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# THE COMMUTATOR AND SOLVABILITY IN A GENERALIZED ORTHOMODULAR LATTICE

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**In this paper we prove in a generalized orthomodular lattice the analog of the following theorem from group theory. For $a$ and $b$ members of a group $G$, let $aba^{-1}b^{-1}$ be the commutator of $a$ and $b$. The set of commutators in $G$ generates a normal subgroup $H$ of $G$ possessing these properties: $G/H$ is Abelian. Moreover, if $K$ is any normal subgroup of $G$ for which $G/K$ is Abelian, then $K \supseteqq H$. Continuing the analogy with group theory, we determine a solvability condition on generalized orthomodular lattices.**

An *orhomodular lattice* is a lattice $L$ with 0 and 1 and with an orthocomplementation $'\colon L \to L$ satisfying the *orthomodular identity*: for $e \leqq f$ in $L$, $f = e \vee (f \wedge e')$. Throughout this paper $L$ shall denote an orthomodular lattice. For $f \in L$ the *Sasaki projection determined by $f$* $\phi_f : L \to L$ by $e\phi_f = (e \vee f') \wedge f$. We say *$e$ commutes with $f$, $ecf$,* when $e\phi_f = e \wedge f$. Basic properties of orthomodular lattices and of their coordinatizing Baer $*$-semigroups are contained in [1, 2].

A lattice ideal $I$ in $L$ is called a *$p$-ideal* if and only if $e \in I$ and $f \in L$ imply $e\phi_f \in I$. Theorem 6, which concerns $p$-ideals in generalized orthomodular lattices, indicates the significance of $p$-ideals in orthomodular lattices.

## 2. The commutator.

For elements $e$ and $f$ of the orthomodular lattice $L$, we define the *commutator* of $e$ and $f$ by

$$[e, f] = (e \vee f) \wedge (e \vee f') \wedge (e' \vee f) \wedge (e' \vee f') .$$

It is easily shown that $ecf$ if and only if $[e, f] = 0$, and that $[e, f] = [e, f'] = [e', f] = [e', f']$.

THEOREM 1. *Let $R$ be a Baer $*$-ring, and let $P'(R)$ denote the orthomodular lattice of closed projections in $R$. Then for*

$$e, f \in P'(R),\ (ef\text{-}fe)'' = [e, f] .$$

In proving the theorem, we shall use the following computation.

LEMMA 2. $[e, f] = (f'ef)'' \vee (e'fe)''$.

*Proof.* $(f'ef)'' = ((f'e)''f)'' = f'\phi_e\phi_f = \{[(f' \vee e') \wedge e] \vee f'\} \wedge f =$

$(f' \vee e') \wedge (e \vee f') \wedge f$, where the last equality holds by the Foulis-Holland theorem [2]—observe that $(f' \vee e')ce$, and $(f' \vee e')cf'$. Similarly, $(e'fe)'' = (f' \vee e') \wedge (e' \vee f) \wedge e$. The following expression is simplified by repeated applications of the Foulis-Holland theorem. We have

$$\begin{aligned}
&(f'ef)'' \vee (e'fe)'' \\
&= [(f' \vee e') \wedge (e \vee f') \wedge f] \vee [(f' \vee e') \wedge (e' \vee f) \wedge e] \\
&\qquad (f' \vee e')c(e \vee f') \wedge f,\ (e' \vee f) \wedge e \\
&= (f' \vee e') \wedge \{[(e \vee f') \wedge f] \vee [(e' \vee f) \wedge e]\} \\
&\qquad (e' \vee f)c(e \vee f') \wedge f,\ e \\
&= (f' \vee e') \wedge [(e' \vee f) \wedge \{[(e \vee f') \wedge f] \vee e\}] \\
&\qquad (e \vee f')cf,\ e \\
&= (f' \vee e') \wedge (e' \vee f) \wedge (e \vee f') \wedge (f \vee e) = [e, f]\ .
\end{aligned}$$

*Proof of theorem.* The element $(ef\text{-}fe)''$ is the smallest closed projection serving as a right identity for $(ef\text{-}fe)$. Equivalently, $(ef\text{-}fe)'$ is the greatest closed projection which serves as a right annihilator for $ef\text{-}fe$. Thus for $k \in P'(R)$, $k \leqq (ef\text{-}fe)'$ if and only if $efk = fek$.

