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ESTIMATES OF POSITIVE CONTRACTIONS

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The purpose of this paper is to obtain an L_p estimate for the supremum of the Cesàro averages of a certain class of positive contractions of L_p. Let $(X, \mathscr{F}, \mu)$ be a measure space, and let T be a linear operator mapping $L_p(X, \mathscr{F}, \mu)$ into itself for p fixed, $1 < p < +\infty$. If there is a constant $c > 0$ such that for each $f \in L_p(X, \mathscr{F}, \mu)$,

$$\int \sup_n |f, (f + Tf)/2, \cdots, (f + Tf + \cdots + T^n f)/n + 1|^p d\mu \leqq c^p \int |f|^p d\mu ,$$

then we say that T admits of a dominated estimate with constant c. In an effort to unify certain results due to A. Ionescu-Tulcea and to E. Stein, a somewhat more general form of the following theorem was obtained earlier: If T is a positive contraction, and if there exists an $h > 0$ a.e., $h \in L_p(X, \mathscr{F}, \mu)$ and $Th = h$, then T admits of a dominated estimate with constant $p/p - 1$. In the present paper, we have extended the theorem, obtaining a slightly more general form of the following: If T is a positive contraction and if for each positive integer n there exists an $h_n > 0$ a.e., $h_n \in L_p(X, \mathscr{F}, \mu)$ and $\| h_n \| = \| T^n h_n \|$, then T admits of a dominated estimate with constant $p/p - 1$.

This result is more widely applicable more directly than the previous theorem, but is not the most general result one might conjecture, that positive contractions admit of a dominated estimate with no further assumptions. In this direction, we have obtained several equivalent formulations of the problem which may help to lead to an answer. In any case, it remains an open problem whether or not positive contractions of $L_p(X, \mathscr{F}, \mu)$, $1 < p < +\infty$, admit of a dominated estimate without the assumption of additional conditions.

2. Main results. Let $(X_1, \mathscr{F}_1, \mu_1)$ and $(X_2, \mathscr{F}_2, \mu_2)$ be two measure spaces and let T be a linear operator mapping $L_p(X_1, \mathscr{F}_1, \mu_1)$ into $L_p(X_2, \mathscr{F}_2, \mu_2)$, p fixed, $1 \leqq p \leqq +\infty$. We say that T is a contraction if its norm is less than or equal to one. We say that T is positive if it maps nonnegative functions to nonnegative functions. We shall omit the phrase almost everywhere, it being understood where applicable.

DEFINITION 2.1. The range set of T, $R(T)$, is the support of Tf, where $f \in L_p(X_1, \mathscr{F}_1, \mu_1)$ and $f > 0$ (it is clear that $R(T) \in \mathscr{F}_2$ and that it is independent of the particular $f > 0$ chosen).

LEMMA 2.1. *Let T be a linear operator of $L_p(X_1, \mathscr{F}_1, \mu_1)$ into $L_p(X_2, \mathscr{F}_2, \mu_2)$, p fixed, $1 < p < +\infty$, and let*

$$h_1 \in L_p(X_1, \mathscr{F}_1, \mu_1),\ h_2 \in L_p(X_2, \mathscr{F}_2, \mu_2)$$

be nonnegative functions, $h_1 > 0$ on $R(T^*)$ and $h_2 > 0$ on $R(T)$. Let P be the linear operator of $L_p(X_1, \mathscr{F}_1, m_1)$ into $L_p(X_2, \mathscr{F}_2, m_2)$ defined by

$$P(f) = \begin{cases} T(fh_1)/h_2 & \text{on } R(T)\ , \\ 0 & \text{otherwise}\ , \end{cases}$$

where the measures m_1 and m_2 are obtained by setting

$$m_1(A_1) = \int_{A_1} h_1^p d\mu_1,\ A_1 \in \mathscr{F}_1 \quad \text{and} \quad m_2(A_2) = \int_{A_2} h_2^p d\mu_2,\ A_2 \in \mathscr{F}_2\ .$$

We have then that

(i) $\|P\| \leqq \|T\|$

and that

(ii) $P^*g = \begin{cases} T^*(gh_2^{p-1})/h_1^{p-1} & \text{on } R(T^*)\ , \\ 0 & \text{otherwise}\ . \end{cases}$

Furthermore, if T is a positive contraction, $\|h_1\| = \|h_2\|$, and $Th_1 = h_2$ on $R(T)$, then we have

(iii) $P^*1 = 1$ *(or equivalently by* (ii), $T^*h_2^{p-1} = h_1^{p-1}$*) on* $R(T^*)$.

