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# CARTAN SUBALGEBRAS OF A LIE ALGEBRA AND ITS IDEALS

DAVID J. WINTER

**The purpose of this paper is to describe, under suitable conditions which are always satisfied at characteristic 0, a close relationship between Cartan subalgebras of a Lie algebra $\mathscr{L}$ and Cartan subalgebras of an ideal $\mathscr{L}'$ of $\mathscr{L}$. Under the conditions referred to, a mapping $\alpha^*$ from the set of Cartan subalgebras of $\mathscr{L}$ onto the set of Cartan subalgebras of $\mathscr{L}'$ is described and the fibres of $\alpha^*$ are determined.**

The main tools for the paper are N. Jacobson's generalization of Engel's Theorem [2; p. 33], and Theorem 5 of [4] which deals with Cartan subalgebras of the Fitting zero space of a derivation of a Lie algebra $\mathscr{L}$. In addition, general material on Lie algebras, to be found in [2], [3], is presupposed.

Throughout this paper, Lie algebras and vector spaces are finite dimensional.

If $V$ is an $\mathscr{N}$-module where $\mathscr{N}$ is a nilpotent Lie algebra over the field $F$, the null and one components of $V$ are denoted $V_0(\mathscr{N})$, $V_*(\mathscr{N})$ respectively [cf. 2; pp. 37–43] and, for $\alpha$ a function from $\mathscr{N}$ into $F$, $V_\alpha(\mathscr{N}) = \{v \in V \mid v(I - \alpha(x))^{\dim V} = 0$ for all $x \in \mathscr{N}\}$.

If $V$ is a vector space (respectively Lie algebra, respectively module for a Lie algebra, over $F$, then the extension $V \otimes_F K$ of $V$ to an extension field $K$ of $F$ is denoted $V_K$.

**2. Cartan subalgebras of a Lie algebra and its ideals.** Throughout this section, $\mathscr{L}$ denotes a Lie algebra over an arbitrary field $F$. The characteristic of $F$ is denoted $p$, $p = 0$ being permissible. Let $\mathscr{L}'$ be an ideal of $\mathscr{L}$ and let the canonical short exact sequence determined by $\mathscr{L}, \mathscr{L}'$ be denoted

$$0 \longrightarrow \mathscr{L}' \xrightarrow{\alpha} \mathscr{L} \xrightarrow{\beta} \overline{\mathscr{L}} = \mathscr{L}/\mathscr{L}' \longrightarrow 0\,,$$

where $\alpha$ is the inclusion mapping. The set of Cartan subalgebras of $\mathscr{L}$ is denoted Cart $\mathscr{L}$. For $\mathscr{H} \in \text{Cart}\,\mathscr{L}$, $(\mathscr{L}')_0(\text{ad}\,(\mathscr{H} \cap \mathscr{L}'))$ is denoted $\alpha^*(\mathscr{H})$. Our main objective is to prove the following theorem.

THEOREM. *Suppose that either $p = 0$, or $p \neq 0$ and $(\text{ad}_{\mathscr{L}'}\mathscr{L}')^p \subset \text{ad}_{\mathscr{L}'}\mathscr{L}'$ and $(\text{ad}_{\mathscr{L}}\mathscr{L})^p \subset \text{ad}_{\mathscr{L}}\mathscr{L}$. Then $\alpha^*(\text{Cart}\,\mathscr{L}) = \text{Cart}\,\mathscr{L}'$ and*

$\alpha^{*-1}(\mathscr{H}') = \operatorname{Cart} \mathscr{L}_0(\operatorname{ad} \mathscr{H}')$ *for* $\mathscr{H}' \in \operatorname{Cart} \mathscr{L}'$.

We defer the proof for the moment, since it is convenient to have the following lemma at our disposal.

LEMMA. *Let $V$ be a vector space over $F$, $\mathscr{L}$ a Lie subalgebra of* $\operatorname{Hom}_F V$. *If the characteristic of $F$ is $p \neq 0$, suppose that $\mathscr{L}$ is closed under $p$-th powers. Let $\mathscr{N}$ be a nilpotent subalgebra of* $\operatorname{Hom}_F V$ *which normalizes $\mathscr{L}$. Suppose that* $\mathscr{L}_0(\operatorname{ad} \mathscr{N})$ *consists of nilpotent transformations of $V$. Then $\mathscr{L}$ consists of nilpotent transformations of $V$.*

*Proof of lemma.* Since $\mathscr{L}_0(\operatorname{ad} \mathscr{N})$ consists of nilpotent transformations and is closed under brackets, $\mathscr{L}_0(\operatorname{ad} \mathscr{N})_K = (\mathscr{L}_K)_0(\operatorname{ad} \mathscr{N}_K)$ consists of nilpotent transformations where $K$ is the algebraic closure of $F$. Moreover, if the characteristic of $F$ is $p \neq 0$, $\mathscr{L}_K$ is closed under $p$-th powers [cf. 2; p. 190]. Thus, we may assume without loss of generality that $F$ is algebraically closed.