Suppose that for some $k \in P'(R)$, $efk = fek$. Then $f'efk = f'fek = 0$ implies that $k = (f'ef)'k$, or $k \leqq (f'ef)'$. Similarly $k \leqq (e'fe)'$, and hence $k \leqq (e'fe)' \wedge (f'ef)' = [e, f]'$. Also, $(ef)[e, f]' = e(f[e, f]') = e(f \wedge [e, f]') = e(f \wedge [(e \wedge f) \vee (e \wedge f') \vee (e' \wedge f) \vee (e' \wedge f')]) = e[(e \wedge f) \vee (e' \wedge f)] = e \wedge [(e \wedge f) \vee (e' \wedge f)] = e \wedge f = fe[e, f]'$. Moreover, for $k \leqq [e, f]'$, then $k = [e, f]'k$ and $efk = ef[e, f]'k = fe[e, f]'k = fek$. Thus we have shown that $efk = fek$ if and only if $k \leqq [e, f]'$. Therefore $(ef\text{-}fe)' = [e, f]'$ and $(ef\text{-}fe)'' = [e, f]$.

LEMMA 3. *For $e, f \in L$, $f\phi_e \leqq f \vee [f, e]$.*

*Proof.* By the Foulis-Holland theorem,

$$f \vee [(f \vee e) \wedge (f \vee e') \wedge (f' \vee e) \wedge (f' \vee e')] = (f \vee e) \wedge (f \vee e')\ .$$

LEMMA 4. *Let $L$ and $X$ be orthomodular lattices.*

( i ) *For an ortho-homomorphism $\phi: L \rightarrow X$ and $c$ a commutator in $L$, $c\phi$ is a commutator in $X$.*

(ii) *For an ortho-epimorphism $\phi: L \rightarrow X$ and $x$ a commutator in $X$, $x = c\phi$ where $c$ is a commutator in $L$.*

(iii) *$X$ is Boolean if and only if* 0 *is the only commutator in $X$.*

*Proof.* Ortho-homomorphisms preserve suprema, infima, and orthocomplements.

THEOREM 5. *Let $L$ be an orthomodular lattice, and let $J$ be the ideal generated by the commutators in $L$. Then $J$ is a $p$-ideal, and $L/J$ is Boolean. Moreover, if $I$ is any $p$-ideal for which $L/I$ is Boolean, then $I \supseteq J$.*

*Proof.* Let $J$ be the ideal generated by the commutators in $L$, i.e.,

$$J = \left\{ y \in L \mid \text{for some commutators } c_1, \cdots, c_n \text{ in } L, y \leqq \bigvee_{i=1}^{n} c_i \right\}.$$

We claim that $J$ is a $p$-ideal. Take any $x \in L$ and $y \leqq \bigvee_{i=1}^{n} c_i$ a finite join of commutators in $L$. Then by Lemma 3, $y\phi_x \leqq (\bigvee_{i=1}^{n} c_i)\phi_x = \bigvee_{i=1}^{n} (c_i\phi_x) \leqq \bigvee_{i=1}^{n} (c_i \vee [c_i, x])$, and hence $y\phi_x \in J$.

To show that $L/J$ is Boolean, use the natural ortho-epimorphism $\phi: L \to L/J$, and apply Lemma 4 (ii). A second application of Lemma 4 completes the proof of the theorem.

**3. Solvability in a generalized orthomodular lattice.** At this point it is impossible to mimic the solvability conditions of group theory [4]. The difficulty is that the $p$-ideals in orthomodular lattices need not be orthomodular lattices. In fact, a $p$-ideal $I$ of $L$ contains a greatest element $d$ if and only if $I = L(0, d)$, where $d$ is central in $L$. In order to generalize both orthomodular lattices and $p$-ideals we make the following

DEFINITION. $G$ is a *generalized orthomodular lattice* (GOML) if and only if

(i) $0 \in G$,

(ii) for every nonzero $a \in G$, $G(0, a) = \{x \in G \mid 0 \leqq x \leqq a\}$ is an orthomodular lattice, and

(iii) for $x \leqq a \leqq b$ in $G$, and for $a$-$x$ and $b$-$x$ the orthocomplements of $x$ in $G(0, a)$ and $G(0, b)$ respectively, $a\text{-}x = (b\text{-}x) \wedge a$.