Proof. We adapt the proof of Lemma 3.1 of [1]. To see that (i) holds,

$$\begin{aligned} \int |Pf|^p\, dm_2 &= \int | T(fh_1)/h_2 |^p h_2^p d\mu_2 \\ &= \int | T(fh_1) |^p d\mu_2 \leqq \|T\|^p \int |fh_1|^p d\mu_1 \\ &= \|T\|^p \int |f|^p dm_1\ , \end{aligned}$$

so that $\|P\| \leqq \|T\|$. To see that (ii) holds,

$$\int gP(f)dm_2 = \int g[T(fh_1)/h_2]h_2^p d\mu_2 = \int T^*(gh_2^{p-1})/h_1^{p-1} f dm_1\ .$$

Finally, to see that (iii) holds, define $\alpha(x)$ by setting $P^*1 = 1 + \alpha(x)$. Then, with $q = p/p - 1$, we have

$$\int 1 dm_2 = \int 1^q dm_2 \geqq \int (P^*1)^q dm_1 = \int (1 + \alpha)^q dm_1$$

since $\|P^*\| \leqq 1$ by (i). Also

$$\int(1+\alpha)^q dm_1 \geqq \int 1dm_1 + q\int \alpha dm_1$$

with strict inequality if $\alpha \neq 0$ on a set of positive measure, since $(1+\alpha(x))^q > 1 + q\alpha(x)$ for each x such that $\alpha(x) \neq 0$. Further, if $Th_1 = h_2$ on $R(T)$, $P(1) = 1$ on $R(T) = R(P)$ and

$$\int P^*(1)dm_1 = \int 1P(1)dm_2 = \int 1dm_2 = \int 1dm_1 + \int \alpha dm_1 \, .$$

If, in addition, we have that $\|h_1\| = \|h_2\|$, then $\int dm_2 = \int dm_1$, and this equation then implies $\int \alpha dm_1 = 0$. This is incompatible with the previous inequalities (if, as we have, $\|h_1\| = \|h_2\|$) unless $\alpha(x) = 0$ on $R(T^*)$ and $P^*1 = 1$ on $R(T^*)$.

REMARK 2.1. Lemma 2.1 includes Lemma 3.1 of [1] as we can see by taking $(X, \mathscr{F}, \mu) = (X_1, \mathscr{F}_1, \mu_1) = (X_2, \mathscr{F}_2, \mu_2)$. That $Th_1 = h_2$, $\|h_1\| = \|h_2\|$ is certainly satisfied if there exists $h > 0$ such that $Th = h$ by taking $h_1 = h_2 = h$. Further, the slight generalization of Lemma 3.1 of [1] which is the one-dimensional version of this result implies the two-dimensional (by taking

$$(X, \mathscr{F}, \mu) = (X_1, \mathscr{F}_1, \mu_1) \oplus (X_2, \mathscr{F}_2, \mu_2) \, ,$$

$T = T_1 \oplus 0$). We have stated in this form primarily because of the application we make of it.

LEMMA 2.2. *Let T be a positive contraction of $L_p(X_1, \mathscr{F}_1, \mu_1)$ into $L_p(X_2, \mathscr{F}_2, \mu_2)$, p fixed, $1 < p < +\infty$, and let $h_1 \in L_p(X_1, \mathscr{F}_1, \mu_1)$, $h_2 \in L_p(X_2, \mathscr{F}_2, \mu_2)$ be nonnegative functions, $h_1 > 0$ on $R(T^*)$, $h_2 > 0$ on $R(T)$ and such that $Th_1 \leqq h_2$ on $R(T)$ and $T^*h_2^{p-1} \leqq h_1^{p-1}$ on $R(T^*)$. Let P be the positive linear operator of $L_p(X_1, \mathscr{F}_1, m_1)$ into $L_p(X_2, \mathscr{F}_2, m_2)$ defined by*

$$Pf = \begin{cases} T(fh_1)/h_2 & \text{on } R(T) \, , \\ 0 & \text{otherwise} \, , \end{cases}$$

where the measures m_1 and m_2 are obtained by setting

$$m_1(A_1) = \int_{A_1} h_1^p d\mu_1, \; A_1 \in \mathscr{F}_1 \quad \text{and} \quad m_2(A_2) = \int_{A_2} h_2^p d\mu_2 \, ,$$

$A_2 \in \mathscr{F}$ (as in Lemma 2.1). Then P is a positive contraction of $L_1(X_1, \mathscr{F}_1, m_1)$ into $L_1(X_2, \mathscr{F}_2, m_2)$ and also of $L_\infty(X_1, \mathscr{F}_1, m_1)$ into $L_\infty(X_2, \mathscr{F}_2, m_2)$.

Proof. It follows by part (i) of Lemma 2.1 that P is a positive contraction of $L_p(X_1, \mathscr{F}_1, m_1)$ into $L_p(X_2, \mathscr{F}_2, m_2)$. It is clear from the definition that $P1 \leqq 1$, implying that P is a positive contraction of $L_\infty(X_1, \mathscr{F}_1, m_1)$.

