

SUBSPACE KERNELS AND MINIMUM PROBLEMS IN HILBERT SPACES WITH KERNEL FUNCTION

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The purpose of this paper is to extend the theory of Hilbert spaces with kernel function to obtain first the kernel function of any subspace described as the intersection of the nullspaces of countably many continuous linear functionals, and secondly the solution of minimum norm to interpolation problems involving countably many linear side conditions. The results are then applied to obtain in §1 a class of pseudoconformally invariant functions in C^n and in §2 further results on the classical interpolation problem involving pointwise evaluation.

Any closed subspace of a Hilbert space with (reproducing) kernel function has itself a reproducing kernel. In §1 (see also [8]) it is shown that the Riesz representation of any bounded linear functional in a Hilbert space with kernel function is obtained by operating with the linear functional on the kernel function itself (see Theorem 1.1). Using this representation, one can display, in terms of the kernel function of the original space, the kernel function of any closed subspace defined as the intersection of the null spaces of at most countably many bounded linear functionals (see Theorems 1.2 and 1.3). By applying this representation to the Bergman space H_B^n of all analytic square integrable functions defined on a domain $B \subset C^n$, one obtains a large class of pseudoconformally invariant functions (see Theorem 1.7 and Note 1.8). As an example are obtained important invariant functions introduced and used by Bergman ([2]—[6]) and others (e.g., [13]).

The solution of minimum problems of the type requiring the infimum of the norm of functions under a finite number of bounded linear side conditions in Hilbert space with kernel function was obtained in [7], by use of the Riesz representation of Theorem 1.1. In §2 of this paper is obtained a necessary and sufficient condition for the existence of the solution to such minimum problems under countably many bounded linear side conditions. When the solution exists, it can be displayed in terms of the kernel function again by use of the Riesz representation of Theorem 1.1 (see Theorem 2.1 and Corollary 2.2). Applying these results one obtains necessary and sufficient conditions for the existence of solutions to a large family of interpolation problems, examples of which are given in the spaces $H_{[|z|<1]}^n$ and the Hardy space $H_2[|z|<1]$ (see Corollary 2.3—Corollary 2.8).

1. **Kernel functions of subspaces.** Let H denote a general Hilbert space with inner product denoted by (f, g) and norm $\|f\| = (f, f)^{1/2}$ where $f, g \in H$. Let H° denote the space of all bounded linear functionals on H . If $\mathcal{L} \in H^\circ$ has the Riesz representation $L \in H$, we denote this correspondence by $\mathcal{L} \sim L$.

Suppose further that H is a Hilbert space of functions $\{f(x)\}$ defined on a base set E and that H has a (reproducing) kernel function $K(x, y)$. Then we denote such a Hilbert space by HK or $HK(E)$.

For fixed y the kernel function $K(x, y)$ is the function providing the Riesz representation in HK of the bounded linear functional $\mathcal{L} \in HK^\circ$ given by $\mathcal{L}(f) = f(y)$ for $f \in HK$. The following theorem (see [8]) shows that not only pointwise evaluation but all bounded linear functionals on HK have a simple representation in HK in terms of the kernel function $K(x, y)$. In fact the representation is obtained by simply applying such a functional to $K(x, y)$ itself:

THEOREM 1.1. *If $\mathcal{L} \in HK^\circ$, then $\mathcal{L} \sim L(x) = \overline{\mathcal{L}_y(K(y, x))}$. (Here the y subscript emphasizes that \mathcal{L} is operating on $K(y, x)$ in HK as a function of y .)*

Proof. $L(x) = (\mathcal{L}(y), K(y, x)) = \overline{(K(y, x), \mathcal{L}(y))} = \overline{\mathcal{L}_y(K(y, x))}$.

