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COVERINGS OF MAPPING SPACES

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The purpose of this paper is to give conditions on a pair of topological spaces (X, B) such that any covering $\rho: E \to B$ induces a covering map

$$\bar{\rho}: E^X \to \bar{\rho}(E^X) \subset B^X$$

where $\bar{\rho}(f) = \rho^\circ f$ and the mapping spaces have the compact-open topology.

This is given in Theorem 1.1. In the classical theory of (connected) coverings over a space B which is connected, locally pathwise connected and semi-locally 1-connected, it is known that to each subgroup $H \subset \pi_1(B, b_0)$ there corresponds a covering projection $\rho: E \to B$ for which

$$\rho_\sharp(\pi_1(E, e_0)) = H$$

for some $e_0 \in \rho^{-1}(b_0)$. Section 2 gives a characterization of those subgroups $H \subset \pi_1(B^X, v)$ which correspond to a mapping covering $\rho: E^X \to B^X$ for some covering $\rho: E \to B$. Section 3 gives partial answers to several questions about mapping coverings, such as when mapping coverings are regular or universal.

1. Mapping coverings. Given a topological space X and a map $\rho: E \to B$, then $\bar{\rho}: E^X \to B^X$, $\bar{\rho}(f) = \rho \circ f$, is continuous if the function spaces of continuous maps E^X and B^X are each given the compact-open topology. In this section we prove

THEOREM 1.1. *Let $\rho: E \to B$ be a covering projection for which E and B are* ANR*'s, and let X be a compact Hausdorff space. Then $\bar{\rho}: E^X \to B^X$ is a covering projection of E^X onto $\bar{\rho}(E^X) \subset B^X$.*

Actually E is automatically an ANR if B is (see § 3), so the hypothesis of (1.1) is just a condition on X and B. We begin the proof of (1.1) by considering two lemmas, the first of which is a result in Spanier [6; 2.5.10].

LEMMA 1.2. *Every Hurewicz fibration with unique path lifting whose base space is locally path connected and semilocally 1-connected and whose total space is locally path connected is a covering projection onto its image.*

Since we eventually want to apply this result to the map $\bar{\rho}: E^X \to$

B^X we need information about the local structure of the function spaces E^X and B^X. If X is a compact metrizable space and Y is an ANR, then the function space Y^X is an ANR ([4; p. 186]) and consequently is locally contractible ([4; p. 96]). We now give a direct proof of this local contractibility of Y^X which does not require the metrizability restriction on X.

LEMMA 1.3. *If X is a compact Hausdorff space and Y is an* ANR, *then the function space Y^X of continuous maps is locally contractible in the compact-open topology.*

Proof. Let the metrizable space Y be considered as a closed subset of a convex set Z in a locally convex topological vector space L ([4; p. 81]). Since Y is an ANR there exists an open neighborhood W of Y in Z together with a retraction $r: W \to Y$, i.e., $r|_Y = 1_Y$.

Given a map $f \in Y^X$ and a neighborhood P of f, we may assume $P = \bigcap_{i=1}^n K(C_i, U_i)$ where $K(C_i, U_i) = \{g \in Y^X: g(C_i) \subset U_i\}$ for each member of the collection $\{C_i\}$ of compact subspaces of X and corresponding member of the collection $\{U_i\}$ of open subsets of Y. Since W is an open subset of a convex set Z in a locally convex topological vector space L, each open covering $\alpha_i = \{r^{-1}(U_i), r^{-1}(Y - f(C_i))\}$ of W admits an open refinement β_i consisting of convex sets. For each $x \in X$ and index i, let $V_{x,i}$ be a member of the covering β_i which contains $f(x)$. Form the convex set $V_x = \bigcap_{i=1}^n V_{x,i}$ for each $x \in X$ and choose by the regularity of X a closed neighborhood $A_x \subset f^{-1}(V_x)$ of x. By the compactness of X, select points $x_1, \cdots, x_m$ of X so that $\{A_j = A_{x_j}: j = 1, \cdots, m\}$ is a collection of closed sets which cover X; then let $V_j = V_{x_j}$ an $V_{j,i} = V_{x_j,i}$. Note that

$$C_i \cap A_j \neq \emptyset \quad \text{implies} \quad V_j \cap Y \subset U_i \,. \tag{1.4}$$

This follows from the facts that $f(A_j) \subset V_j \cap Y$ and that V_j is contained in $V_{j,i}$, a member of the covering β_i which refines α_i.

We define an open neighborhood of

$$f \in Y^X \quad \text{by} \quad K = \bigcap_{j=1}^m K(A_j, V_j \cap Y) \,.$$

Our first claim for K is that it lies in P, i.e., that for $g \in K$, $g(C_i) \subset U_i (i = 1, \cdots, n)$. Since $\{A_j\}_{j=1}^m$ is a cover of X, we need merely to show that $g(C_i \cap A_j) \subset U_i$ for all i, j. If $C_i \cap A_j = \emptyset$, the result holds trivially; if $C_i \cap A_j \neq \emptyset$, it follows from the relations

$$g(C_i \cap A_j) \subset g(A_j) \subset V_j \cap Y \subset U_i \,,$$

the last being due to (1.4).