We have that P^* is a positive contraction of $L_q(X_2, \mathscr{F}_2, m_2)$ into $L_q(X_1, \mathscr{F}_1, m_1)$ $(q = p/p - 1)$ and by part (ii) of Lemma 2.1 that $P^*1 \leqq 1$ and therefore that P^* is a positive contraction of $L_\infty(X_2, \mathscr{F}_2, m_2)$ into $L_\infty(X_1, \mathscr{F}_1, m_1)$. An application of the Riesz convexity theorem then implies that P is a contraction of $L_1(X_1, \mathscr{F}_1, m_1)$ into $L_1(X_2, \mathscr{F}_2, m_2)$.

We next state the dominated ergodic theorem (see [1] for an outline of its proof with the constant given here).

LEMMA 2.3. (*Dominated ergodic theorem.*) *Let T be a contraction of $L_1(X, \mathscr{F}, \mu)$ into itself and of $L_\infty(X, \mathscr{F}, \mu)$ into itself. Then T admits of a dominated estimate with constant $p/p - 1$.*

THEOREM 2.1. *Let $(X_i, \mathscr{F}_i, \mu_i)$, $i = 1, 2, \cdots, n$ be measure spaces, and let T_i be a positive contraction of*

$$L_p(X_i, \mathscr{F}_i, \mu_i) \quad \text{into} \quad L_p(X_{i+1}, \mathscr{F}_{i+1}, \mu_{i+1})$$

for $i = 1, 2, \cdots, n-1$. Let $h_i \in L_p(X_i, \mathscr{F}_i, \mu_i)$ be nonnegative functions, $i = 1, 2, \cdots, n$ such that

(i) $h_i > 0$ on $R(T_i^*)$, $i = 1, \cdots, n-1$, $h_i > 0$ on $R(T_{i-1})$, $i = 2, \cdots, n$,

(ii) $T_i h_i \leqq h_{i+1}$, on $R(T_i)$,

(iii) $T_i^* h_{i+1}^{p-1} \leqq h_i^{p-1}$, on $R(T_i^*)$,

for $i = 1, 2, \cdots, n-1$. Then we have

$$\begin{aligned}\sum_{i=1}^{n} \int \sup | f_i, (f_i + T_{i-1}f_{i-1})/2, \cdots ,\\ (f_i + T_{i-1}f_{i-1} + \cdots + T_{i-1}T_{i-2} \cdots T_1 f_1)/i |^p \, d\mu_i\\ \leqq (p/p-1)^p \sum_{i=1}^{n} \int | f_i |^p \, d\mu_i \, ,\end{aligned}$$

where $f_i \in L_p(X_i, \mathscr{F}_i, \mu_i)$.

Proof. Let $(X, \mathscr{F}, \mu) = \bigoplus_{i=1}^{n} (X_i, \mathscr{F}_i, \mu_i)$. A function f on X can be written as an n-tuple of functions, $f = (f_1, \cdots, f_n)$ where f_i is a function on X_i, $i = 1, 2, \cdots, n$. The norm of f is given by $\| f \| = (\sum_{i=1}^{n} \| f_i \|^p)^{1/p}$.

We define a positive contraction T of $L_p(X, \mathscr{F}, \mu)$ into $L_p(X, \mathscr{F}, \mu)$ by setting

$$T(f_1, \cdots, f_n) = (0, T_1 f_1, \cdots, T_{n-1} f_{n-1}) \, .$$

That T is a contraction follows from

$$\| T(f_1, \cdots, f_n) \|^p = \sum_{i=1}^{n-1} \| T_i f_i \|^p \leqq \sum_{i=1}^{n-1} \| f_i \|^p \leqq \| f \|^p .$$

We next calculate T^*g. Let

$$f \in L_p(X, \mathscr{F}, \mu) \quad \text{and let} \quad T^*(g) = (g_1^*, \cdots, g_n^*) .$$

Then

$$\begin{aligned}\int T^*(g) f d\mu &= \int (g_1^*, \cdots, g_n^*)(f_1, \cdots, f_n) d\mu \\ &= \sum_{i=1}^{n-1} \int g_i^* f_i d\mu_i = \int T(f) g = \sum_{i=1}^{n-1} \int (T_i f_i) g_{i+1} d\mu_{i+1} \\ &= \sum_{i=1}^{n-1} \int T_i^*(g_{i+1}) f_i d\mu_i .\end{aligned}$$

Since f is arbitrary, it follows that $T^*(g) = (T_1^* g_2, \cdots, T_{n-1}^* g_n, 0)$.

