A structure theorem is given for all locally compact rings such that $x$ belongs to the closure of $\{x^n: n \geq 2\}$, in particular, all such rings are commutative, a result which extends a well-known theorem of Jacobson. Similarly we show the commutativity of semisimple locally compact rings satisfying topological analogues of properties studied by Herstein.

Jacobson has shown that a ring is commutative if for every $x$ there is some $n(x) \geq 2$ such that $x^{n(x)} = x$ [5, Th. 1, p. 212]. Herstein has generalized this result, and certain of his and other generalizations are of interest here. A ring is commutative if (and only if) for all $x$ and $y$ there is some $n(x, y) \geq 2$ such that $(x^{n(x,y)} - x)y = y(x^{n(x,y)} - x)$ [4, Th. 2]; a ring is commutative if (and only if) for all $x$ and $y$ there is some $n(x, y) \geq 2$ such that $xy - yx = (xy - yx) x^{n(x,y)}$ [3, Th. 6]; a semisimple ring is commutative if (and only if) for all $x$ and $y$ there is some $n(x, y) \geq 1$ such that $x y^{n(x,y)} y = y x^{n(x,y)}$ [4, Th. 1] or if for all $x$ and $y$ there are $n, m \geq 1$ such that $x y^n = y'' x^n$ [1, Lemma 1]. The investigation of analogous conditions for topological rings is the major concern of this paper.

1. A topological analogue of Jacobson’s condition. If $x^n = x$ for some $n \geq 2$, then an inductive argument shows that $x^{k(n-1)+1} = x$ for all $k \geq 1$. A possible topological analogue of Jacobson’s condition would thus be that for every $x$ there is some $n(x) \geq 2$ such that \( \lim_{k \to \infty} x^{k(n(x)-1)+1} = x \). But this implies that $x^{n(x)} = x$, since

$$x^{n(x)} = x^{n(x)-1}x = x^{n(x)-1} \lim_k x^{k(n(x)-1)+1} = \lim_k x^{k+1(n(x)-1)+1} = x.$$ 

Thus all topological rings having this property have Jacobson’s property and hence are commutative.

A less trivial analogue of Jacobson’s condition is that for every $x$ in the topological ring $A$, $x$ belongs to the closure of $\{x^n: n \geq 2\}$. In our investigation of these rings, rings with no nonzero topological nilpotents play an important role. Recall that an element $x$ of a topological ring is a topological nilpotent if $\lim_n x^n = 0$. We shall prove that a locally compact ring has no nonzero topological nilpotents if and only if it is the topological direct sum of a discrete ring having no nonzero nilpotents and a ring $B$ that is the local direct sum of a family of discrete rings having no nonzero nilpotents with respect to finite subfields. From this it is easy to derive a structure theorem for locally compact rings.
having the topological analogue of Jacobson's property mentioned above.

**Lemma 1.** If \( A \) is a locally compact ring with no nonzero topological nilpotents, then \( A \) is totally disconnected.

**Proof.** The connected component \( C \) of zero in \( A \) is a closed ideal of \( A \) and so is itself a connected locally compact ring with no nonzero topological nilpotents. By hypothesis, \( C \) is not annihilated by any of its nonzero elements, for if \( xc = 0 \), then \( x^2 = 0 \), so \( x = 0 \). Thus \( C \) is a finite-dimensional algebra over the real numbers (cf. [6, Th. III]). As the radical of a finite-dimensional algebra is nilpotent, \( C \) is a semisimple algebra. If \( C \neq 0 \), then by Wedderburn's Theorem, \( C \) has an identity \( e \), and clearly \((1/2)e\) would then be a nonzero topological nilpotent contrary to our hypothesis. Thus \( C = 0 \), and so \( A \) is totally disconnected.

**Lemma 2.** A compact ring \( A \) has no nonzero topological nilpotents if and only if \( A \) is the Cartesian product of finite fields.