EXAMPLE. In the case of evaluation of the derivative at z in H_B^1 , $\mathcal{L}: f \rightarrow f'(z)$, then

$$\overline{\mathcal{L}_y(K(y, \bar{x}))} = \left. \frac{\partial K(y, \bar{x})}{\partial y} \right|_{y=z} = \left. \frac{\partial K(x, \bar{y})}{\partial \bar{y}} \right|_{y=z}.$$

NOTATION. Suppose $\mathcal{L}^m \in H^\circ$, $m = 1, 2, \dots, n$ are linearly independent and that $\mathcal{L}^m \sim L_m$, $m = 1, 2, \dots, n$. Then denote by D the determinant of the $n \times n$ matrix $Q = \{(\overline{L_k}, \overline{L_m})\}_{n \times n}$. Denote by $D_w^v(R)$ the determinant of the $(n + 1) \times (n + 1)$ matrix $\begin{pmatrix} R & v \\ w & Q \end{pmatrix}$, where $v = (v_1, \dots, v_n)$ is a row vector and $w = (w_1, \dots, w_n)^T$ is a column vector. Note also that in the case of HK , $(L_k, L_m) = \mathcal{L}_x^m(\overline{\mathcal{L}_y^k(K(y, x))})$.

THEOREM 1.2. *Suppose $\mathcal{L}^m \in HK^\circ$, $m = 1, 2, \dots, n$ are linearly independent. Then, letting $\mathcal{L}(t) = (\mathcal{L}_z^1 K(z, t), \dots, \mathcal{L}_z^n K(z, t))$, we have that $D_{\overline{\mathcal{L}(y)}}(\overline{K(x, y)})/D$ is the kernel function $k_n(x, y)$ of the subspace $H_n = \{f \in HK; \mathcal{L}^r f = 0, r = 1, \dots, n\}$.*

Proof. Let $k_n^*(x, y) = D_{\overline{\mathcal{L}(y)}}(\overline{K(x, y)})/D$. For each fixed y , k_n^* equals $K(x, y) + \sum_{r=1}^n c_r \overline{\mathcal{L}_z^r K(z, x)}$ for some constants c_1, \dots, c_n and

thus belongs to HK . For each r , $\mathcal{L}_x^r k_n^*(x, y) = 0$, since

$$\mathcal{L}_x^r (D_{\mathcal{L}(y)}^{\overline{\mathcal{L}(x)}}(K(x, y)))$$

is a determinant of a matrix with the 1st and r^{th} rows equal. Finally if $f \in H_n$, then $(f(x), k_n^*(x, y)) = f(y) + \sum_{r=1}^n c_r \mathcal{L}^r f = f(y)$. So $k_n^*(x, y) \equiv k_n(x, y)$.

THEOREM 1.3. *Suppose $\mathcal{L}^m \in HK^\circ$, $m = 1, 2, \dots$ are linearly independent. Let $k_n(x, y)$ denote the kernel function of the subspace $H_n = \{f \in HK; \mathcal{L}^r f = 0, r = 1, \dots, n\}$. Then $\lim_{n \rightarrow \infty} k_n(x, y)$ gives the kernel function $k(x, y)$ of the subspace $H_\infty = \{f \in HK; \mathcal{L}^r f = 0, r=1, 2, \dots\}$, where in each variable the convergence is both in the norm and pointwise uniform on each set E^1 on which $K(z, z)$ is bounded.*

Proof. Let $\mathcal{L}^r \sim L_r, r = 1, 2, \dots$. If S is the subspace of HK spanned by the $\{L_r\}$, then S is the orthogonal complement of H_∞ in HK . Now orthogonalize $\{L_1, L_2, \dots\}$ by the Gram-Schmidt process to get $\{M_1, M_2, \dots\}$. Thus $K(x, y) = k(x, y) + \sum_{r=1}^\infty M_r(x)\overline{M_r(y)}$, and $k_n(x, y) = k(x, y) + \sum_{r=n+1}^\infty M_r(x)\overline{M_r(y)}$, as is easily checked. Thus,

$$\begin{aligned} \|k_n(x, y) - k(x, y)\|^2 &= \|k_n(x, y)\|^2 - (k_n(x, y), k(x, y)) \\ &\quad - (k(x, y), k_n(x, y)) + \|k(x, y)\|^2 = k_n(y, y) - k(y, y) \\ &= \sum_{r=n+1}^\infty |M_r(y)|^2 \rightarrow 0 \end{aligned}$$

for each fixed $y \in E$. Furthermore, we have

$$\begin{aligned} |k_n(x, y) - k(x, y)|^2 &= |(k_n(z, y) - k(z, y), K(z, x))|^2 \\ &\leq \|K(z, x)\|^2 \|k_n(z, y) - k(z, y)\|^2 \\ &= K(x, x)[k_n(y, y) - k(y, y)]. \end{aligned}$$

Hence for y fixed $k_n(x, y) \rightarrow k(x, y)$ pointwise uniformly on each set E^1 on which $K(z, z)$ is bounded. Since

$$\overline{k_n(x, y) - k(x, y)} = k_n(y, x) - k(y, x),$$

the same statements hold for fixed x .