We may apply Lemma 2.2 to each pair

$$L_p(X_i, \mathscr{F}_i, \mu_i), L_p(X_{i+1}, \mathscr{F}_{i+1}, \mu_{i+1})$$

(or to $(X, \mathscr{F}, \mu)$ and $(h_1, h_2, \cdots, h_n)$ each taken twice) to obtain that the operator

$$Pr = (0, P_1 r_1, \cdots, P_{n-1} r_{n-1})$$

of

$$L_p(X, \mathscr{F}, m) = \bigoplus_{i=1}^{n} (X_i, \mathscr{F}_i, m_i)$$

into itself is a contraction of $L_1(X, \mathscr{F}, m)$ into $L_1(X, \mathscr{F}, m)$ and also of $L_\infty(X, \mathscr{F}, m)$ into $L_\infty(X, \mathscr{F}, m)$, where

$$P_i r_i = \begin{cases} T_i(r_i h_i)/h_{i+1} & \text{on } R(T_i) , \\ 0 \text{ otherwise} , \end{cases}$$

and

$$m_i(A_i) = \int_{A_i} h_i^p d\mu_i, \; A_i \in \mathscr{F}_i .$$

An application of Lemma 2.3 then yields

$$\begin{aligned}&\int \sup | r, r + Pr/2, \cdots, (r + \cdots + P^{k-1} r)/k, \cdots |^p dm \\ &\quad \leqq (p/p - 1)^p \int | r |^p dm .\end{aligned}$$

Now

$$(r + Pr + \cdots P^{k-1}r)/k$$

$$= \begin{cases} (r_1/k, (r_2 + P_1 r_1)/k, \cdots , \\ \quad (r_n + P_{n-1}r_{n-1} + \cdots + P_{n-1} \cdots P_{n-k+1}r_{n-k+1})/k) \\ \quad \text{for } k < n - 1 , \\ (r_1/k, (r_2 + P_1 r_1)/k, \cdots , \\ \quad (r_n + P_{n-1}r_{n-1} + \cdots + P_{n-1} \cdots P_1 r_1)/k) \\ \quad \text{for } k \geqq n - 1 , \end{cases}$$

and, therefore, Lemma 2.3 implies that

$$\sum_{i=1}^{n} \int \sup | r_i, (r_i + P_{i-1}r_{i-1})/2, \cdots , \\ (r_i + P_{i-1}r_{i-1} + \cdots + P_{i-1} \cdots P_1 r_1)/i |^p \, dm_i \\ \leqq (p/p - 1)^p \sum_{i=1}^{n} \int | r_i |^p \, dm_i .$$

The theorem follows from this on setting

$$r_i = f_i/h_i, \; dm_i = h_i^p d\mu_i$$

and

$$P_{i-1} \cdots P_{i-j} r_{i-j} = (T_{i-1} \cdots T_{i-j} r_{i-j} h_{i-j})/h_{i-j} .$$

LEMMA 2.4. *Let $(X_1, \mathscr{F}_1, \mu_1)$ and $(X_2, \mathscr{F}_2, \mu_2)$ be measure spaces, and let T be a positive contraction of $L_p(X_1, \mathscr{F}_1, \mu_1)$ into $L_p(X_2, \mathscr{F}_2, \mu_2)$. Let $f \in L_p(X_1, \mathscr{F}_1, \mu_1)$ and $g \in L_p(X_1, \mathscr{F}_1, \mu_1)$ be nonnegative functions such that f vanishes outside the support of g. Then Tf vanishes outside the support of Tg.*

Proof. Follows at once by contradiction.

LEMMA 2.5. *Let $(X_1, \mathscr{F}_1, \mu_1)$ and $(X_2, \mathscr{F}_2, \mu_2)$ be measure spaces, and let T be a positive contraction of $L_p(X_1, \mathscr{F}_1, \mu_1)$ into $L_p(X_2, \mathscr{F}_2, \mu_2)$. Let $f, g \in L_p(X_1, \mathscr{F}_1, \mu_1)$ be nonnegative and g such that $\| g \| = \| Tg \|$. If f vanishes on the support of g, then Tf vanishes on the support of Tg.*

Proof. Let E denote the support of g and F the support of Tg. Suppose Tf does not vanish of F. Then we can find a subset A of $F \cap \{x: Tf > 0\}$ having positive measure and a positive number α such that $\alpha Tg < Tf$ on A. We have for any $\beta > 0$

$$\beta^p \| g \|^p + \| f \|^p = \| \beta g + f \|^p \geqq \| \beta Tg + Tf \|^p \\ > \| (\beta Tg + \alpha Tg)\psi_A \|^p + \| \beta Tg \psi_{X-A} \|^p \\ = (\beta + \alpha)^p \| Tg\psi_A \|^p + \| \beta Tg \psi_{X-A} \|^p .$$

Thus, since $\| Tg \| = \| g \|$,

$$\beta^p \| (Tg)\psi_A \|^p + \| f \|^p > (\alpha + \beta)^p \| (Tg)\psi_A \|^p .$$

For β sufficiently large,

$$(\alpha + \beta)^p = \beta^p\left(1 + \frac{\alpha}{\beta}\right)^p > \beta^p\left(1 + p\frac{\alpha}{\beta}\right) = \beta^p + p\beta^{p-1}\alpha$$

implying

$$\| f \|^p > p\beta^{p-1}\alpha \| Tg \cdot \psi_A \|^p$$

for β sufficiently large, which is impossible since $\| f \|^p < +\infty$.