**Proof.** Necessity: By Lemma 1, \( A \) is totally disconnected. Thus the radical \( J(A) \) of \( A \) is topologically nilpotent [11, Th. 14], and hence is the zero ideal. Thus \( A \) is a compact semisimple ring, and so \( A \) is topologically isomorphic to the Cartesian product of a family of finite simple rings [11, Th. 16]. A finite simple ring is a matrix ring over a finite field, and unless the matrix ring is just the finite field itself, it will have nonzero nilpotent elements. Thus as \( A \) has no nonzero nilpotents, \( A \) is topologically isomorphic to the Cartesian product of a family of finite fields. Sufficiency: Clearly zero is the only topological nilpotent in the Cartesian product of a family of finite fields.

**Lemma 3.** If \( A \) is a ring with no nonzero nilpotents, then every idempotent is in the center of \( A \).

**Proof.** If \( e \) is an idempotent and if \( a \in A \), an easy calculation shows that \((ae - eae)^2 = 0\), hence \( ae - eae = 0 \). Similarly, \( ea = eae \) and thus \( ae = ea \).

We recall that the local direct sum of a family \((A_r)_{r \in \Gamma}\) of topological rings with respect to open subrings \((B_r)_{r \in \Gamma}\) is the subring of the Cartesian product \( \Pi_r A_r \) consisting of all \((a_r)\) such that \( a_r \in B_r \) for all but finitely many \( r \), topologized by declaring all neighborhoods of zero in the topological ring \( \Pi_r B_r \) to be a fundamental system of neighborhoods of zero in the local direct sum. It is easy to see that the local direct sum equipped with this topology is indeed a topological ring.
THEOREM 1. A locally compact ring \( A \) has no nonzero topological nilpotents if and only if \( A \) is the topological direct sum of a discrete ring having no nonzero nilpotents and a ring \( B \) (possibly the zero ring) that is topologically isomorphic to the local direct sum of a family of discrete rings having no nonzero nilpotents with respect to finite subfields.

Proof. Necessity: As \( A \) is totally disconnected by Lemma 1, \( A \) contains a compact open subring \( F \) [7, Lemma 4]. By Lemma 2, \( F \) is topologically isomorphic to the product of finite fields. Consequently there exists in \( F \) a summable orthogonal family \( (e_\gamma)_{\gamma \in \Gamma} \) of idempotents such that \( F e_\gamma \) is a finite field and \( \sum_{\gamma \in \Gamma} e_\gamma = e \), the identity of \( F \).

By Lemma 3, \( e \) is in the center of \( A \), so \( A e \) and \( A(1 - e) = \{a - ae: a \in A\} \) are ideals. The continuous mappings \( a \mapsto ae \) and \( a \mapsto (a - ae) \) are the projections from \( A \) onto \( A e \) and \( A(1 - e) \). Thus \( A \) is the topological direct sum of \( A e \) and \( A(1 - e) \). As \( e \) is the identity of \( F \), \( F \) is open, \( A(1 - e) \) is discrete and hence has no nonzero nilpotents.

As \( F \) is open and as \( A e \cap F = F e_\gamma \), a finite field, \( A e_\gamma \) is discrete and is an ideal as \( e_\gamma \) is in the center of \( A \). Consequently \( A e_\gamma \) has no nonzero nilpotents. It will therefore suffice to show that \( B = A e \) is topologically isomorphic to the local direct sum of the discrete rings \( A e_\gamma \), with respect to the finite subfields \( F e_\gamma \).

Let \( B' \) be the local direct sum of the \( A e_\gamma \)'s with respect to the \( F e_\gamma \)'s. Let \( K: b \mapsto (be_\gamma) \in \Pi_\gamma A e_\gamma \). Clearly \( b \mapsto be_\gamma \) is a continuous homomorphism for each \( \gamma \), hence \( K \) is a continuous homomorphism from \( B \) into \( \Pi_\gamma A e_\gamma \). If \( b \in B \), then \( (be_\gamma) \) is summable and \( \sum \gamma be_\gamma = b(\sum \gamma e_\gamma) = be = b \). Therefore as \( F \) is open in \( B \), \( be_\gamma \in F \) for all but finitely many \( \gamma \in \Gamma \). Thus \( K(B) \subseteq B' \).