Our second claim on K is that it is contractible rel f in P. Because Y is contained in Z, a convex subset of a topological vector space, we can define a continuous function $H: Y^X \times I \times X \to Z$ by $H(g, t, x) = tf(x) + (1-t)g(x)$. Since on the member A_j of the covering $\{A_j\}_{j=1}^m$ both f and $g \in K$ take values in the convex subset $V_j \subset W$, it follows that $H(K \times I \times X) \subset W$ and therefore the composition $r^\circ H$: $K \times I \times X \to W \to Y$ is well defined. The associated map $h: K \times I \to Y^X$ given by $h(g, t)(x) = r(H(g, t, x))$ takes values in $P \subset Y^X$ since $r(H(K \times I \times (C_i \cap A_j))) \subset r(H(K \times I \times A_j)) \subset r(V_j)$ and the latter is contained in $r(r^{-1}(U_i)) = U_i$ when $C_i \cap A_j \neq \emptyset$. Thus $h: K \times I \to P$ is a homotopy rel f from the inclusion $K \subset P$ to the constant map $K \to f \in P$. This shows that Y^X is locally contractible.

Proof of Theorem 1.1. We first show that if $\rho: E \to B$ is a covering projection and X is a compact Hausdorff space, then $\bar{\rho}: E^X \to B^X$ (and hence $\bar{\rho}: E^X \to \bar{\rho}(E^X)$) is a Hurewicz fibration with unique path lifting. For a homotopy $h_t: Z \to B^X$ of a map $h_0: Z \to B^X$ which lifts to a map $g_0: Z \to E^X$, the associated map $h'_t: Z \times X \to B$ is a homotopy of the associate $h'_0: Z \times X \to B$ which lifts to $g'_0: Z \times X \to E$. Since $\rho: E \to B$ is a Hurewicz fibration, the homotopy h'_t lifts to a homotopy $g'_t: Z \times X \to E$ of g'_0, and therefore the associate $g_t: Z \to E^X$ is a homotopy of g_0 which is a lifting of h_t. This shows that $\bar{\rho}: E^X \to B^X$ is a Hurewicz fibration.

If $\omega, \gamma: I \to E^X$ are paths in E^X which cover the same path $\alpha: I \to B^X$ and $\omega(0) = \gamma(0)$, then their associates $\omega', \gamma': I \times X \to E$ agree on the subspace $0 \times X$ of $I \times X$ and they are liftings of the associate $\alpha': I \times X \to B$. Since a covering map has the unique lifting property for connected spaces, the fact that $0 \times X$ meets each component of $I \times X$ implies that $\omega' = \gamma'$ and hence $\omega = \gamma$. This shows that $\bar{\rho}: E^X \to B^X$ has unique path lifting.

In view of Lemma 1.2 the proof that $\bar{\rho}: E^X \to \bar{\rho}(E^X)$ is a covering projection is complete once it is shown that E^X and $\bar{\rho}(E^X)$ are locally path connected and $\bar{\rho}(E^X)$ is semilocally 1-connected. Since E^X is locally contractible by (1.3) the condition on E^X is trivial; since B^X is also locally contractible the conditions on $\bar{\rho}(E^X) \subset B^X$ follows from the fact that the image of a Hurewicz fibration is the union of path components of the base space.

There are two convenient corollaries of Theorem 1.1. In the first, the notation $(Y^X)_f$ is used for the path component of the function space Y^X containing $f: X \to Y$.

COROLLARY 1.5. *If, in addition to the hypotheses of* (1.1), $v': X \to E$ *is a lifting of* $v: X \to B$, *then* $\bar{\rho}: (E^X)_{v'} \to (B^X)_v$ *is a covering* *projection.*

COROLLARY 1.6. *If, in addition to the hypotheses of* (1.1), X *is locally path connected and* Hom $(\pi_1(X, x_0), \pi_1(B, b_0)) = 0$ *for every* $x_0 \in X$, $b_0 \in B$, *then* $\bar{\rho}: E^X \to B^X$ *is a covering projection.*

Corollary 1.5 is immediate. In (1.6) we are asserting that the additional hypotheses imply the surjectivity of $\bar{\rho}: E^X \to B^X$. Since $\rho: E \to B$ is a covering projection a necessary and sufficient condition that a map $f: (Y, y_0) \to (B, b_0)$ with connected locally path connected domain have a lifting $(Y, y_0) \to (E, e_0)$ is that in $\pi_1(B, b_0)$, $f_\# \pi_1(Y, y_0) \subset \rho_\# \pi_1(E, e_0)$. Thus the hypothesis Hom $(\pi_1(X, x_0), \pi_1(B, b_0)) = 0$, for every $x_0 \in X$, $b_0 \in B$, implies that a map $f: X \to B$ has a lifting on each (path) component of X. Because the components of a locally path connected space are open and closed, liftings on the components of X determine a lifting on all of X. Thus $\bar{\rho}$ is surjective.