LEMMA 2.6. *Let $(X, \mathscr{F}, \mu)$ be a measure space, and let T be a positive contraction of $L_p(X, \mathscr{F}, \mu)$. Suppose that for some n, there exist n positive functions $h_{1n}, \cdots, h_{nn}$ in $L_p(X, \mathscr{F}, \mu)$ such that*

$$Th_{kn} \leqq h_{k+1n} \text{ on } R(T)$$
$$T^*h_{k+1n}^{p-1} \leqq h_{kn}^{p-1} \text{ on } R(T^*)$$

for $k = 1, \cdots, n - 1$. Then, for $f \in L_p(X, \mathscr{F}, \mu)$, we have

$$\frac{1}{n}\sum_{i=1}^{n}\int \sup | f, (f + Tf)/2, \cdots, (f + \cdots + T^{i-1}f)/i |^p d\mu$$
$$\leqq (p/p - 1)^p \int | f |^p d\mu .$$

Proof. The lemma follows at once from Theorem 2.1 if we let $(X_i, \mathscr{F}_i, \mu_i) = (X, \mathscr{F}, \mu)$, $T_i = T$, and $f_i = f$, $i = 1, \cdots, n$.

LEMMA 2.7. *Let $(X, \mathscr{F}, \mu)$ be a measure space, and let T be a positive contraction of $L_p(X, \mathscr{F}, \mu)$. Suppose that for some $n \geqq 1$, there exists a positive function $h_n \in L_p(X, \mathscr{F}, \mu)$ such that $\| T^n h_n \| = \| h_n \|$. Then there exist $n + 1$ positive functions $h_{1n}, \cdots, h_{n+1n}$* in *$L_p(X, \mathscr{F}, \mu)$ such that*

$$Th_{kn} = h_{k+1n} \text{ on } R(T) ,$$
$$T^*h_{k+1n}^{p-1} = h_{kn}^{p-1} \text{ on } R(T^*) ,$$

for $k = 1, \cdots, n$.

Proof. We let $h_{1n} = h_n$, and $h_{k+1n} = Th_{kn} + \psi_{cR(T)}h_{kn}$, $k = 1, \cdots, n$. It is clear from the definition that the functions $h_{1n}, \cdots, h_{n+1n}$ are positive and that $Th_{kn} = h_{k+1n}$ on $R(T)$, $k = 1, \cdots, n$. It also follows easily from the definition of T^* that

$$T^*h_{k+1n}^{p-1} = T^*(Th_{kn})^{p-1},\ k = 1, \cdots, n\ .$$

If we show that $\| h_{kn} \| = \| Th_{kn} \|$, $k = 1, \cdots, n$, then we would have also shown that $T^*h_{k+1n}^{p-1} = h_{kn}^{p-1}$ on $R(T^*)$, $k = 1, \cdots, n$, by part (iii) of Lemma 2.1, since $T^*h_{k+1n}^{p-1} = T^*(Th_{kn})^{p-1}$, $k = 1, \cdots, n$ on $R(T^*)$. To see that $\| h_{kn} \| = \| Th_{kn} \|$, $k = 1, \cdots, n$, we'll prove a little more, that $\| h_{kn} \| = \| T^ih_{kn} \|$, $i = 1, \cdots, n - k + 1$, $k = 1, \cdots, n$. We proceed by induction. That $\| h_{1n} \| = \| T^ih_{1n} \|$, $i = 1, \cdots, n$ holds by hypothesis. Next, we suppose $\| h_{k(0)n} \| = \| T^ih_{k(0)n} \|$, $i = 1, \cdots, n - k(0) + 1$ for $k(0) < n$. We write $h_{k(0)+1n} = Th_{k(0)n} + \psi_{cR(T)}h_{k(0)n}$, and since the supports of $Th_{k(0)n}$ and of $\psi_{cR(T)}h_{k(0)n}$ are disjoint, and since by the induction hypothesis $\| Th_{k(0)n} \| = \| T^ih_{k(0)n} \|$, $i = 1, \cdots, n - k(0) + 1$ it follows by Lemma 2.5 that $T^iTh_{k(0)n}$ and $T\psi_{cR(T)}h_{k(0)n}$ have disjoint support, and hence that $T^i\psi_{R(T)}h_{k(0)n}$ and $T^i_{cR(T)}h_{k(0)n}$ have disjoint support, $i = 1, \cdots, n - k(0)$ (since the supports of $Th_{k(0)n}$ and of $\psi_{R(T)}h_{k(0)n}$ are the same). That $T^i\psi_{R(T)}h_{k(0)n}$ and $T^i\psi_{cR(T)}h_{k(0)n}$ have disjoint support, $i = 1, \cdots, n - k(0)$ and the induction hypothesis imply that