The mapping \( K \) is an isomorphism onto \( K(B) \), since if \( x \in B \) and if \( xe_\gamma = 0 \) for all \( \gamma \in \Gamma \), then \( x = xe = x(\sum \gamma e_\gamma) = \sum \gamma xe_\gamma = 0 \). Let \( y_\beta \in F e_\beta \), and let \( x_\gamma = 0 \) for all \( \gamma \neq \beta \), \( x_\beta = y_\beta \); then \( (x_\gamma) = K(y_\beta) \in K(F) \) since \( (e_\gamma) \gamma \) is an orthogonal family. Thus \( K(F) \) contains a dense subring of \( \Pi_\gamma F e_\gamma \), and hence \( K(F) = \Pi_\gamma F e_\gamma \) as \( K(F) \) is compact. As the restriction of \( K \) to \( F \) is thus a continuous isomorphism from compact \( F \) onto \( \Pi_\gamma F e_\gamma \), \( F \) is topologically isomorphic to \( \Pi_\gamma F e_\gamma \) under \( K \).

Thus it suffices to show that \( K(B) \cong B' \), for \( K \) is then, by the definition of the local direct sum, a topological isomorphism from \( B \) onto \( B' \). If \( (b_\gamma e_\gamma) \in B' \), then \( b_\gamma e_\gamma \in F e_\gamma \) for all but finitely many \( \gamma \), say \( \gamma_1, \ldots, \gamma_n \). Call this set \( \Gamma_1 \) and let \( \Gamma - \Gamma_1 = \Gamma_2 \). Thus \( \sum_{\gamma \in \Gamma_1} b_\gamma e_\gamma \in B \) and \( b_\gamma e_\gamma \in F \) for all \( \gamma \in \Gamma_2 \). Hence as \( F \) is topologically isomorphic to \( \Pi_\gamma F e_\gamma \), \( b' = \sum_{\gamma \in \Gamma_2} b_\gamma e_\gamma \in B \). Thus \( b = b' + \sum_{\gamma \in \Gamma_1} b_\gamma e_\gamma \in B \), and \( b_\gamma = b_\gamma e_\gamma \), so \( K(b) = (b_\gamma e_\gamma) \). The sufficiency is clear.
We will call a ring $A$ a Jacobson ring if given any $x \in A$ there is an $n(x) \geq 2$ such that $x^{n(x)} = x$. All Jacobson rings are commutative [5, Th. 1, p. 212], and in extending this result to topological rings we give the following definition, noting that it reduces to Jacobson's condition in the discrete case.

**Definition.** A topological ring $A$ is a $J$-ring if for each $x \in A, x$ belongs to the closure of $\{x^n : n \geq 2\}$.

**Lemma 4.** If $A$ is a $J$-ring, then $A$ has no nonzero topological nilpotents.

*Proof.* If $\lim_n x^n = 0$, then since $x$ belongs to the closure of $\{x^n : n \geq 2\}$, we conclude that $x = 0$.

**Theorem 2.** A locally compact ring $A$ is a $J$-ring if and only if $A$ is the topological direct sum of a discrete Jacobson ring and a ring $B$ which is topologically isomorphic to the local direct sum of a family of discrete Jacobson rings with respect to finite subfields.

*Proof.* Necessity: By Theorem 1 and Lemma 4, $A$ is the topological direct sum of a discrete ring $C$ and a ring $B$ which is topologically isomorphic to the local direct sum of a family of discrete rings with respect to finite subfields. As each of these rings is an ideal of $A$, each is a discrete $J$-ring and so is a Jacobson ring.

Sufficiency: Let $B$ be the local direct sum of a family of discrete Jacobson rings $B_\gamma, \gamma \in \Gamma$ with respect to finite subfields $F_\gamma, \gamma \in \Gamma$. Let $(x_\gamma) \in B$ and let $U$ be a neighborhood of zero in $B$. Then we may assume that there is a finite subset $\Delta$ of $\Gamma$ such that $x_\gamma \in F_\gamma$ for all $\gamma \in \Delta$ and $U = \prod_{\gamma \in \Delta} G_\gamma$, where $G_\gamma = F_\gamma$ for all $\gamma \in \Delta$. For each $\gamma \in \Delta$, let $n(\gamma) > 1$ be such that $x_\gamma^{n(\gamma)} = x_\gamma$. Let $n = 1 + \prod_{\gamma \in \Delta} (n(\gamma) - 1)$. An inductive argument shows that $x_\gamma^n = x_\gamma$ for all $\gamma \in \Delta$. Hence $(x_\gamma)^n = (x_\gamma) \in U$. Thus $B$ is a $J$-ring, and consequently $A$ is also a $J$-ring.