2. **Subgroups of $\pi_1(B^X, v)$ realizable by mapping coverings.** In this section X will always represent a connected finite CW complex of dim $\leqq n$, B a path connected simple ANR, and $v: X \to B$ a selected map. For convenience in stating the main theorem of this section, we define $K_B = \ker \{r_\#: \pi_1(B^X, v) \to \pi_1(B^{X^0}, r(v))\}$, where $r: B^X \to B^{X0}$ is the map induced by restriction to the 0-skeleton X^0 of X, and we define $e_{x_0}: B^X \to B$ to be the evaluation map $e_{x_0}(f) = f(x_0)$ at $x_0 \in X^0$.

If $\rho: E \to B$ is a covering projection, it follows that E is an ANR (see § 3) so that by (1.5) $\bar{\rho}: (E^X)_{v'} \to (B^X)_v$ is a covering projection for each lifting $v': X \to E$ of $v: X \to B$. We say a subgroup $G \subset \pi_1(B^X, v)$ can be *realized by a mapping covering* if there exists a covering projection $\rho: E \to B$ with fundamental group $(e_{x_0})_\#(G)$ (that is, $\rho_\# \pi_1(E, e_0) = (e_{x_0})_\#(G)$) and a lifting $v': X \to E$ of $v: X \to B$ such that the covering projection $\bar{\rho}: (E^X)_{v'} \to (B^X)_v$ has fundamental group G (that is, $\bar{\rho}_\# \pi_1(E^X, v') = G \subset \pi_1(B^X, v)$). When $v: X \to B$ is homotopic to the constant map, it follows from [2; 6.1] that the condition on the fundamental group of ρ is a consequence of that on the fundamental group of $\bar{\rho}$.

THEOREM 2.1. *A subgroup* $G \subset \pi_1(B^X, v)$ *can be realized by a mapping covering if and only if* $G \supset K_B$ *and* $e_{x_0\#}(G) \supset v_\#(\pi_1(X, x_0))$.

COROLLARY 2.2. *When* X *is simply connected, a subgroup* $G \subset \pi_1(B^X, v)$ *can be realized by a mapping covering if and only if it contains* K_B.

COROLLARY 2.3. *Let* $\pi_1(B) = 0$ *for* $1 < i \leqq n$. *Then a subgroup* $G \subset \pi_1(B^X, v)$ *can be realized by a mapping covering if and only if* $G \supset K_B = H^n(X; \pi_{n+1}(B))$ *and* $(e_{x_0})_\#(G) \supset v_\#(\pi_1(X, x_0))$.

EXAMPLE. Let $X = S^2, E = S^3, B = P^3$, the 3-dimensional real projective space. P^3 is a topological group $(SO(3)) \Rightarrow P^3$ is simple. Let $\rho: S^3 \rightarrow P^3$ be the antipodal identification map. The hypothesis of 2.2 and 2.3 are satisfied for $n = 2$. Thus the only subgroups of $\pi_1(P^{3S^2}, v)$ realized by a mapping covering are those containing $K_B = H^2(S^2, \pi_3(P^3)) \approx Z$.

Let $c: S^2 \rightarrow P^3$ be the constant map to $p_0 \in P^3$. It follows easily from the spectral sequence in [3] and Theorem 6.1 of [2] that the sequence below is split exact

$$0 \longrightarrow H^2(S^2: \pi_3(P^3)) \longrightarrow \pi_1(P^{3S^2}, c) \xrightarrow{r_\#} \pi_1(P^3, c) \longrightarrow 0$$

where $H^2(S^2; \pi_3(P^3)) \approx K_{p^3}$ and $r_\#$ is induced by the restriction map r. P^3 is a topological group $\Rightarrow P^{3S^2}$ is a topological group $\Rightarrow \pi_1(P^{3S^2}, c)$ is abelian $\Rightarrow \pi_1(P^{3S^2}, c) \approx Z \oplus Z_2$.

Thus the only subgroups of $\pi_1(P^{3S^2}, c)$ which are realizable by a mapping covering are $Z \oplus \{0\}$ and $Z \oplus Z_2$ which correspond to $\bar{\rho}: S^{3S^2} \rightarrow P^{3S^2}$ and $I: P^{3S^2} \rightarrow P^{3S^2}$.

We give the proof of (2.1) after a few preliminary propositions. The first involves exact couples of Federer [3] and is easily proved from the data given there.