$$\| T^i\psi_{cR(T)}h_{k(0)n} \| = \| \psi_{cR(T)}h_{k(0)n} \|,\ i = 1, \cdots, n - k(0)\ .$$

That $T^iTh_{k(0)n}$ and $T^i\psi_{cR(T)}h_{k(0)n}$ have disjoint support, $i=1, \cdots, n-k(0)$, implies that

$$\begin{aligned} \| T^ih_{k(0)+1n} \| &= \| T^iTh_{k(0)n} \| + \| T^i\psi_{cR(T)}h_{k(0)n} \| \\ &= \| T^iTh_{k(0)n} \| + \| \psi_{cR(T)}h_{k(0)n} \|\ , \end{aligned}$$

$i = 1, \cdots, n - k(0)$, since

$$\| T^i\psi_{cR(T)}h_{k(0)n} \| = \| \psi_{cR(T)}h_{k(0)n} \|,\ i=1, \cdots, n - k(0)\ .$$

This equation implies by the induction hypothesis, that

$$\begin{aligned} \| T^ih_{k(0)+1n} \| &= \| Th_{k(0)n} \| + \| \psi_{cR(T)}h_{k(0)n} \| \\ &= \| h_{k(0)+1n} \|\ , \end{aligned}$$

$i = 1, \cdots, n - k(0)$, from which the lemma follows, since $n - k(0) = n - (k(0) + 1) + 1$.

LEMMA 2.8. *Let $(X, \mathscr{F}, \mu)$ be a measure space, and let T be a positive contraction of $L_p(X, \mathscr{F}, \mu)$. Suppose that for some n there exists a positive function h_n such that $\| T^nh_n \| = \| h_n \|$. Then, for $f \in L_p(X, \mathscr{F}, \mu)$ we have*

$$\begin{aligned} &\frac{1}{n+1}\sum_{i=1}^{n+1}\int \sup | f, (f + Tf)/2, \cdots, (f + \cdots + T^{i-1}f)/2 |^p\, d\mu \\ &\leqq (p/p-1)^p \int | f |^p\, d\mu\ . \end{aligned}$$

THEOREM 2.2. *Let $(X, \mathscr{F}, \mu)$ be a measure space, and let T be a positive contraction of $L_p(X, \mathscr{F}, \mu)$. Suppose that for each n there exists a positive function h_n such that $\|T^n h_n\| = \|h_n\|$. Then T admits of a dominated estimate with constant $p/p-1$.*

Proof. Follows at once from Lemma 2.8.

3. Equivalent formulations. In this section, we obtain various sets of necessary and sufficient conditions for a positive contraction of L_1 to admit of a dominated estimate, and same related results.

DEFINITION 3.1. Let $(X, \mathscr{F}, \mu)$ be a measure space. For

$$1 \leqq p(1) \leqq p(2) \leqq +\infty \ ,$$

we define the class of positive contractions $\mathscr{A}(p_1, p_2)$ setting $\mathscr{A}(p_1, p_2) = \{T\colon T \text{ is a positive contraction of } L_p(X, \mathscr{F}, \mu),\ p(1) \leqq p \leqq p(2)\}$.

THEOREM 3.1. *If $1 < p(0) < +\infty$, then the following are equivalent:*

(i) *If $T \in \mathscr{A}(1, p(0))$, then T (regarded as a contraction of $L_{p(0)}(X, \mathscr{F}, \mu)$) admits of a dominate estimate with constant $c(p(0))$.*

(ii) *If $T \in \mathscr{A}(p(0), p(0))$, then T admits of a dominated estimate with constant $c(p(0))$.*

(iii) *If $T \in \mathscr{A}(p(0), +\infty)$, then T (regarded as a contraction of $L_{p(0)}(X, \mathscr{F}, \mu)$) admits of a dominated estimate with constant $c(p(0))$.*

Proof. Part (a), (i) implies (ii). Let $T \in \mathscr{A}(p(0), p(0))$, and suppose $\|T\| = \delta < 1$. Then $\|T^*\| = \delta < 1$ and there exists a positive function $\hbar \in L_{q(0)}(X, \mathscr{F}, \mu)$, with $q(0) = p(0)/p(0) - 1$, such that $T^*\hbar \leqq \hbar$ (to see this, let $f \in L_{q(0)}(X, \mathscr{F}, \mu)$ be an arbitrary positive function, and let $\hbar = \sum_{k=0}^{\infty} T^* f$). If $h = (\hbar)^{1/p(0)-1}$ then

$$h \in L_{p(0)}(X, \mathscr{F}, \mu) \quad \text{and} \quad T^* h^{p(0)-1} \leqq h^{p(0)-1} \ .$$