Note 1.4. If HK is separable then any closed subspace of HK is of the form H_∞ in Theorem 1.3.

EXAMPLE 1.5. If B is a bounded domain in C^n , let

$$K = K_{B(u_r, m)}(z, \bar{w})$$

denote the kernel function of $H_B^n \cap [f_{r^\nu}(u^\nu) = 0, u^\nu \in B, \nu = 1, 2, \dots, m]$, where

$$f_{r^\nu}(z) = \frac{\partial^{r^\nu} f}{\partial z^{r^\nu}} = \frac{\partial^{r^\nu} f(z)}{\partial z_1^{r_1^\nu} \dots \partial z_n^{r_n^\nu}}$$

with $r_1^\nu + \dots + r_n^\nu = r^\nu \geq 0$. Then by Theorem 1.2,

$$K_{B(u_r, m)}(z, \bar{w}) = \frac{\begin{vmatrix} K(z, \bar{w}) & K_{\bar{r}1}(z, \bar{w}^1) & \dots & K_{\bar{r}m}(z, \bar{w}^m) \\ K_{r1}(u^1, \bar{w}) & K_{r1\bar{r}1}(u^1, \bar{w}^1) & \dots & K_{r1\bar{r}m}(u^1, \bar{w}^m) \\ \vdots & \vdots & & \vdots \\ K_{r^m}(u^m, \bar{w}) & K_{r^m\bar{r}1}(u^m, \bar{w}^1) & \dots & K_{r^m\bar{r}m}(u^m, \bar{w}^m) \end{vmatrix}}{\begin{vmatrix} K_{r1\bar{r}1}(u^1, \bar{w}^1) & \dots & K_{r1\bar{r}m}(u^1, \bar{w}^m) \\ \vdots & & \vdots \\ K_{r^m\bar{r}1}(u^m, \bar{w}^1) & \dots & K_{r^m\bar{r}m}(u^m, \bar{w}^m) \end{vmatrix}}.$$

If $K_{B(u_r, \infty)}$ denotes the kernel function of

$$H_B^n \cap [f_{r^\nu}(u^\nu) = 0, u^\nu \in B, \nu = 1, 2, \dots],$$

then $K_{B(u_r, \infty)}(z, \bar{w}) = \lim_{m \rightarrow \infty} K_{B(u_r, m)}(z, \bar{w})$, by Theorem 1.3.

LEMMA 1.6. *Let $z^* = z^*(z)$ be a pseudoconformal transformation of a domain B onto a domain B^* in C^n . Then any closed subspace HK of H_B^n is taken onto a closed subspace H^*K^* of $H_{B^*}^n$ by the isometry T given by $Tf(z^*) = f(z(z^*))((\partial z)/(\partial z^*))$ and $K^*(z^*, \bar{w}^*) = K(z, \bar{w})((\partial z)/(\partial z^*))(\overline{(\partial w)/(\partial w^*)})$.*

Proof. In [7] it was shown that if $T: H_1K_1(E_1) \rightarrow H_2K_2(E_2)$ is a (surjective) isometry, then $K_2(w, z) = T_x(\overline{T_y(K_1(y, x))})_z$.