Now $\mathscr{L} = \sum \mathscr{L}_\alpha(\operatorname{ad} \mathscr{N})$ and $V = \sum V_\beta(\mathscr{N})$. For all $\alpha, \beta$, we have $V_\beta(\mathscr{N})\mathscr{L}_\alpha(\operatorname{ad} \mathscr{N}) \subset V_{\beta+\alpha\circ\operatorname{ad}}(\mathscr{N})$ [cf. 2; p. 63]. Thus, if the characteristic of $F$ is 0, $\mathscr{L}_\alpha(\operatorname{ad} \mathscr{N})$ consists of nilpotent transformations for all $\alpha$: for $\alpha = 0$ by hypothesis and for $\alpha \neq 0$ by the above observation. Suppose next that the characteristic of $F$ is $p \neq 0$. Let $x \in L_\alpha(\operatorname{ad} \mathscr{N})$. Then $x^p \in \mathscr{L} \cap (\operatorname{Hom}_F V)_0(\operatorname{ad} \mathscr{N}) = \mathscr{L}_0(\operatorname{ad} \mathscr{N})$, for if $t$ is the semi-simple part of an element $y$ of $\mathscr{N}$; $t \operatorname{ad} x = -\alpha(y)x$ so that $0 = t(\operatorname{ad} x)^2 = \cdots = t(\operatorname{ad} x)^p = [t, x^p]$. Thus, $x^p$, hence $x$, is nilpotent. Thus, the $\mathscr{L}_\alpha(\operatorname{ad} \mathscr{N})$ again consist of nilpotent transformations for all $\alpha$. We now can apply [2; p. 33] to the weakly closed set $\cup \mathscr{L}_\alpha(\operatorname{ad} \mathscr{N})$ of nilpotent transformations. This implies that the Lie algebra generated by $\cup \mathscr{L}_\alpha(\operatorname{ad} \mathscr{N})$, namely $\mathscr{L}$ itself, consists of nilpotent transformations.

*Proof of theorem.* We first show that $\alpha^*(\operatorname{Cart} \mathscr{L}) \subset \operatorname{Cart} \mathscr{L}'$. Thus, let, $\mathscr{H} \in \operatorname{Cart} \mathscr{L}$. Then $\mathscr{H} \cap \mathscr{L}' = \mathscr{L}_0(\operatorname{ad} \mathscr{H}) \cap \mathscr{L}' = (\mathscr{L}')_0(\operatorname{ad} \mathscr{H})$. Now $\mathscr{N} = \operatorname{ad} \mathscr{H}|_{\mathscr{L}'}$ is a nilpotent Lie algebra of derivations of $\mathscr{L}'$ and $\mathscr{H} \cap \mathscr{L}'$ is trivially a Cartan subalgebra of $(\mathscr{L}')_0(\mathscr{N}) = \mathscr{H} \cap \mathscr{L}'$. Thus, Theorem 5 of [4] applies and shows that $(\mathscr{L}')_0(\operatorname{ad}(\mathscr{H} \cap \mathscr{L}')) = \alpha^*(\mathscr{H})$ is a Cartan subalgebra of $\mathscr{L}'$.

Next suppose that $\mathscr{H}' \in \operatorname{Cart} \mathscr{L}'$ and that $\mathscr{N} \in \operatorname{Cart} \mathscr{L}_0(\operatorname{ad} \mathscr{H}')$. Since $\mathscr{L}_0(\operatorname{ad} \mathscr{H}')$ normalizes $\mathscr{L}_0(\operatorname{ad} \mathscr{H}') \cap \mathscr{L}' = (\mathscr{L}')_0(\operatorname{ad} \mathscr{H}') = \mathscr{H}'$, we have:

$$(1) \qquad \mathscr{N} \text{ normalizes } \mathscr{H}'.$$

In view of (1), we have $\mathscr{H}' = \mathscr{H}_0' \oplus \mathscr{H}_*'$ where $\mathscr{H}_0' = (\mathscr{H}')_0(\operatorname{ad} \mathscr{N})$