We may then define a transformation P of $L_{p(0)}(X, \mathscr{F}, m)$ by setting, for $f \in L_{p(0)}(X, \mathscr{F}, m)$

$$P(f) = T(f \cdot h)/h$$

where m is the measure given by $m(A) = \int_A h^{p(0)} d\mu$.

It follows by part (i) of Lemma 2.1 (with $h_1 = h_2 = h$, and

$$(X_1, \mathscr{F}_1, \mu_1) = (X_2, \mathscr{F}_2, \mu_2) = (X, \mathscr{F}, \mu))$$

that $\|P\|_{p(0)} \leqq \|T\|_{p(0)}$. Part (ii) of Lemma 2.1 implies that $P^*1 \leqq 1$, and since P is positive, it follows by the Riesz convexity theorem that $P \in \mathscr{A}(1, p(0))$ (with respect to the measure space $(X, \mathscr{F}, m)$). Since there is a set isomorphism of $(X, \mathscr{F}, m)$ into $(X, \mathscr{F}, \mu)$ (it may be necessary to choose h with a sufficiently small norm), we have by assumption that P admits of a dominated estimate with constant $c(p(0))$. Since we have that $P^n f = T^n(fh)/h$, and since $g \in L_{p(0)}(X, \mathscr{F}, \mu)$ implies $g/h \in L_{p(0)}(X, \mathscr{F}, m)$, we have that T admits of a dominated estimate with constant $c(p(0))$ if P does, and this finishes the proof of part (a) under the assumption that $\| T \| = \delta < 1$.

Next, suppose $\| T \| = 1$. Define $T_\alpha = \alpha T$ for $0 < \alpha \leqq 1$. For $g \in L_{p(0)}(X, \mathscr{F}, \mu)$, let g_α^* denote $\sup_n | (g + \cdots + T_\alpha^n g)/n + 1 |$. If $g \geqq 0$, then $\| g_\alpha^* \|_{p(0)}$ converges monotonically to $\| g_1^* \|_{p(0)}$, and the general case then follows from the special case, $\| T \| < 1$, by the Lebesgue monotone convergence theorem.

Part (b), (iii) $\Rightarrow$ (ii). Let $T \in \mathscr{A}(p(0), p(0))$ and assume $\| T \| = \delta < 1$. Let h be a positive function in $L_{p(0)}(X, \mathscr{F}, \mu)$ such that $Th \leqq h$. Define

$$Pf = T(fh)/h$$

for $f \in L_{p(0)}(X, \mathscr{F}, m)$ where $m(A) = \int_A h^{p(0)} d\mu$. Then P is a contraction of $L_{p(0)}(X, \mathscr{F}, m)$ since $\| P \|_{p(0)} \leqq \| T \|_{p(0)}$ by part (i) of Lemma 2.1. We also have $P(1) \leqq 1$, and hence $P \in \mathscr{A}(p(0), +\infty)$ (with respect to $(X, \mathscr{F}, m)$) again by the isomorphism result used in part (a) of the proof, we have that P admits of a dominated estimate with constant $c(p(0))$, by assumption. In a similar way to that of part (a), we have that T admits of a dominated estimate with constant $c(p(0))$ if P does, and the general case ($\| T \| \leqq 1$) follows from the special case by taking limits of $T_\alpha = \alpha T$ as in part (a).

Part (c), (ii) implies (i) and (iii). This case is trivial since

$$\mathscr{A}(1, p(0)) \subset \mathscr{A}(p(0), p(0)) ,$$

and since $\mathscr{A}(p(0), +\infty) \subset \mathscr{A}(p(0), p(0))$.

THEOREM 3.2. *If for each*

$$(X, \mathscr{F}, \mu),\ T \in \mathscr{A}(p(0), p(0)),\ 1 < p(0) < +\infty ,$$

admits of a dominated estimate with constant $c(p(0), f, T, (X, \mathscr{F}, \mu))$, then there exists a constant $c(p(0))$, independent of T and of f (and of $(X, \mathscr{F}, \mu)$) such that each $T \in \mathscr{A}(p(0), p(0))$ will admit of a dominated estimate with constant $c(p(0))$.