As all Jacobson rings are commutative we have the following analogue of Jacobson's Theorem:

**Corollary.** A locally compact $J$-ring is commutative.

**Theorem 3.** A locally compact ring $A$ is a Jacobson ring if and only if there exists $N \geq 2$ such that $A$ is the topological direct sum of a discrete Jacobson ring and a ring $B$ that is topologically isomorphic to the local direct sum of a family of discrete Jacobson rings with respect to finite subfields of order $\leq N$. 
Proof. Necessity: Let \( |B_\gamma| \) be the order of \( B_\gamma \). By Theorem 2 it suffices to show that \( \sup |B_\gamma| < +\infty \). If \( \sup |B_\gamma| = +\infty \), then there exists \( (x_\gamma) \in \prod_{\gamma} B_\gamma \) such that the orders of the \( x_\gamma \)'s are unbounded. Consequently for no \( n \) does \( x_\gamma^n = x_\gamma \) for all \( \gamma \), i.e., for no \( n \) does \( (x_\gamma)^n = (x_\gamma) \).

Sufficiency: Let \( (A_\gamma)_{\gamma \in \Gamma} \) be a family of discrete Jacobson rings with finite subfields \( B_\gamma \) such that \( |B_\gamma| \leq N \) for all \( \gamma \). Let \( (x_\gamma) \) be in the local direct sum of the \( A_\gamma \)'s with respect to the \( B_\gamma \)'s. There exists a finite subset \( A \) of \( \Gamma \) such that if \( \gamma \in A \), \( x_\gamma \in B_\gamma \). Since each \( A_\gamma \) is a Jacobson ring, for \( \gamma \in A \) there is \( n(\gamma) \) such that \( x_\gamma^n = x_\gamma \).

If \( x_\gamma^{n(\gamma)} = x_\gamma \), an inductive argument shows that \( x_\gamma^{k(n(\gamma) - 1) + 1} = x_\gamma \) for all \( k \). If \( x_\gamma \in B_\gamma \), then \( |B_\gamma| \leq N \), so since \( |B_\gamma| - 1 < N \), \( x_\gamma^{k + k(N)} = x_\gamma \) for all \( k \). Let \( n = 1 + [(N!) \prod_{\gamma \in A} (n(\gamma) - 1)] \). Then \( x_\gamma^n = x_\gamma \) for all \( \gamma \), i.e., \( (x_\gamma)^n = (x_\gamma) \).

2. Analogues of four of Herstein's results. An analogue for topological rings of the first of Herstein's conditions that are mentioned above is that for all \( x \) and \( y \), \( xy - yx \) is in the closure of \( \{x^ny - yx^n \mid n \geq 2\} \), and we say such a topological ring is an \( H_1 \)-ring. An analogue of the second of Herstein's conditions is that for all \( x \) and \( y \), \( xy - yx \) is in the closure of \( \{(xy - yx)^n \mid n \geq 2\} \), and we say such a topological ring is an \( H_2 \)-ring. (If \( (xy - yx)^{n(x,y)-1} = xy - yx \) for all \( k \geq 1 \); hence another topological analogue is the assumption that for each \( x, y \in A \), there exists \( n(x,y) \geq 2 \) that \( \lim_k (xy - yx)^{k(n(x,y)-1)} = xy - yx \); however by an argument similar to that of the first paragraph of § 1, this condition implies that \( (xy - yx)^{n(x,y)} = xy - yx \).) Similarly an analogue of the third of Herstein's conditions is that for all \( x, y \) in \( A \), \( \lim_n x^ny - yx^n = 0 \), and we say such topological rings are \( H_3 \)-rings, just as we will call \( H_3 \)-rings those topological rings in which for all \( x, y \) there is an \( m(x, y) \geq 1 \) such that \( \lim_n x^ny^{m(x,y)} - y^{m(x,y)}x^n = 0 \). We shall prove that those \( H_3 \)-rings which are semisimple and locally compact are commutative, \( i = 1, 2, 3, 4 \).