PROPOSITION 2.4. *Let $\rho: W \rightarrow Z$ be a map between path connected simple spaces. Then ρ induces a map*

$$\rho^i: \mathscr{C}^i(X, W, f) \longrightarrow \mathscr{C}^i(X, Z, \rho \circ f)$$

of the ith Federer exact couples.. Furthermore, there is a commutative diagram

$$\begin{array}{ccc} E^2_{p,q}(W) & \xrightarrow{\rho^2} & E^2_{p,q}(Z) \\ \gamma_W \downarrow & & \downarrow \gamma_Z \\ H^q(X; \pi_{p+q}(W)) & \xrightarrow{(\rho_\#)^*} & H^q(X; \pi_{p+p}(Z)) \end{array}$$

where γ is an isomorphism onto if $p > 0$ and into if $p = 0$.

PROPOSITION 2.5. *Let $\rho: W \rightarrow Z$ be a covering projection between path connected simple spaces. Then for the map*

$$\rho^i: \mathscr{C}^i(X, W, f) \longrightarrow \mathscr{C}^i(X, Z, \rho \circ f)$$

of the i-th Federer exact couple,

$$\rho^i: E^i_{p,q}(W) \longrightarrow E^i_{p,q}(Z) \qquad (i \geqq 2)$$

is an isomorphism for all (p, q) satisfying either (a) *if $p \geqq 1$, then*

$p+q>1$ or (b) *if $p=0$, then $q \geqq i$.*

Proof. We proceed by induction on $i \geqq 2$. Since $p\colon W \to Z$ is a covering projection, $\rho_\#\colon \pi_j(W) \to \pi_j(Z)$ is an isomorphism for $j \geqq 2$ and a monomorphism for $j=1$. Then in the commutative diagram of (2.4) γ_W, γ_Z, and $(\rho_\#)^*$ are isomorphisms for $p+q \geqq 2$, $p \geqq 1$, hence ρ^2 is an isomorphism here. For $p=0$, $q \geqq 2$, γ_W, γ_Z are injective and $(\rho_\#)^*$ is bijective; consequently ρ^2 is injective. That ρ^2 is also surjective when $p=0, q \geqq 2$ follows from the definition of $\mathscr{C}^1(X)$ in [3] and the following statement which has the same proof as that of [6, 7.6.22].

(2.6) Let $q \geqq 2$ and let $h\colon X^q \to Z$ be given such that $h \mid X^{q-1} = p \circ f \mid X^{q-1}$. Then since $\rho_\#\colon \pi_{q-1}(W) \to \pi_{q-1}(Z)$ is injective and $\rho_\#\colon \pi_q(W) \to \pi_q(Z)$ is surjective, there exists $h'\colon X^q \to E$ such that

$$h' \mid X^{q-1} = f \mid X^{q-1} \quad \text{and} \quad \rho \circ h' \simeq h(\operatorname{rel} X^{q-1})\,.$$

We now assume that (2.5) holds for $i = k-1 \geqq 2$, i.e., ρ^{k-1} is an isomorphism if (a) $p \geqq 1$ and $p+q>1$, or (b) $p=0$ and $q \geqq k-1$. If $p \geqq 1$ and $p+q>1$, $(q \geqq 0)$, then in

$$\begin{array}{c} E^k_{p,q}(W) = \dfrac{\ker\{d\colon E^{k-1}_{p,q}(W) \longrightarrow E^{k-1}_{p-1,q+k-1}(W)\}}{\operatorname{im}\{d\colon E^{k-1}_{p+1,q-k+1}(W) \longrightarrow E^{k-1}_{p,q}(W)\}} \\ \Big\downarrow \rho^k \\ E^k_{p,q}(Z) = \dfrac{\ker\{d\colon E^{k-1}_{p,q}(Z) \longrightarrow E^{k-1}_{p-1,q+k-1}(Z)\}}{\operatorname{im}\{d\colon E^{k-1}_{p+1,q-k+1}(Z) \longrightarrow E^{k-1}_{p,q}(Z)\}} \end{array}$$

we have $E^{k-1}_{p,q}(W) \approx E^{k-1}_{p,q}(Z)$ by case (a) of the induction hypothesis; $E^{k-1}_{p-1,q+k-1}(W) \approx E^{k-1}_{p-1,q+k-1}(Z)$ when $p \geqq 2$ by case (a) and when $p=1$ by case (b); and $E^{k-1}_{p+1,q-k+1}(W) \approx E^{k-1}_{p+1,q-k+1}(Z)$ because if $q<k-1$ then both are zero, and if $q \geqq k-1$ then case (a) of the induction hypothesis applies. Thus case (a) of (2.5) holds for ρ^k.

To show that case (b) of (2.5) holds for ρ^k suppose that index $p=0$. Here we must show that

$$\begin{array}{l} E^k_{0,q}(W) = E^{k-1}_{0,q}(W)/\operatorname{im}\{d\colon E^{k-1}_{1,q-k+1}(W) \longrightarrow E^{k-1}_{0,q}(W) \\ \Big\downarrow \rho^k \\ E^k_{0,q}(Z) = E^{k-1}_{0,q}(Z)/\operatorname{im}\{d\colon E^{k-1}_{1,q-k+1}(Z) \longrightarrow E^{k-1}_{0,q}(Z)\} \end{array}$$

is an isomorphism for $q \geqq k$. This is obvious since then $E^{k-1}_{0,q}(W) \approx E^{k-1}_{0,q}(Z)$ by case (b) of the induction hypothesis and $E^{k-1}_{1,q-k+1}(W) \approx E^{k-1}_{1,q-k+1}(Z)$ by case (a).