and $\mathscr{H}'_* = (\mathscr{H}')_*(\mathrm{ad}\,\mathscr{N})$. Note that $\mathscr{H}'_0 = \mathscr{H}' \cap \mathscr{N}$ since $\mathscr{N} \in$ Cart $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$ and $\mathscr{H}' \subset \mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$. Let $V = (\mathscr{L}')_0(\mathrm{ad}\,\mathscr{H}'_0)$. Since $\mathscr{H}'_0 \subset \mathscr{H}'$ and $\mathscr{H}'_0 \subset \mathscr{N}$, $V$ is stable under ad $\mathscr{H}'$ and ad $\mathscr{N}$ [cf. 2; p. 58]. Now we prepare the way for applying the above lemma to $(V, \mathrm{ad}\,\mathscr{H}'|_V, \mathrm{ad}\,\mathscr{N}|_V)$. Thus, note that ad $\mathscr{H}'|_V$ is a subalgebra of $\mathrm{Hom}_F V$ normalized by the nilpotent subalgebra ad $\mathscr{N}|_V$ and that, if the characteristic of $F$ is $p \neq 0$, ad $\mathscr{H}'|_V$ is closed under $p$-th powers. (In fact, $\mathrm{ad}_{\mathscr{L}'}\mathscr{H}'$ is closed under $p$-th powers since $\mathrm{ad}_{L'}\mathscr{L}'$ is closed under $p$-th powers and since $\mathscr{H}'$ is a Cartan subalgebra of $\mathscr{L}'$: for $x \in \mathscr{H}'$, $(\mathrm{ad}\,x)^p = \mathrm{ad}\,y$ for some $y \in \mathscr{L}'$, and $y \in \mathscr{H}'$, since $\mathscr{H} \supset \mathscr{H}\,(\mathrm{ad}\,x)^p = [\mathscr{H}, y]$). Moreover

$$(\mathrm{ad}\,\mathscr{H}'|_V)_0(\mathrm{ad}\,(\mathrm{ad}\,\mathscr{N}|_V)) = \mathrm{ad}\,\mathscr{H}'_0|_V$$

and ad $\mathscr{H}'_0|_V$ consists of nilpotent transformations by the definition of $V$. Thus, by the lemma, ad $\mathscr{H}'|_V$ consists of nilpotent transformations. Thus, $(\mathscr{L}')_0(\mathrm{ad}\,\mathscr{H}'_0) = ((\mathscr{L}')_0(\mathrm{ad}\,\mathscr{H}') = \mathscr{H}'$. We therefore have:

$$\mathscr{H}' = (\mathscr{L}')_0(\mathrm{ad}\,(\mathscr{H}' \cap \mathscr{N})) . \tag{2}$$

We show that (2) implies $\mathscr{H}' = (\mathscr{L}')_0(\mathrm{ad}\,(\mathscr{H} \cap \mathscr{L}))$ for substable $\mathscr{H} \in$ Cart $\mathscr{L}$. Thus, let $\mathscr{H} = \mathscr{L}_0(\mathrm{ad}\,\mathscr{N})$. Then $\mathscr{H} \in$ Cart $\mathscr{L}$, by Theorem 5 of [4], since $\mathscr{N}$ is a Cartan subalgebra of $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$. Since $\mathscr{H}' \cap \mathscr{N} \subset \mathscr{L}' \cap \mathscr{H}$, (2) implies that

$$\alpha^*(\mathscr{H}) = (\mathscr{L}')_0(\mathrm{ad}\,(\mathscr{L}' \cap \mathscr{H})) \subset (\mathscr{L}')_0(\mathrm{ad}\,(\mathscr{H}' \cap \mathscr{N})) = \mathscr{H}' .$$

Thus, since $\alpha^*(\mathscr{H}) \in$ Cart $\mathscr{L}'$, by the preceding paragraph, $\alpha^*(\mathscr{H}) = \mathscr{H}'$, by the maximal nilpotency of Cartan subalgebras. Thus, we have:

$$\mathscr{L}_0(\mathrm{ad}\,\mathscr{N}) \in \text{Cart}\,\mathscr{L} \text{ and } \alpha^*(\mathscr{L}_0(\mathrm{ad}\,\mathscr{N})) = \mathscr{H}' . \tag{3}$$

We have $\alpha^*(\text{Cart}\,\mathscr{L}) \subset \text{Cart}\,\mathscr{H}'$, from the first paragraph. Thus, it follows from (3) that $\alpha^*(\text{Cart}\,\mathscr{L}) = \text{Cart}\,\mathscr{L}'$. Note, however, that the existence of $\mathscr{N} \in$ Cart $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$ is used for this conclusion. But ad $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$ is a linear Lie $p$-algebra for $p \neq 0$, as the null component of the linear Lie $p$-algebra ad $\mathscr{L}$ with respect to the subalgebra ad $\mathscr{H}'$. Thus, ad $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$ has a Cartan subalgebra, by [3; p. 121], so that $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$ has a Cartan subalgebra.