Lemma 5. All idempotents in an \( H_i \)-ring, \( i = 1, 2, 3, 4 \), commute.

Proof. Let \( e \) and \( f \) be idempotents in such a ring \( A \). Then \( (efe - ef)^2 = 0 \), so \( (efe - ef)^ne - e(efe - ef)^n : n \geq 2 \) = \( 0 \). Therefore, if \( A \) is an \( H_i \)-ring, then \( (efe - ef)e - e(efe - ef) = 0 \), so

\[
0 = (efe - ef)e = e(efe - ef) = efe - ef.
\]

If \( A \) is an \( H_x \)-ring, then \( (ef)e - (ef) = efe - ef = 0 \) since \( efe - ef \) is in the closure of \( \{(ef)e - (ef)\}^n : n \geq 2 \) = \( 0 \). Similarly in either case
efe = fe, so ef = fe. As 0 = \lim_n e^n f - f e^n = \lim_n e^n f^n - f^n e^n = ef - fe, the assertion also holds for \( H_3 \) and \( H_4 \)-rings.

Since it is clear that all subrings and quotient rings determined by closed ideals of \( H_i \)-rings are \( H_i \)-rings, \( i = 1, 2, 3, 4 \), and since all idempotents in such rings commute, we see that the following is applicable.

**LEMMA 6.** Let \( P \) be a property of Hausdorff topological rings such that:

1. If \( A \) is a Hausdorff topological ring with property \( P \), then every subring of \( A \) has property \( P \) and \( A/B \) has property \( P \) where \( B \) is any closed ideal of \( A \).
2. If \( A \) has property \( P \), then all idempotents in \( A \) commute.

If \( A \) is a locally compact primitive ring with property \( P \), then \( A \) is a division ring.

**Proof.** Since \( A \) is a semisimple ring, \( A \) is the topological direct sum of a connected ring \( B \) and a totally disconnected ring \( C \), where \( B \) is a semisimple algebra over \( R \) of finite dimension [7, Th. 2]. As \( A \) is primitive, either \( A = B \) or \( A = C \). In the former case \( A \) is a matrix ring since it is primitive, and so has idempotents which do not commute unless it is a division ring.

It suffices, therefore, to consider the case in which \( A \) is totally disconnected. We shall first prove the assertion under the additional assumption that \( A \) is a \( Q \)-ring (i.e., the set of quasi-invertible elements is a neighborhood of zero). We may consider \( A \) to be a dense ring of linear operators on a vector space \( E \) over a division ring \( D \). If \( E \) is not one-dimensional, then \( E \) has a two-dimensional subspace \( M \) with basis \( \{z_1, z_2\} \). Let \( B = \{a \in A: a(M) \subseteq M\} \), and let

\[
N = \{a \in A: a(M) = (0)\} = K_1 \cap K_2
\]

where \( K_i = \{a \in A: a(z_i) = 0\} \), \( i = 1, 2 \).

There exists \( u \in A \) such that \( u(z_i) = z_i \), and hence \( x - xu \in K_i \), for all \( x \in A \). If \( v \in K_i \), then there exists \( w \in A \) such that \( wv(z_i) = z_i \), so as \( u = wv + (u - wv) \) and \( u - wv \in K_i \), \( A = Au + K_i = Av + K_i \). Therefore \( K_i \), and similarly \( K_2 \), is a regular maximal left ideal, an observation of the referee that simplifies the proof. Hence \( K_1 \) and \( K_2 \) are closed (cf. [11, Th. 2]), so \( N \) is a closed ideal of \( B \). By hypothesis \( B/N \) is therefore a Hausdorff topological ring having property \( P \). Thus all idempotents in \( B/N \) commute; but \( B/N \) is isomorphic to the ring of all linear operators on \( M \), a ring containing idempotents which do not commute. Hence \( E \) is one-dimensional and \( A \) is a division ring.

Next we shall show that \( A \) is necessarily a \( Q \)-ring, from which
the result follows by preceding. As $A$ is totally disconnected $A$ has a compact open subring $D$ [7, Lemma 4]. If $D = J(D)$, the radical of $D$, then $D$ and hence $A$ are $Q$-rings. Assume therefore that $J(D) \subset D$. We shall show that $D/J(D)$ is a finite ring and hence is discrete.