Before giving the proof of Theorem 2.1, we prove two lemmas.

LEMMA 2.7. *If $\rho\colon E \to B$ is a covering projection with E a path connected simple space and $v'\colon X \to E$ is a lifting of $v\colon X \to B$, then there is a commutative ladder*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_E & \longrightarrow & \pi_1(E^X, v') & \xrightarrow{e_{x_0}} & E^{\infty}_{1,0}(E) & \longrightarrow & 0 \\ & & \phi\Big\downarrow & & \Big\downarrow(\bar{\rho})_\# & & \Big\downarrow\rho^{\infty} & & \\ 0 & \longrightarrow & K_B & \longrightarrow & \pi_1(B^X, v) & \xrightarrow{e_{x_0}} & E^{\infty}_{1,0}(B) & \longrightarrow & 0 \end{array}$$

in which the rows are exact and ϕ is an isomorphism.

Proof. By a theorem on page 351 of [3], the images of

$$r_\#: \pi_1(E^X, v') \longrightarrow \pi_1(E^{X^0}, r(v')),\ r_\#: \pi_1(B^X, v) \longrightarrow \pi_1(B^{X^0}, r(v))$$

can be identified with the subgroups

$$E^{\infty}_{1,0}(E) \subset E^2_{1,0}(E) = H^0(X, \pi_1(E)) = \pi_1(E),$$
$$E^{\infty}_{1,0}(B) \subset E^2_{1,0}(B) = H^0(X, \pi_1(B)) = \pi_1(B).$$

by means of diagonal homomorphisms:

$$\left\{\begin{array}{l} \Delta: \pi_1(E, v'(x_0)) \longrightarrow \pi_1(E^{X^0}, r(v')) \\ \Delta: \pi_1(B, v(x_0)) \longrightarrow \pi_1(B^{X^0}, r(v)) \end{array}\right\}.$$

The identification process is natural in E, B and so there is a commutative ladder as indicated.

To show that ϕ is an isomorphism we consider the following normal chains (see [3, p. 351]) for $\pi_1(E^X, v'), \pi_1(B^X, v)$ and maps induced by $\bar{\rho}$

$$(2.8)\qquad \begin{array}{ccc} \pi_1(E^X, v') & \xrightarrow{\bar{\rho}_\#} & \pi_1(B^X, v) \\ \cup & & \cup \\ H_0 & \longrightarrow & G_0 \\ \cup & & \cup \\ H_1 & \longrightarrow & G_1 \\ \cup & & \cup \\ \vdots & & \vdots \\ \cup & & \cup \\ H_{n-1} & \longrightarrow & G_{n-1} \\ \cup & & \cup \\ 0 & & 0 \end{array}$$

given by

$$\begin{array}{l} H_i = \ker\{r_\#: \pi_1(E^X, v') \longrightarrow \pi_1(E^{X^i}, r(v'))\} \\ G_i = \ker\{r_\#: \pi_1(B^X, v) \longrightarrow \pi_1(B^{X^i}, r(v))\} \end{array} \qquad (i = 0, 1, \cdots, n).$$

Thus we must show $K_E = H_0 \approx G_0 = K_B$. By [3, p. 351], there are isomorphisms

$$\text{(2.9)} \qquad \frac{H_i}{H_{i+1}} \approx E^{\infty}_{1,i+1}(E), \frac{G_i}{G_{i+1}} \approx E^{\infty}_{1,i+1}(B) \qquad (i = 0, \cdots, n-1)$$

which can be shown to be compatible with the homomorphisms induced by $\bar{\rho}$. Since $E^{\infty}_{1,i}(E) = E^{k}_{1,i}(E)$ and $E^{\infty}_{1,i}(B) = E^{k}_{1,i}(B)$ for $k > \max(i, \dim X - i)$, Proposition (2.5) implies that

$$\rho^{\infty}\colon E^{\infty}_{1,i}(E) \longrightarrow E^{\infty}_{1,i}(B)$$

is an isomorphism for $i \geqq 1$. Via induction and the five lemma, these isomorphisms together with those of (2.9) imply that all but the top homomorphism of the ladder (2.8) are isomorphisms.

LEMMA 2.10. *Let $\rho\colon E \to B$ be a covering projection with E a path connected (simple) space. If in the commutative diagram*

$$\begin{array}{ccccccc}
E^{\infty}_{1,0}(E) \subset E^{2}_{1,0}(E) & \overset{\gamma}{\approx} & H^0(X, \pi_1(E)) & = & \pi_1(E) \\
\downarrow \rho^{\infty} \qquad \downarrow \rho^2 & & \downarrow (\rho_\sharp)^* & & \downarrow \rho_\sharp \\
E^{\infty}_{1,0}(B) \subset E^{2}_{1,0}(B) & \overset{\gamma}{\approx} & H^0(X, \pi_1(B)) & = & \pi_1(B)
\end{array}$$

$\rho_\sharp(\pi_1(E)) \subset E^{\infty}_{1,0}(B)$, then $E^{\infty}_{1,0}(E) = \pi_1(E)$.

