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EXISTENCE OF A SPECTRUM FOR NONLINEAR TRANSFORMATIONS

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Denote by S a complex (nondegenerate) Banach space. Suppose that T is a transformation from a subset of S to S. A complex number λ is said to be in the resolvent of T if $(\lambda I - T)^{-1}$ exists, has domain S and is Fréchet differentiable (i.e., if p is in S there is a unique continuous linear transformation $F = [(\lambda I - T)^{-1}]'(p)$ from S to S so that

$$\lim_{q \to p} \| q - p \|^{-1} \| (\lambda I - T)^{-1} q - (\lambda I - T)^{-1} p - F(q - p) \| = 0)$$

and locally Lipschitzean everywhere on S. A complex number is said to be in the spectrum of T if it is not in the resolvent of T.

Suppose in addition that the domain of T contains an open subset of S on which T is Lipschitzean.

THEOREM. *T has a (nonempty) spectrum.*

If T is a continuous linear transformation from S to S, then the notion of resolvent and spectrum given here coincides with the usual one ([1], p. 209, for example). Such a transformation T is, of course, Lipschitzean on all of S and hence the above theorem gives as a corollary the familiar result that a continuous linear transformation on a complex Banach space has a spectrum.

The set of all complex numbers is denoted by C.

LEMMA. *Suppose that $d > 0$, p is in S, Q is a transformation from a subset of S to S, D is an open set containing p which is a subset of the domain Q, Q is Lipschitzean on D and $(I - cQ)^{-1}$ exists and has domain S if c is in C and $|c| < d$. Then,*

$$\lim_{c \to 0} (I - cQ)^{-1} p = p .$$

Proof. Denote by M a positive number so that $\| Qr - Qs \| \leqq M \| r - s \|$ if r and s are in D. Suppose $\varepsilon > 0$. Denote by δ a number so that $0 < \delta < \min(\varepsilon, 1/2)$ and $\{q \in S : \| q - p \| \leqq \delta\}$ is a subset of D. Denote by δ' a positive number so that $\delta'(\max(M, \| Qp \|)) < \delta/2$. Denote by c a member of C so that $|c| < \min(\delta', d)$. Denote $(I - cQ)^{-1} p$ by q, denote p by q_0 and $p + cQq_{n-1}$ by q_n, $n = 1, 2, \cdots$.

Then, $\| q_1 - q_0 \| = \| p + cQq_0 - q_0 \| = |c| \, \| Qq_0 \| < \delta/2$. Suppose that k is a positive integer so that

$$\| q_m - q_{m-1} \| < (\delta/2)^m, \ m = 1, 2, \cdots, k .$$

Then $\| q_m - p \| \leqq \sum_{j=0}^{m-1} \| q_{j+1} - q_j \| \leqq \sum_{j=0}^{m-1} (\delta/2)^{j+1} < \delta$, $m = 0, 1, \cdots, k$ and hence

$$\begin{aligned} \| q_{k+1} - q_k \| &= \| cQq_k - cQq_{k-1} \| \\ &\leqq |c| M \| q_k - q_{k-1} \| \\ &\leqq |c| M(\delta/2)^k \leqq (\delta/2)^{k+1} . \end{aligned}$$

Hence $\| q_n - q_{n-1} \| \leqq (\delta/2)^n$, $n = 1, 2, \cdots$ and therefore $q_1, q_2, \cdots$ converges to a point r of S. Note that $\| q_{n+1} - p \| \leqq \sum_{j=0}^{n} (\delta/2)^{j+1} < \delta$, $n = 1, 2, \cdots$ so that $\| r - p \| \leqq \delta$ and hence r is in D. But $\| r - (p + cQr) \| = \| (r - q_{n+1}) + (p + cQq_n) - (p + cQr) \| \leqq \| r - q_{n+1} \| + |c| \| Qq_n - Qr \| \leqq \| r - q_{n+1} \| + |c| M \| q_n - r \| \to 0$ as $n \to \infty$. Hence $r = p + cQr$, i.e., $(I - cQ)r = p$, i.e., $r = (I - cQ)^{-1}p = q$. Hence, $\| (I - cQ)^{-1}p - p \| \leqq \delta < \varepsilon$. This proves the lemma.