Finally, we suppose that $\mathscr{H}' \in$ Cart $\mathscr{L}'$. Let $\mathscr{H} \in \alpha^{*^{-1}}(\mathscr{H}')$. Then $\mathscr{H} \subset \mathscr{L}_0(\mathrm{ad}\,(\mathscr{H} \cap \mathscr{L}'))$, so that $\mathscr{H}$ normalizes

$$\mathscr{H}' = \mathscr{L}_0(\mathrm{ad}\,(\mathscr{H} \cap \mathscr{L}')) \cap \mathscr{L}' .$$

Thus, $\mathscr{H} \subset \mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$, so that $\mathscr{H} \in$ Cart $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$. Suppose, conversely, that $\mathscr{N} \in$ Cart $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$. By (3), $\mathscr{L}_0(\mathrm{ad}\,\mathscr{N}) \in$ Cart $\mathscr{L}$

and $\alpha^*(\mathscr{L}_0(\text{ad}\,\mathscr{N})) = \mathscr{H}'$. Thus, by first part of this paragraph, $\mathscr{L}_0(\text{ad}\,\mathscr{N}) \subset \mathscr{L}_0(\text{ad}\,\mathscr{H}')$. But then

$$\mathscr{L}_0(\text{ad}\,\mathscr{N}) = \mathscr{L}_0(\text{ad}\,\mathscr{H}'))_0(\text{ad}\,\mathscr{N}) = \mathscr{N}\,,$$

since $\mathscr{N} \in \text{Cart}\,\mathscr{L}_0(\text{ad}\,\mathscr{H}')$ [cf. 2; p. 57–58]. Thus, $\mathscr{N}$ is a Cartan subalgebra of $\mathscr{L}$, by [2; p. 57–58]. Now

$$\alpha^*(\mathscr{N}) = \alpha^*(\mathscr{L}_0(\text{ad}\,\mathscr{N})) = \mathscr{H}'\,,$$

by (3), and $\mathscr{N} \in \alpha^{*^{-1}}(\mathscr{H}')$. Thus, $\alpha^{*^{-1}}(\mathscr{H}') = \text{Cart}\ \mathscr{L}_0(\text{ad}\,\mathscr{H}')$.

We now turn to two related results. The first is concerned with the fibres of $\alpha^*$. The second is concerned with the relations between the sequences

$$\text{Cart}\,\mathscr{L}' \xleftarrow{\alpha^*} \text{Cart}\,\mathscr{L} \xleftarrow{i} \text{Cart}\,\mathscr{L}_0(\text{ad}\,\mathscr{H}')$$

$$\text{Cart}\,\beta^{-1}(\overline{\mathscr{H}}) \xrightarrow{i} \text{Cart}\,\mathscr{L} \xrightarrow{\beta_*} \text{Cart}\,\overline{\mathscr{L}}$$

where $i$ is inclusion, $\mathscr{H} \in \text{Cart}\,\mathscr{L}$, $\mathscr{H}' = \alpha^*(\mathscr{H})$, $\overline{\mathscr{H}} = \beta(\mathscr{H})$ and $\beta_*$ is defined by $\beta_*(\mathscr{J}) = \beta(\mathscr{J})$ for $\mathscr{J} \in \text{Cart}\,\mathscr{L}$.

PROPOSITION 1. *Let the hypothesis be as in the theorem, and let* $\mathscr{H} \in \text{Cart}\,\mathscr{L}$, $\mathscr{H}' \in \text{Cart}\,\mathscr{L}'$. *Then the following conditions are equivalent.*

(1) $\mathscr{H}' = \alpha^*(\mathscr{H})$;

(2) $\mathscr{H}$ *normalizes* $\mathscr{H}'$;

(3) $\mathscr{H} \cap \mathscr{L}' \subset \mathscr{H}'$.