Proof. We will show by induction on $k \geqq 2$ that $E^{k}_{1,0}(E) = E^{2}_{1,0}(E)$, or equivalently, that $d^k_E\colon E^k_{1,0}(E) \to E^k_{0,k}(E)$ is zero. By (2.4) there is a commutative diagram

$$\begin{array}{ccccc}
\pi_1(E) \approx E^2_{1,0}(E) \supset E^k_{1,0}(E) & \xrightarrow{d^k_E} & E^k_{0,k}(E) \\
\downarrow \rho_\sharp \qquad \downarrow \rho^2_{1,0} \qquad \downarrow \rho^k_{1,0} & & \downarrow \rho^k_{0,k} \\
\pi_1(B) \approx E^2_{1,0}(B) \supset E^k_{1,0}(B) & \xrightarrow{d^k_B} & E^k_{0,k}(B)
\end{array}$$

in which $\rho^k_{0,k}$ is an isomorphism for $k \geqq 2$ by (2.5) and $\rho^k_{1,0}$ is a monomorphism for $k \geqq 2$ because $\rho_\sharp$ is a monomorphism. In order to prove d^k_E is zero for $k \geqq 2$, we need only show $\ker d^k_B \supset \operatorname{im} \rho^k_{1,0}$ for $k \geqq 2$. We proceed by induction on $k \geqq 2$. For $k = 2$, the statement follows from the relations

$$\ker d^2_B \supset E^{\infty}_{1,0}(B) \supset \rho_\sharp(\pi_1(E)) = \operatorname{im} \rho^2_{1,0}\ .$$

If we assume $\ker d^j_B \supset \operatorname{im} \rho^j_{1,0}$ for $j < k$, $k \geqq 2$, then $E^2_{1,0}(E) = E^k_{1,0}(E)$ and hence $\rho_\sharp(\pi_1(E)) = \operatorname{im} \rho^k_{1,0}$. Then we have the relation

$$\ker d^k_B \supset E^{\infty}_{1,0}(B) \supset \rho_\sharp(\pi_1(E)) = \operatorname{im} \rho^k_{1,0}$$

which completes the proof by induction.

Proof of Theorem 2.1. Suppose that $G \subset \pi_1(B^X, v)$ can be realized by a mapping covering, i.e., there exists a covering projection $\rho: E \to B$ and a lifting $v': X \to E$ of $v: X \to B$ such that $\bar{\rho}_{\#}(\pi_1(E^X, v') = G$ and $\rho_{\#}(\pi_1(E, v'(x_0)) = (e_{x_0})_{\#}(G)$. Since we may assume that E is path connected and simple, Lemma 2.7 is applicable. If follows from commutativity of the diagram given there and the surjectivity of ϕ that

$$\bar{\rho}_{\#}(\pi_1(E^X, v')) \supset \bar{\rho}_{\#}(K_E) = K_B, \text{ i.e., } G \supset K_B .$$

Furthermore, we have relations

$$(e_{x_0})_{\#}(G) = \rho_{\#}(\pi_1(E, v'(x_0)) \supset v_{\#}(\pi_1(X, x_0)) .$$

Conversely, suppose given $G \supset K_B$ and $G' = (e_{x_0})_{\#}(G) \supset v_{\#}(\pi_1(X, x_0))$. Using the subgroup $G' \subset E^{\infty}_{1,0}(B) \subset \pi_1(B, v(x_0))$, it is possible to construct a covering projection $\rho: E \to B$ with E a path connected simple space such that $\rho_{\#}(\pi_1(E, e_0)) = G'$. Since $v_{\#}(\pi_1(X, x_0)) \subset G' \subset \rho_{\#}(\pi_1(E, e_0))$, there exists a lifting $v': (X, x_0) \to (E, e_0)$ of $v: X \to B$. Then by Lemma (2.7) there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_E & \longrightarrow & \pi_1(E^X, v') & \xrightarrow{(e_{x_0})_{\#}} & E^{\infty}_{1,0}(E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\rho}_{\#}} & & \downarrow{\scriptstyle \rho^{\infty}} & & \\ 0 & \longrightarrow & K_B & \longrightarrow & \pi_1(B^X, v) & \xrightarrow{(e_{x_0})_{\#}} & E^{\infty}_{1,0}(B) & \longrightarrow & 0 \end{array}$$

in which ϕ is an isomorphism and im $\rho^{\infty} = G' = (e_{x_0})_{\#}(G)$ since $E^{\infty}_{1,0}(E) = \pi_1(E)$ by Lemma (2.10). It follows from some diagram chasing that $\bar{\rho}_{\#}(\pi_1(E^X, v')) = G$.