The radical, $J(D)$, of $D$ is closed [8, Th. 1], $D/J(D)$ is compact semisimple ring and thus $D/J(D)$ is topologically isomorphic to the Cartesian product of a family $(F_\gamma)_{\gamma \in \Gamma}$ of finite simple rings with identities $(f_\gamma)_{\gamma \in \Gamma}$ [11, Th. 16]. As $J(D)$ is topologically nilpotent [11, Th. 14], $D$ is suitable for building idempotents [12, Lemma 4] (cf. [11, Lemma 12]). Suppose that $\Gamma$ has more than one element, say $\{\alpha, \beta\} \subseteq \Gamma$. Then there are nonzero orthogonal idempotents $e_\alpha, e_\beta$ in $D$ such that $e_\alpha + J(D), e_\beta + J(D)$ correspond, respectively, under the isomorphism to $(/\alpha)$, $(/\beta)$ if $\gamma \neq \alpha, \beta$ and $/\gamma = /$; let $\phi$ be the canonical mapping $x \mapsto x + J(D)$ from $D$ onto $D/J(D)$. As $(/\alpha) + (/\beta)$ annihilates the open neighborhood $\bigcap_{\gamma \in \Gamma} G_\gamma$ of zero where $G_\alpha = \{0\}, G_\beta = \{0\}$, and $G_\gamma = F_\gamma$ for $\gamma \neq \alpha, \beta$, we conclude that $\phi(e_\alpha + e_\beta)$ annihilates a neighborhood $V$ of zero in $D/J(D)$. Consequently $U = \phi^{-1}(V)$ is a neighborhood of zero in $D$, and $(e_\alpha + e_\beta)U(e_\alpha + e_\beta) \subseteq J(D)$ (cf. [7, proof of Th. 11]). Therefore as $(e_\alpha + e_\beta)U(e_\alpha + e_\beta) = U \cap (e_\alpha + e_\beta)A(e_\alpha + e_\beta), (e_\alpha + e_\beta)U(e_\alpha + e_\beta)$ is a neighborhood of zero in $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ consisting of quasi-invertable elements, so $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a $Q$-ring.

Lemma 7. If $A$ is an $H_i$-ring, $i = 1, 2, 3, 4$ and if $A$ is a locally compact division ring, then $A$ is a field.

Proof. If $A$ is discrete and is an $H_i$-ring $(i = 1, 2, 3, 4)$ then $A$ is commutative [3, Th. 2; 4, Th. 1; 3, Th. 1; 1, Lemma 1].

If $A$ is not discrete, then $A$ has a nontrivial absolute value giving its topology, and $A$ is a finite-dimensional algebra over its center, on which the absolute value is nontrivial [10, Th. 8].

If $A$ is an $H_i$-ring and $x$ is nonzero in $A$, then there exists some nonzero $z$ in the center of $A$ such that $|z| < 1/|x|$. Thus $|xz| < 1$, so $\lim_n (xz)^n = 0$. Hence for any $y \in A$, $\lim_n (xz)^ny - y(xz)^n = 0$, so as $(xz)y - y(xz)$ is in the closure of $\{(xz)^ny - y(xz)^n: n \geq 2\}, 0 = (xz)y - y(xz) = x(yz - yx)$. Hence $xy = yx$, as $z \neq 0$. Thus $A$ is commutative.

If $A$ is an $H_i$-ring and if $x, y \in A$ satisfy $xy - yx \neq 0$, then there exists some nonzero $z$ in the center such that $|z| < 1/|xy - yx|$. Thus
\[ |(xz)y - y(xz)| < 1, \text{ so } \lim_n [(xz)y - y(xz)]^n = 0. \text{ Hence } 0 = (xz)y - y(xz) = (xy - yx)z, \text{ so } xy - yx = 0 \text{ as } z \neq 0, \text{ a contradiction. Thus } A \text{ is commutative.} \]

Assume that \( A \) is an \( H_5 \)-ring. As \( A \) is a division ring, \( A \) is either totally disconnected or connected [7, Th. 2].