*Proof.* If $\mathscr{H}' = \alpha^*(\mathscr{H})$, then $\mathscr{H} \subset \mathscr{L}_0(\text{ad}\,(\mathscr{L}' \cap \mathscr{H}))$ and $\mathscr{H}$ normalizes $\mathscr{H}' = \mathscr{L}_0(\text{ad}(\mathscr{L}' \cap \mathscr{H})) \cap \mathscr{L}'$. Thus, (1) implies (2). If $\mathscr{H}$ normalizes $\mathscr{H}'$, then $\mathscr{H} \subset \mathscr{L}_0(\text{ad}\,\mathscr{H}')$ and

$$\mathscr{H} \cap \mathscr{L}' \subset (\mathscr{L}')_0(\text{ad}\,\mathscr{H}') = \mathscr{H}'\,.$$

Thus, (2) implies (3). Suppose, finally, that $\mathscr{H} \cap \mathscr{L}' \subset \mathscr{H}'$. Then $\mathscr{H}' = (\mathscr{L}')_0(\text{ad}\,\mathscr{H}') \subset (\mathscr{L}')_0(\text{ad}(\mathscr{H} \cap \mathscr{L}')) = \alpha^*(\mathscr{H})$. But $\alpha^*(\mathscr{H}) \in \text{Cart}\,\mathscr{L}'$ and $\mathscr{H}'$ is, a Cartan subalgebra of $\mathscr{L}'$, maximal nilpotent in $\mathscr{L}'$. Thus, $\mathscr{H}' = \mathscr{L}(\mathscr{H})^*(\mathscr{H})$. Thus (3) implies (1), and the conditions (1)–(3) are equivalent.

PROPOSITION 2. *Let the hypothesis be as in the theorem. Let* $\mathscr{H} \in \text{Cart}\,\mathscr{L}$, $\mathscr{H}' = \alpha^*(\mathscr{H})$, $\overline{\mathscr{H}} = \beta(\mathscr{H})$. *Then* $\mathscr{H}$ *normalizes* $\mathscr{H}'$, $\text{Cart}\,\mathscr{L}$ *contains* $\text{Cart}\,(\mathscr{H} + \mathscr{H}')$, $\text{Cart}\,\mathscr{L}_0(\text{ad}\,\mathscr{H}')$ *and* $\text{Cart}\,\beta^{-1}(\overline{\mathscr{H}})$, *and* $(\text{Cart}\,\mathscr{L}_0(\text{ad}\,\mathscr{H}')) \cap \text{Cart}\,\beta^{-1}(\overline{\mathscr{H}}) = \text{Cart}\,(\mathscr{H} + \mathscr{H}')$.

*Proof.* $\mathscr{H}$ normalizes $\mathscr{H}'$, by Proposition 1. Since

$$\beta^{-1}(\overline{\mathscr{H}}) = \mathscr{H} + \mathscr{S}', \ \mathscr{L}_0(\mathrm{ad}\,\mathscr{H}') \cap \beta^{-1}(\overline{\mathscr{H}}) = \mathscr{H} + \mathscr{H}' .$$

Thus, it suffices to show that Cart $\mathscr{L}$ contains Cart $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$, Cart $\beta^{-1}(\overline{\mathscr{H}})$ and Cart $(\mathscr{H} + \mathscr{H}')$. The first two sets are contained in Cart $\mathscr{L}$ – Cart $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$ by the theorem and Cart $\beta^{-1}(\overline{\mathscr{H}})$ by [1]. Thus, it remains only to show that

$$\mathrm{Cart}\,(\mathscr{H} + \mathscr{H}') \subset \mathrm{Cart}\,\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}') .$$

But $\mathscr{H}' = \mathscr{L}_0(\mathrm{ad}\,\mathscr{H}') \cap \mathscr{L}'$ is an ideal of $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$ and $\mathscr{H} \in$ Cart $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$. Thus, Cart $(\mathscr{H} + \mathscr{H}') \subset$ Cart $\mathscr{H}_0(\mathrm{ad}\,\mathscr{H}')$, by [1], since $\mathscr{H} + \mathscr{H}'$ is the preimage in $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')$ of the Cartan subalgebra $(\mathscr{H} + \mathscr{H}')/\mathscr{H}'$ of $\mathscr{L}_0(\mathrm{ad}\,\mathscr{H}')/\mathscr{H}'$.

## REFERENCES

1. D. W. Barnes, *On Cartan subalgebras of Lie algebras*, Math. Zeit. **101** (1967), 350-355.
2. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.
3. G. Seligman, *Modular Lie algebras*, Springer, New York, 1967.
4. D. J. Winter, *On groups of automorphisms of Lie algebras*, J. of Algebra **8** (1968).

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