3. Miscellaneous questions. Many questions arise concerning mapping coverings. In this section we consider certain ones and give partial answers.

(a) Is a covering space of an ANR an ANR?

(b) If G is a properly discontinuous group of homeomorphisms acting on an *ANR* E, is E/G, the orbit space of G, an ANR?

(c) If $\rho: E \to B$ is a covering, what is card $(\bar{\rho}^{-1}(f)), f \in \bar{\rho}(E^X)$?

(d) If $\rho: E \to B$ is regular, then is $\bar{\rho}: E^X \to B^X$?

(e) When does a fiber $\bar{\rho}^{-1}(f)$ lie in a single path component of E^X, i.e., when are all the lifts of f homotopic?

(*f*) When is $\bar{\rho}: E^X \to B^X$ universal?

For convenience, throughout this section we assume that B is an ANR and X is a compact Hausdorff space.

(a) Since a covering space of an ANR is locally homeomorphic to an ANR, it is an ANR provided it is metrizable (see [4], III, 7.9 and 8.7). So Question (a) now reduces to a consideration of metrizability.

THEOREM 3.1. *If $\rho: E \to B$ is a covering with B metrizable, then E is metrizable.*

The proof utilizes the characterization of T_0 spaces which are metrizable due to A. H. Stone (see [1], page 196).

COROLLARY 3.2. *Every covering* of an ANR *is an* ANR.

(b) Since $\bar{\rho}: E \to E/G$ is a covering projection, the question, as in (a), reduces to one of metrizability.

THEOREM 3.3. *If a finite group of homeomorphisms G acts on a metric space E without fixed points, then E/G is metrizable.*

This again follows from Stone's characterization.

COROLLARY 3.4. *G finite, acting without fixed points on an* ANR $E \Rightarrow E/G$ *is an* ANR.

(c) If X is connected and locally pathwise connected, then $f: (X, x_0) \to (B, b_0)$ has a (unique) lift to $f^*: (X, x_0) \to (E, e_0)$, when $e_0 \in \rho^{-1}(b_0)$, if $f_\#(\pi_1(X, x_0)) \subset \rho_\#(\pi_1(E, e_0))$. If E is path connected and nonempty, the cardinality of $\bar{\rho}^{-1}(f)(f \in \bar{\rho}(E^X))$ reduces to the following question: How many conjugate subgroups of $\rho_\#(\pi_1(E, e_0))$ contain $f_\#(\pi_1(X, x_0))$?

THEOREM 3.5. *Let $\rho: E \to B$ be a regular covering such that E is connected. For any $f \in \bar{\rho}(E^X)$,* card $\bar{\rho}^{-1}(f) =$ card $\rho^{-1}(b_0)$, $b_0 \in B$.

Proof. E, B ANR $\Rightarrow E, B$ are locally pathwise connected. E is connected $\Rightarrow E, B$ are path connected. ρ is regular $\Rightarrow$ that the group G of covering transformations $\approx \pi_1(B, \rho(e_0))/\rho_\#(\pi_1(E, e_0)) \leftrightarrow \rho^{-1}(b_0)$. Then $f \in \bar{\rho}(E^X) \Rightarrow \exists f^*: X \to E \ni f = \rho \circ f^*$. Then

$$\rho^{-1}(f) = \{g \circ f^* \mid g \in G\} \longleftrightarrow \rho^{-1}(b_0)$$

because G acts transitively on $\rho^{-1}(b_0)$ and any lift of f is determined uniquely by the image of a single point.

With the same hypotheses as 3.5, we can show that any two path components of E^X lying over $(B^X)_v$ are homeomorphic. Specifically,

COROLLARY 3.6. *If v', v'' are any two lifts of $v: X \to B$, then* $(E^X)_{v'} \approx (E^X)_{v''}$.

Proof. ρ regular $\Rightarrow \exists$ a covering transformation $v: E \to E \ni r_0 v' = v''$. Then $\bar{r}: E^X \to E^X$ is a covering transformation of $E^X \ni \bar{r}(v') = v''$. Thus $\bar{r}: (E^X)v' \approx (E^X)v''$.

For example, let $X = S^2 = E$ and $B = P^2$, the real projective plane. Let $a\colon S^2 \to S^2$ denote the antipodal map, $i\colon S^2 \to S^2$, the identity. $i \not\simeq a$ because $\deg(i) = 1$ and $\deg(a) = -1$. $\therefore (S^{2^{S^2}})_i \neq (S^{2^{S^2}})_a$ but if $\rho\colon S^2 \to P^2$ is the antipodal identification, then $\rho \circ a = \rho \circ i = \rho\colon S^2 \to P^2$. Thus $(S^{2^{S^2}})_a \approx (S^{2^{S^2}})_i$ as components of $S^{2^{S^2}}$ lying over $(P^{2^{S^2}})_\rho$.