\textbf{Case 1.} \( A \) is totally disconnected. Then the topology of \( A \) is given by a nonarchimedean absolute value. Suppose \( A \) is not commutative. Then as \( A \) is a finite-dimensional and hence an algebraic extension of its center \( C \), there exists some \( x \in C \) having minimal degree \( m > 1 \) over \( C \). Let \( y \) be arbitrary in \( A \), and assume that for no \( 1 \leq i \leq m - 1 \), does \( x^i y = yx^i \). Hence \( x^i y - yx^i \neq 0, 1 \leq i \leq m - 1 \), and we claim \( \{x^i y - yx^i : 1 \leq i \leq m - 1\} \) is a linearly independent set over \( C \).

Suppose \( \sum_{i=1}^{m-1} \beta_i (x^i y - yx^i) = 0 \), where \( \beta_i \in C \), and let \( z = \sum_{i=0}^{m-1} \beta_i x^i \). Then \( zy = yz \). By the definition of \( m \), either \( z \in C \) on \( z \) has degree \( \geq m \) over \( C \). Suppose \( z \notin C \). Then \( C[x] \) has dimension \( m \) over \( C \), so \( m \) is the degree of \( z \) as \( z \in C[x] \). Therefore \( C[x] = C[z] \), so as \( zy = yz \), every element of \( C[x] \) commutes with \( y \), contrary to our assumption. Thus \( z \in C \); let \( -\beta_0 = z \). Then \( \sum_{i=0}^{m-1} \beta_i x^i = 0 \), so \( \beta_i = 0, 0 \leq i \leq m - 1 \) since \( \{1, x, \ldots, x^{m-1}\} \) is linearly independent over \( C \).

Since \( x \) is algebraic of degree \( m \) over the center \( C \) of \( A \), there exist \( \alpha_i \in C, 0 \leq i \leq m - 1 \), such that \( x^n = \sum_{i=0}^{m-1} \alpha_i x^i \); thus for all \( n \geq m \), there exist \( \alpha_{i,n} \in C, 0 \leq i \leq m - 1 \), such that \( x^n = \sum_{i=0}^{m-1} \alpha_{i,n} x^i \). We may also assume that \( |x| > 1 \), since all our assumption on \( x \) are true for any \( \lambda x, \lambda \in C^* \). We note that there is therefore some \( r \) such that \( |x|^r \geq |\alpha_i|, 0 \leq i \leq m - 1 \).

Since \( x^n = \sum_{i=0}^{m-1} \alpha_{i,n} x^i \),

\[ x^i y - yx^n = \sum_{i=1}^{m-1} \alpha_{i,n} (x^i y - yx^i); \]

so \( \lim_n x^n y - yx^n = 0 \) if and only if \( \lim_n \alpha_{i,n} = 0, 1 \leq i \leq m - 1 \).

Since \( |x^n| \leq \max \{|\alpha_{i,n}| \cdot |x|^i: 0 \leq i \leq m - 1\} \), if \( |\alpha_{i,n}| < 1, 1 \leq i \leq m - 1 \), then \( |x|^n \leq |\alpha_{i,n}|. \) Let \( r_0 \) be such that \( |x|^{r_0} > |x| + 1 \). Since \( \lim_n \alpha_{i,n} = 0, 1 \leq i \leq m - 1 \), there exists \( n_0 > r + r_0 \) such that \( |\alpha_{i,n}| < 1 \), for all \( n \geq n_0 \) and all \( i \) such that \( 1 \leq i \leq m - 1 \). But for any \( n > n_0 \),

\[ x^{n+1} = \sum_{i=0}^{m-2} \alpha_i x^i + x^{m-1} + \sum_{i=0}^{m-1} \alpha_i x^i; \]

so

\[ |\alpha_{i,n+1}| = |\alpha_{i,n} + \alpha_{m-1,n} \alpha_i| \geq |\alpha_{i,n}| - |\alpha_{m-1,n} \alpha_i| \geq |x|^r - |\alpha_i| \geq |x|^{r+r_0} - |x|^r = |x|^r \cdot (|x|^{r_0} - 1) > 1. \]