As an immediate consequence of Lemma 1.6, we have

THEOREM 1.7. *If H_1K_1 and H_2K_2 are any two closed subspaces of H_B^n , then $I(K_1; K_2; z, \bar{w}) = K_1(z, \bar{w})/K_2(z, \bar{w})$ is a pseudoconformally invariant function. (That is, if $H_i^*K_i^* = T(H_iK_i)$, $i = 1, 2$, where the isometry T is induced by a pseudoconformal transformation from B onto B^* , then $K_1^*(z^*, \bar{w}^*)/K_2^*(z^*, \bar{w}^*) = K_1(z, \bar{w})/K_2(z, \bar{w})$).*

Note 1.8. If H_i is given as the intersection of the nullspaces of the linearly independent bounded linear functionals $\mathcal{L}^{i, m}$, $m = 1, 2, \dots$,

$i = 1, 2$, then we can write $I(\mathcal{L}^{1,1}, \mathcal{L}^{1,2}, \dots; \mathcal{L}^{2,1}, \mathcal{L}^{2,2}, \dots; z, \bar{w})$ for $I(K_1, K_2; z, \bar{w})$, where $\mathcal{L} \in HK^\circ \rightarrow \mathcal{L}^* \in (H^*K^*)^\circ$ under the isometry T .

COROLLARY 1.9. *If K denotes the Bergman kernel function of the space H_B^1 , then*

$$J_B(z) = \frac{\begin{vmatrix} K(z, \bar{z}) & K_{\bar{z}}(z, \bar{z}) \\ K_z(z, \bar{z}) & K_{z\bar{z}}(z, \bar{z}) \end{vmatrix}}{K^3(z, \bar{z})}$$

is a scalar-valued conformal invariant.

Proof. By Theorem 1.7 and Note 1.8, $I(D_z; ; z, \bar{z})$, where $D_z f = f'(z)$ in H_B^1 , is a conformal invariant. But by Theorem 1.2,

$$I(D_z; ; z, \bar{z}) = \frac{\begin{vmatrix} K(z, \bar{z}) & K_{\bar{z}}(z, \bar{z}) \\ K_z(z, \bar{z}) & K_{z\bar{z}}(z, \bar{z}) \end{vmatrix}}{[K_{z\bar{z}}(z, \bar{z})K(z, \bar{z})]}.$$

Now $D_z \in (H_B^1)^\circ \rightarrow D_{z^*} \in (H_{B^*}^1)^\circ$, where $z \rightarrow z^*$ and

$$D_{z^*}(g) = g'(z^*)\left(\frac{dz^*}{dz}\right)^2 + g(z^*)\frac{d^2z^*}{dz^2}.$$

This follows from the fact that if $g(z^*) = f(z(z^*))dz/dz^*$, then $f(z) = g(z^*(z))dz^*/dz$ and $f'(z) = g'(z^*)((dz^*)/(dz))^2 + g(z^*)(d^2z^*)/(dz^2)$. So $K(z, \bar{z}) = K^*(z^*, \bar{z}^*)(dz^*)/(dz)(d\bar{z}^*)/(d\bar{z})$,

$$K_z(z, \bar{z}) = D_{z^*}K^*(z^*, \bar{z}^*)\frac{d\bar{z}^*}{d\bar{z}},$$

and $K_{z\bar{z}}(z, \bar{z}) = D_{z^*}(D_{\bar{z}^*}K^*(z^*, \bar{z}^*))$. But

$$\left| \frac{K^*(z^*, \bar{z}^*) \left| \frac{dz^*}{dz} \right|^2 \overline{D_{\bar{z}^*}K^*(z^*, \bar{z}^*)} \frac{dz^*}{dz}}{D_{z^*}K^*(z^*, \bar{z}^*) \frac{d\bar{z}^*}{d\bar{z}} D_{z^*}(D_{\bar{z}^*}K^*(z^*, \bar{z}^*))} \right| = \left| \frac{K^*(z^*, \bar{z}^*) K_{\bar{z}^*}(z^*, \bar{z}^*)}{K_{z^*}(z^*, \bar{z}^*) K_{z^*\bar{z}^*}(z^*, \bar{z}^*)} \right| \left| \frac{dz^*}{d\bar{z}} \right|^6.$$

Hence $I'(D_z; ; z, \bar{z}) = I(D_z; ; z, \bar{z})K_{z\bar{z}}(z, \bar{z})/K^2(z, \bar{z})$ is a conformal invariant. Shows that $J_B(z) = J_{B^*}(z^*)$, i.e., $J_B(z)$ is a conformal invariant.

Similarly one obtains the following generalization of Corollary 1.9.