Proof of theorem. Suppose the statement of the theorem is false. Then T has an inverse since if not, 0 would be in the spectrum of T. Denote by D an open set on which T is defined and is Lipschitzean. Denote by p a point of D different from $-T(0)$.

Define $f(\lambda)$ to be $(\lambda I - T)^{-1}p$ for all λ in C. Suppose b is in C. If q is in S and different from p denote

$$(1/\| q - p \|)\{[bI - T)^{-1}q - (bI - T)^{-1}p] - [(bI - T)^{-1}]'(p)(q - p)\}$$

by $L(q)$. Denote by $L(p)$ the zero element of S and note that $\lim_{p \to p} L(q) = L(p)$ since $(bI - T)^{-1}$ is Fréchet differentiable at p. Denote $(bI - T)^{-1}$ by Q. If λ is in C, then

$$(\lambda I - T) = [I - (b - \lambda)(bI - T)^{-1}](bI - T)$$

and, since both $(\lambda I - T)^{-1}$ and $(bI - T)^{-1}$ exist and have domain S, it follows that $[I - (b - \lambda)(bI - T)^{-1}]^{-1} = [I - (b - \lambda)Q]^{-1}$ has the same properties and $(\lambda I - T)^{-1} = Q[I - (b - \lambda)Q]^{-1}$.

Hence, if λ is in C,

$$\begin{aligned} f(\lambda) - f(b) &= Q[I - (b - \lambda)Q]^{-1}p - Qp \\ &= Q'(p)[[I - (b - \lambda)Q]^{-1}p - p] \\ &\quad + \| [I - (b - \lambda)Q]^{-1}p - p \| L([I - (b - \lambda)Q]^{-1}p) . \end{aligned}$$

But $[I - (b - \lambda)Q]^{-1}p - p = (b - \lambda)Q[I - (b - \lambda)Q]^{-1}p$ so

$$\begin{aligned} &(\lambda - b)^{-1}[f(\lambda) - f(b)] \\ &\quad = -Q'(p)Q[I - (b - \lambda)Q]^{-1}p \\ &\qquad + (|b - \lambda|/(\lambda - b)) \| Q[I - (b - \lambda)Q]^{-1}p \| \\ &\qquad \times L([I - (b - \lambda)Q]^{-1}p) \to -Q'(p)Qp \end{aligned}$$

as $\lambda \to b$ since $\lim_{\lambda \to b}[I - (b - \lambda)Q]^{-1}p = p$. Hence,

$$f'(b) = -[(bI - T)^{-1}]'(p)(bI - T)^{-1}p \, .$$

Now $\lim_{c\to 0}(I - cT)^{-1}p = p$. Denote by δ a positive number so that if $|c| \leqq \delta$, then $\|(I - cT)^{-1}p\| \leqq \|p\| + 1$. Then if λ is in C and $|\lambda| \geqq 1/\delta$, $\|f(\lambda)\| = \|(\lambda I - T)^{-1}p\| = |1/\lambda| \, \|(I - (1/\lambda)T)^{-1}p\| \leqq \delta(\|p\| + 1)$. Hence f is bounded. So, by Liouville's theorem ([1], p. 129, for example), f is constant, i.e., there is a point q in S such that if λ is in C, $(\lambda I - T)^{-1}p = f(\lambda) = q$, and so $\lambda q = p + Tq$. Hence it must be that $q = 0$, i.e., $p = -T(0)$, a contradiction. This establishes the theorem.

The author considers it likely that the statement of the theorem is true if the condition (in the definition of resolvent) that $(\lambda I - T)^{-1}$ be locally Lipschitzean is dropped.

Reference

1. K. Yosida, *Functional analysis*, Academic Press, New York, 1965.

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