(d) The answer is probably no in general, although the authors have not been able to construct a counterexample. We prove the following

THEOREM 3.7. *Let $\rho\colon E \to B$ be a covering such that E, B are simple, path-connected* ANR's. *Let X be a finite CW complex. Then the covering projection*

$$\bar{\rho}\colon E^X \longrightarrow B^X$$

is a regular covering onto $\bar{\rho}(E^X)$.

Proof. As in § 2, the following is a commutative ladder of exact sequences $\ni \bar{\rho}_\sharp$ and ρ^∞ are injective:

$$\begin{array}{ccccccc} & & \pi_1(E^X, v') & \xrightarrow{f'} & E^\infty_{1,0}(E) & & \\ & \nearrow & \Big\downarrow {\scriptstyle \bar{\rho}_\sharp} & & \Big\downarrow {\scriptstyle \rho^\infty} & \searrow & \\ 0 \longrightarrow K & & & & & & 0 \\ & \searrow & \pi_1(B^X, v) & \xrightarrow{f} & E^\infty_{1,0}(B) & \nearrow & \end{array}$$

where

$$\begin{array}{ccc} E^\infty_{1,0}(E) & \subset & \pi_1(E, v'(x_0)) \\ {\scriptstyle \rho^\infty}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho_\sharp} \\ E^\infty_{1,0}(B) & \subset & \pi_1(B, v(x_0)) \end{array}$$

commutes.

E, B are simple $\Rightarrow E^\infty_{1,0}(E), E^\infty_{1,0}(B)$ are abelian. We will show that $\bar{\rho}_\sharp(\pi_1(E^X, v'))$ is a normal subgroup of $\pi_1(B^X, v)$ for any $v, v' \ni \rho \circ v' = v$. Choose $x \in \bar{\rho}_\sharp(\pi_1(E^X)), b \in \pi_1(B^X)$. Then $f(bxb^{-1}) = f(b)f(x)f(b)^{-1} = f(x)$ since $E^\infty_{1,0}(B)$ is abelian $\Rightarrow bxb^{-1}x^{-1} = k \in K \Rightarrow bxb^{-1} = kx \in \bar{\rho}_\sharp(\pi_1(E^X))$.

$$\therefore \bar{\rho}_\sharp(\pi_1(E^X, v')) \lhd \pi_1(B^X, v) \quad \text{for any} \quad v, v' \ni p \circ v' = v\,.$$

Theorem 12 on page 74 of [6] $\Rightarrow \bar{\rho}\,|_{(E^X)_{v'}}\colon (E^X)_{v'} \to (B^X)_v$ is a regular covering for each $v' \in \bar{\rho}^{-1}(v)$. Fix $v' \in \bar{\rho}^{-1}(v)$. Suppose $v'' \in \bar{\rho}^{-1}(v)$ but $(E^X)_{v'} \neq (E^X)_{v''}$. Then by 3.6 $\exists$ a homeomorphism

$$\bar{r}\colon (E^X)_{v'} \to (E^X)_{v''} \ni \bar{r}(v') = v'' \quad \text{and} \quad \bar{\rho} \circ \bar{r} = \bar{\rho}\,.$$

Hence a loop at v in B^X lifts to a loop at v' if and only if it lifts to a loop at v''. Therefore $\bar{\rho}\colon E^X \to \bar{\rho}(E^X)$ is a regular covering.

(e) We quote a result essentially due to Serre, [5], Proposition 3, page 479.

PROPOSITION 3.8. *If G is a path-connected, locally path connected, and semilocally 1-connected H-space, then each covering transformation on any connected covering space E of G is homotopic to the identity map $i: E \to E$.*

COROLLARY 3.9. *If $\rho: E \to B$ is a covering $\ni B$ is an H-space, then $\bar{\rho}^{-1}((B^X)_v)$ is path-connected.*

(*f*) This question only makes sense when we are considering $(B^X)_v$. Let us ask: When is $(E^X)_{v'}$ a universal covering over $(B_X)_v$, where $\rho \circ v' = v$?

THEOREM 3.10. *If X is a CW complex of* $\dim \leqq n$ *and E is an n-connected space, then $\pi_1(E^X, v) = 0$ for all $v \in E^X$.*

The proof follows easily from Federer's spectral sequence [3].

BIBLIOGRAPHY

1. J. Dugundi, *Topology*, Allyn and Bacon, 1966.
2. M. Dyer, *Two term conditions in π-exact couples*, Canad. J. Math. **19** (1967), 1263-1288.
3. H. Federer, *A study of function spaces by spectral sequences* Trans. Amer. Math. Soc. **82** (1956), 340-361.
4. S. -T. Hu, *Theory of retracts*, Wayne State Univ. Press, 1965.
5. J. -P. Serre, *Homologie singuliere des espaces fibres*, Ann. of Math. **54** (1951), 425-505.
6. E. Spanier, *Algebraic topology*, McGraw-Hill, 1966.

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