COROLLARY 1.10. *If K denotes the Bergman kernel function of the space H_B^n , then*

$$J_B(z) = J_B(z_1, \dots, z_n) = \frac{\begin{vmatrix} K(z, \bar{z}) & K_{\bar{z}_1}(z, \bar{z}) & \dots & K_{z_n}(z, \bar{z}) \\ K_{z_1}(z, \bar{z}) & K_{z_1\bar{z}_1}(z, \bar{z}) & \dots & K_{z_1\bar{z}_n}(z, \bar{z}) \\ \vdots & \vdots & \ddots & \vdots \\ K_{z_n}(z, \bar{z}) & K_{z_n\bar{z}_1}(z, \bar{z}) & \dots & K_{z_n\bar{z}_n}(z, \bar{z}) \end{vmatrix}}{K^{n+2}(z, \bar{z})}$$

is a scalar-valued conformal invariant.

2. Minimum problems involving infinitely many side conditions.

THEOREM 2.1. *If $\mathcal{L}^m \in H^\circ$, $m = 1, 2, \dots$, are linearly independent, consider $\mu = \inf \|f\|$ under the side conditions $\mathcal{L}^m(f) = a_m$, $m = 1, 2, \dots$. Then the problem has a solution if and only if $\lim_{n \rightarrow \infty} \|f_n\| = M < \infty$, where f_n yields $\inf \|f\|$ under the side conditions $\mathcal{L}^m(f) = a_m$, $m = 1, 2, \dots, n$. Moreover, if a solution f exists it is unique and is given by $f = \lim_{n \rightarrow \infty} f_n$, where $\|f\| = M$.*

Proof. In [7] we showed that if $\mathcal{L}^m \sim L_m$, $m = 1, 2, \dots, n$, then $f_n = -D_a^L(0)/D$, where $L = (L_1, \dots, L_n)$ and $a = (a_1, \dots, a_n)^T$.

If $\lim_{n \rightarrow \infty} \|f_n\| = M < \infty$, then

$$\|f_n - f_m\|^2 = \|f_n\|^2 + \|f_m\|^2 - 2\text{Re}(f_n, f_m).$$

But if $m \geq n$, we have that $(f_n, f_m) = \|f_n\|^2$, since (f_n, f_m) is obtained by replacing the first row in the determinant form of f_n by

$$0(L_1, L_m) \dots (L_n, f_m) = 0 \overline{\mathcal{L}^1} f_m \dots \overline{\mathcal{L}^n} f_m = 0 \bar{a}_1 \dots \bar{a}_n.$$

Thus $\|f_n - f_m\|^2 = \|f_m\|^2 - \|f_n\|^2$. So $\{f_n\}$ is a Cauchy sequence converging to $f \in H$ and $\|f\| = \lim_{n \rightarrow \infty} \|f_n\| = M$. Clearly $\mathcal{L}_m f = \lim_{n \rightarrow \infty} \mathcal{L}^m f_n = a_m$. Suppose $\mathcal{L}^m(g) = a_m$, $m = 1, 2, \dots$. Consider $u = g - f$. Then $\mathcal{L}^m(u) = 0$, $m = 1, 2, \dots$, and thus

$$(u, f) = (u, \lim_{n \rightarrow \infty} f_n) = \lim_{n \rightarrow \infty} (u, f_n) = \lim_{n \rightarrow \infty} (u, \sum_{k=1}^n c_k^{(n)} L_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \overline{c_k^{(n)}} \mathcal{L}_k u = 0.$$

Then $\|g\|^2 = \|f\|^2 + \|u\|^2$ and $\|g\| \geq \|f\|$, while $\|g\| = \|f\| \Leftrightarrow u = g - f = 0$.

On the other hand if the problem has a solution f , then

$$\|f_1\| \leq \dots \leq \|f_n\| \leq \dots \leq \|f\| < \infty,$$

which implies that $\lim_{n \rightarrow \infty} \|f_n\| = M < \infty$. Then the argument above applies to show that in fact $f = \lim_{n \rightarrow \infty} f_n$ and so $\|f\| = M$.