Proof. Note that we have used an obvious extension of a dominated estimate with constant dependent on the particular function used. Suppose to the contrary. Then there exists a sequence of spaces $(X_n, \mathscr{F}_n, \mu_n)$ and a sequence $\{f_n\}$ of positive functions with f_n in $L_{p(0)}(X_n, \mathscr{F}_n, \mu_n)$ such that $\|f_n\|^{p(0)} = 1/2^n, n = 1, 2, \cdots$, and a sequence $\{T_n\}$ of positive contractions in $\mathscr{A}(p(0), p(0))$ such that $\|T_n\| \leqq 1$, and such that

$$\int \sup_j |f_n, (f_n + T_n f_n)/2, \cdots, (f_n + \cdots + T_n^{j-1} f_n)/j|^{p(0)} du_n$$
$$\geqq 4^n \int |f_n|^{p(0)} d\mu_n .$$

Let $(Y, \mathscr{B}, m) = \bigoplus_{n=1}^{\infty} (X_n, \mathscr{F}_n, \mu_n)$, and let $T = \bigoplus_{n=1}^{\infty} T_n$. We have then, with $f = \sum_{n=1}^{\infty} f_n$, that

$$\int \sup_j |f, (f + Tf)/2, \cdots, (f + \cdots + T^{j-1}f)/j|^{p(0)} dm = +\infty$$

contradicting the hypothesis (a simple modification of the proof shows that we may restrict attention to a single measure space if it is non-atomic, by taking $(X_n, \mathscr{F}_n, \mu_n) = (X, \mathscr{F}, \mu/2^n)$, since $\bigoplus_{n=1}^{\infty} (X_n, \mathscr{F}_n, \mu_n)$ is then isomorphic $(X, \mathscr{F}, \mu)$).

THEOREM 3.3. *Let T be a positive contraction of $L_{p(0)}(X, \mathscr{F}, \mu)$, $1 < p(0) < +\infty$, and let f be a function in $L_{p(0)}(X, \mathscr{F}, \mu)$ such that $\|f\| = 1$ and such that*

$$\int \sup_j |f, (f + Tf)/2, \cdots, (f + \cdots + T^j f)/j + 1|^{p(0)} = +\infty .$$

Then for each $k > 0$, there exists a positive contraction $T(k)$ of $L_p(X, \mathscr{F}, \mu)$ such that

$$\sup_j |g_k, (g_k + T(k)g_k/2, \cdots, (g_k + \cdots T^j(k)g_k)/j + 1|^{p(0)} \geqq k$$

for a function g_k with $\|g_k\| = 1$.

Proof. Let $M(f, T) = \sup_j |f, \cdots, (f + \cdots + T^j f)/j + 1|^{p(0)}$. Since $\|M(f, T)\| = +\infty$, for each $k > 0$ there exists a positive isometry P of $L_{p(0)}(X, \mathscr{F}, \mu)$ such that $P(M(f, T)) \geqq k$. The lemma then follows if we define $T(k) = PTP^{-1}$ and $g_k = Pf$, on noting that

$$P(M(f, T)) = \sup_j |Pf, (Pf + PTf)/2, \cdots, (Pf + \cdots + PT^j f)/j + 1|^{p(0)} .$$

This may be seen in the case that $(X, \mathscr{F}, \mu)$ is the denumerable union of Lebesgue spaces, since isometries of $L_{p(0)}(X, \mathscr{F}, \mu)$, in this case, admit of the following representation:

$$Pf = f(\tau x) \cdot r(x)$$

where τ is an invertible measurable transformation, and where $r(x)$ is the nonnegative measurable function given by $r(x) = [d(\tau\mu)/d\mu]^{1/p(0)}$ (see, e.g., [1], p. 5). In the general case, we may see the equality on noting that each σ-finite subspace is set isomorphic to a denumberable union of Lebesgue spaces.

THEOREM 3.4. *If there exists a constant* $c(p(0))$ *such that*

$$T \in \mathscr{A}(p(0), p(0)), 1 < p(0) < +\infty$$

admits of a dominated estimate with constant $c(p(0))$ *then the limit*

$$\lim (1/n + 1) \sum_{k=0}^{n} T^k f$$

exists for each $f \in L_{p(0)}(X, \mathscr{F}, \mu)$.

Proof. The mean ergodic theorem for reflexive spaces gives us that

$$L_p(X, \mathscr{F}, \mu) = \{\overline{f: f = g - Tg}\} \oplus \{f: f = TF\} .$$

The dominated estimate then shows that the limit exists provided that it exists for fixed functions, which is trivial, and provided it exists for functions of the form $g - Tg$. In order to establish this fact, we need to show that $\lim_{n \to \infty} T^n g/n = 0$. This may be seen on noting that

$$\int \sum_{n=1}^{\infty} | T^n f/n |^{p(0)} d\mu = \sum_{n=1}^{n} \int | T^n f/n |^{p(0)} d\mu < +\infty ,$$

as has been pointed out by M. A. Akcoglu.

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