a contradiction. Hence \( A \) is commutative.
Case 2. A is connected. Then the center C of A contains the real number field R, A is finite-dimensional over R, so the degree of each element of A over R is less than or equal to 2, and the topology is given by an absolute value. Suppose \( x \in C \). Then \( \deg x = 2 \); let \( x = \alpha_1 + \alpha_2 \alpha \), and for each \( n \geq 2 \), let \( x^n = \alpha_{1,n} + \alpha_{2,n} \alpha \), where \( \alpha_{1,n}, \alpha_{2,n} \in R \). As before we may assume that \( |x| > 1 \). Let \( r \) be such that \( |x|^r > \max (|\alpha_1|, |\alpha_2|) \). Let \( y \in A \) be such that \( xy \neq yx \). Then \( 0 = \lim_n (x^n y - yx^n) = \lim_n \alpha_{3,n}(xy - yx) \), so \( \lim_n \alpha_{3,n} = 0 \). Let \( n_0 > r \) be such that \( |\alpha_{3,n}| < 1 \) for all \( n \geq n_0 \). But if \( n \geq n_0 \) is such that \( |x|^n > 3 |x|^r \), then

\[
|x|^n = |\alpha_{1,n} + \alpha_{2,n} \alpha| \leq |\alpha_{1,n}| + |\alpha_{2,n}| |x| < |\alpha_{1,n}| + |x|,
\]

so \( |x^n| - |x| < |\alpha_{1,n}| \). As

\[
x^{n+1} = \alpha_{1,n} x + \alpha_{2,n} (\alpha_1 + \alpha_2 x) = \alpha_{3,n} \alpha_1 + (\alpha_{1,n} + \alpha_{2,n} \alpha_2) x,
\]

\[
|\alpha_{2,n+1}| = |\alpha_{1,n} + (\alpha_{1,n}) \alpha_2| \geq |\alpha_{1,n}| - |\alpha_{2,n}| |\alpha_2|.
\]

Hence \( |\alpha_{3,n+1}| \geq (|x|^n - |x|) - |x|^r \geq 3|x|^r - |x|^r - |x|^r = |x|^r > 1 \), a contradiction. Hence A is commutative.

Finally let A be an \( H_i \)-ring. If for all \( x \) and \( y \), \( \lim_n x^n y - yx^n = 0 \), then A is an \( H_i \)-ring and so a field; so assume there are \( x \) and \( y \) in A such that \( \lim_n x^n y - yx^n \neq 0 \). Let \( W = \{w \in A: \lim_n x^n w - wx^n = 0\} \). Clearly W is a division subring of A, and since \( y \in W \), W is a proper division subring. By hypothesis, for all \( a \in A \) there is an \( r \geq 1 \) such that \( a^r \in W \); thus A is a field [2, Th. B].

**Theorem 4.** All \( H_i \)-rings that are locally compact and semisimple are commutative, \( i = 1, 2, 3, 4 \).

**Proof.** P is a primitive ideal of such a ring A if and only if \( P = (B: A) \) (by definition \( (B: A) = \{x \in A: Ax \subseteq B\} \)) where B is a regular maximal to left ideal [5, Corollary to Proposition 2, p. 7]. Let \( e \in A \) be such that \( x - ex \in B \) for all \( x \in A \). If \( x \in (B: A) \), then \( ex \in B \), so \( x \in B \). Hence \( (B: A) \subseteq B \).

If \( B \) is closed, then \( (B: A) \) is closed for if \( (x_n) \) is a directed set of elements of \( (B: A) \) converging to \( x \), then for all \( a \in A \), \( ax_n \in B \), whence \( ax = \lim ax_n \in B \).

As A is semisimple, \( (0) = \bigcap \{B: B \text{ is a closed regular maximal left ideal}\} \supseteq \bigcap \{P: P \text{ is a closed primitive ideal}\} \) [8, Th. 1]. By Lemma 6 and 7, \( A/P \) is a field if \( P \) is a closed primitive ideal. Thus for all \( x, y \in A \), \( xy - yx \in P \), so \( xy - yx \in \bigcap \{P: P \text{ is a closed primitive ideal}\} = (0) \).
REFERENCES


Received August 16, 1968. The results in this paper are taken from the author's doctoral dissertation, written at Duke University under Professor Seth Warner.

DUKE UNIVERSITY