. If M is semi-locally-connected at a point x, and x cuts M weakly, then x separates M [7]. The corollary to Theorem 3 indicates that the set of separating points in M is nowhere dense. Since the only other weak cut points in M, are in the complement of a dense open subset of M, the set consisting of all points which cut M weakly must be nowhere dense.

THEOREM 6. *If M is semi-locally connected but not connected im kleinen at any of its points, then the set of weak cut points in M is nowhere dense and countable.*

Proof. Let C denote the set of points which cut weakly in M. Each point of C separates M [7]. The fact that C is nowhere dense follows immediately from Theorem 3.

M is not regular at any point of C [6, Th. 76, p. 129]. Since all but countably many separating points of M have order 2, it follows that C must be countable [6, Th. 33, p. 292].

It is easily seen that the proof of Theorem 6 can be modified to allow M to be connected im kleinen on a countable set and not semi-locally-connected at any point of a countable set.

By joining together in a suitable fashion infinitely many copies of the continuum in [1, Example 2], it is possible to construct a compact plane continuum which is semi-locally-connected and not connected im kleinen at any of its points but which has a countably infinite set of separating points.

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