COROLLARY 2.2. *If $H = HK$ in Theorem 2.1, then the solution is given by*

$$(1) \quad f(x) = - \lim_{n \rightarrow \infty} \frac{\begin{vmatrix} 0 & \overline{\mathcal{L}_z^1(K(z, x))} \cdots \overline{\mathcal{L}_z^n(K(z, x))} \\ a_1 & & & \\ \vdots & & Q_n & \\ a_n & & & \end{vmatrix}}{|Q_n|},$$

and

$$\|f\|^2 = - \lim_{n \rightarrow \infty} \frac{\begin{vmatrix} 0 & \overline{a_1} \cdots \overline{a_n} \\ a_1 & & & \\ \vdots & & Q_n & \\ a_n & & & \end{vmatrix}}{|Q_n|},$$

where $Q_n^* = \{\overline{\mathcal{L}_z^r(\mathcal{L}_z^s(K(z, x)))}\}_{n \times n}$. Here the convergence is in the norm as well as being pointwise uniform on every set on which $K(x, x)$ is bounded.

Proof. Again, as in [7], if $\mathcal{L}^m \in HK^\circ$, $m = 1, \dots, n$, are linearly independent then $\mu = \inf \|f\|$ under the side conditions $\mathcal{L}^m(f) = a_m$, $m = 1, \dots, n$, has the unique solution $f(x) = -D_{\overline{a^z(x)}}(0)/D$ with $\mu^2 = -D_{\overline{a^z}}(0)/D$, where $\mathcal{L}(x) = (\mathcal{L}_z^1(K(z, x)), \dots, \mathcal{L}_z^n(K(z, x)))$. The corollary then follows from Theorem 2.1.

COROLLARY 2.3. *If $\mathcal{L}^m \in HK^\circ$, $m = 1, 2, \dots$ are linearly independent, then there exists an f in HK such that $\mathcal{L}^m(f) = a_m$, $m = 1, 2, \dots$, if and only if*

$$- \lim_{n \rightarrow \infty} \frac{\begin{vmatrix} 0 & \overline{a_1} \cdots \overline{a_n} \\ a_1 & & & \\ & & Q_n & \\ a_n & & & \end{vmatrix}}{|Q_n|} < \infty.$$

If a solution exists, the one of minimum norm is then given by formula (1).

Proof. There exists such a function f if and only if there exists one of minimum finite norm in HK , and the statement follows from Corollary 2.2.

COROLLARY 2.4. *Let $\{z_n\}$ be any sequence of points in the ball $B = \{|z| < 1\} \subset C^k$, and let $\{w_n\}$ be any sequence of complex numbers. Then a necessary and sufficient condition that there exist a square-integrable analytic function f on B such that $f(z_n) = w_n, n = 1, 2, \dots$, is that*

$$\begin{aligned}
 a_z(w) &= a_{z_1, z_2, \dots}(w_1, w_2, \dots) \\
 (2) \quad &= - \lim_{n \rightarrow \infty} \frac{\begin{vmatrix} 0 & \overline{w_1} & & \dots & \overline{w_n} \\ & 1 & & & 1 \\ w_1 & \frac{1}{(1 - z_1 \cdot \overline{z_1})^{k+1}} & \dots & \frac{1}{(1 - z_1 \cdot \overline{z_n})^{k+1}} \\ \vdots & \vdots & & & \vdots \\ w_n & \frac{1}{(1 - z_n \cdot \overline{z_1})^{k+1}} & \dots & \frac{1}{(1 - z_n \cdot \overline{z_n})^{k+1}} \end{vmatrix}}{\begin{vmatrix} \frac{1}{(1 - z_1 \cdot \overline{z_1})^{k+1}} & \dots & \frac{1}{(1 - z_1 \cdot \overline{z_n})^{k+1}} \\ \vdots & & \vdots \\ \frac{1}{(1 - z_n \cdot \overline{z_1})^{k+1}} & \dots & \frac{1}{(1 - z_n \cdot \overline{z_n})^{k+1}} \end{vmatrix}} < \infty,
 \end{aligned}$$

where if $z_j = (z_j^1, \dots, z_j^k), z_i \cdot \overline{z_j} = \sum_{r=1}^k z_i^r \overline{z_j^r}$. If a solution exists, the solution $f(z)$ of minimum norm is obtained by replacing $\overline{w_j}$ by $k! / (\pi^k (1 - z \cdot \overline{z_j})^{k+1})$ in formula (2), and this convergence is in the norm and pointwise uniform on compact subsets of B .

Proof. The Hilbert space H_B^k of all square-integrable analytic functions on B has the Bergman kernel function $K(z, t) = k! / (\pi^k (1 - z \cdot \overline{t})^{k+1})$. Then $\mathcal{L}_z^r(\mathcal{L}_z^s(\overline{K(z, x)})) = K(z_r, z_s) = k! / (\pi^k (1 - z_r \cdot \overline{z_s})^{k+1})$, and the corollary follows from Corollary 2.3, and the fact that $K(z, z)$ is continuous on B .

DEFINITION 2.5. A sequence of points $\{z_n\}$ in the ball $B = \{|z| < 1\} \subset C^k$ is called an H - S interpolating sequence if for each sequence of complex numbers $w = \{w_n\} \in S$, there exists an $f \in H$ such that $f(z_n) = w_n$.

COROLLARY 2.6. $\{z_n\}$ is an H_B^k - S interpolating sequence if and only if $a_z(w)$ is finite for each $w \in S$.

COROLLARY 2.7. *Let $\{z_n\}$ be any sequence of points in the unit disk $\{|z| < 1\}$, and let $\{w_n\}$ be any sequence of complex numbers. Then a necessary and sufficient condition that there exist an analytic function f , with square integrable boundary values on the circle, such that $f(z_n) = w_n, n = 1, 2, \dots$, is that*

$$(3) \quad b_z(w) = - \lim_{n \rightarrow \infty} \frac{\begin{vmatrix} 0 & \overline{w_1} & \cdots & \overline{w_n} \\ w_1 & \frac{1}{1 - z_1 \overline{z_1}} & \cdots & \frac{1}{1 - z_1 \overline{z_n}} \\ \vdots & \vdots & & \vdots \\ w_n & \frac{1}{1 - z_n \overline{z_1}} & \cdots & \frac{1}{1 - z_n \overline{z_n}} \end{vmatrix}}{\begin{vmatrix} \frac{1}{1 - z_1 \overline{z_1}} & \cdots & \frac{1}{1 - z_1 \overline{z_n}} \\ \vdots & & \vdots \\ \frac{1}{1 - z_n \overline{z_1}} & \cdots & \frac{1}{1 - z_n \overline{z_n}} \end{vmatrix}} < \infty .$$

If a solution exists, the solution of minimum norm is obtained by replacing $\overline{w_j}$ by $1/(2\pi(1 - z\overline{z_j}))$ in formula (3), and this convergence is in the norm and pointwise uniform on compact subsets of $\{|z| < 1\}$.

Proof. The kernel function of H_2 is the Szegő kernel $K(z, t) = 1/(2\pi(1 - z\overline{t}))$ and the same argument applies as in Corollary 2.4.

COROLLARY 2.8. $\{z_n\}$ is an H_2 - S interpolating sequence if and only if $b_z(w)$ is finite for each $w \in S$.

BIBLIOGRAPHY

1. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337-404.
2. S. Bergman, *Über die Kernfunktion eines Bereiches und ihr Verhalten am Rande*, J. Reine Angew. Math. **169** (1933), 1-42, and **172** (1934), 89-128.
3. ———, *Sur les fonctions orthogonales de plusieurs variables complexes*, Mém. des Sciences Math. **106** (1947).
4. ———, *Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes*, Mém. des Sciences Math. **108** (1948).
5. ———, *The kernel function and conformal mapping*, Math. Surveys **5**, Amer. Math. Soc., New York, 1950.
6. ———, *Distinguished sets of invariants in the theory of pseudo-conformal transformations*, J. D'Analyse Math. **13**, Jerusalem, 1964.
7. B. Chalmers, *Linear transformations in Hilbert Spaces with kernel function* (to appear)
8. P. J. Davis, *Interpolation and approximation*, Blaisdell, 1963.
9. B. Epstein, *Orthogonal families of analytic functions*, New York, 1964.
10. K. Hoffman, *Banach Spaces of analytic functions*, Prentice-Hall New Jersey, 1962.
11. H. Meschkowski, *Hilbertsche Raume mit Kernfunktion*, Berlin, 1962.
12. H. S. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. (1961), 513-532.

13. G. Springer, *Pseudo-conformal transformations onto circular domains*, Duke Math. J. **18** (1951), 411-424.

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