M. F. Janowitz [5] has shown that every GOML $G$ can be embedded as a $p$-ideal in an orthomodular lattice $L$. If $G$ is not already an orthomodular lattice then $G$ is embedded as a prime ideal in $L$, i.e., for $a \in L$ either $a \in G$ or $a' \in G$. Let $G$ be a GOML, and let $G \leqq L$ be the Janowitz embedding. For any $e, f \in L$, since $G$ is prime in $L$, then $[e, f] \in G$. Thus the $p$-ideal generated by the cummutators in $L$ is a subset of $G$. The following theorem clarifies this. For elements $e, f \in G$ we define the *generalized Sasaki projection* by $e\Psi_f = \{e \vee [(e \vee f) - f]\} \wedge f$, the Sasaki projection in $G(0, e \vee f)$. An ideal $I$ of $G$ is called a *p-ideal* of $G$ when $I$ is closed under all generalized Sasaki projections. For elements $e, f \in G$ we say that $e$ *is perspective to $f$ via $t$*, written $e \sim_p f$, if and only if for some

$t \in G$, $e \vee t = f \vee t$ and $e \wedge t = f \wedge t = 0$.

THEOREM 6. *Let $I$ be an ideal of $G$, and let $G \leqq L$ be the Janowitz embedding. These conditions are equivalent.*

(i) *For $e \in I$, $f \in G$ and $e \sim_p f$, then $f \in I$.*
(ii) *$I$ is a $p$-ideal of $G$.*
(iii) *$I$ is a $p$-ideal of $L$.*
(iv) *For $e \in I$, $f \in L$ and $e \sim_p f$, then $f \in I$.*
(v) *$I$ is the kernel of a (unique) congruence on $L$.*
(vi) *$I$ is the kernel of a (unique) congruence on $G$.*

*Proof.* (i) $\Rightarrow$ (ii). Let $e \in I$ and $f \in G$. A computation shows that $e\Psi_f \sim_p f\Psi_e$ via $(e \vee f) - e\Psi_f$. Since $f\Psi_e \leqq e$, then $f\Psi_e \in I$, and by (i) $e\Psi_f \in I$.

(ii) $\Rightarrow$ (iii). Let $e \in I$ and $f \in L$. If $f \in G$, we are finished. Otherwise, $f' \in G$ and it follows that $e \vee f' \in G$ and $e\phi_f = (e \vee f') \wedge f \in G$. By (ii), $e\Psi_{e\phi_f} \in I$. But

$$\begin{aligned} e\Psi_{e\phi_f} &= [e \vee [(e \vee e\phi_f) - e\phi_f]\} \wedge e\phi_f = \{e \vee [(e\phi_f)' \wedge (e \vee e\phi_f)]\} \wedge e\phi_f \\ &= [e \vee (e\phi_f)'] \wedge [e \vee e\phi_f] \wedge e\phi_f \\ &= [e \vee (e' \wedge f) \vee f'] \wedge e\phi_f = e\phi_f \,. \end{aligned}$$

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) are well known [3].

(v) $\Rightarrow$ (vi). The restriction of the congruence on $L$ to $G$ is a congruence. Notice that the congruence preserves relative orthocomplements. The uniqueness stems the fact in any relatively complemented lattice with 0, every ideal is the kernel of at most one congruence [3].

(vi) $\Rightarrow$ (i). Suppose that $\theta$ is a congruence on $G$ with $\ker \theta = I$. Let $e \in I$ and $f \in G$ with $e \sim_p f$ via $t \in G$. The $e\theta 0$ implies $e \vee t\theta t$, or f $\vee\, t\theta t$. It follows that $f = (f \vee t) \wedge f\theta t \wedge f = 0$. Hence $f \in I$.

The Janowitz embedding and Theorem 6 furnish an immediate generalization of Theorem 5.

THEOREM 7. *Let $G$ be a* GOML, *and let $J$ be the commutator $p$-ideal in $G$. Then $G/J$ is distributive. Moreover, if $I$ is a $p$-ideal of $G$ for which $G/I$ is distributive, then $I \supseteqq J$.*

We are now in a position to discuss solvability of GOML. Let $G$ be a GOML, let $G_1$ be the $p$-ideal generated by the commutators in $G$, and for $n > 1$ let $G_n$ be the $p$-ideal generated by the commutators in $G_{n-1}$. A GOML $G$ will be called *solvable* if and only if for some $n$ $G_n = \{0\}$.

LEMMA 8. *Let $J$ be a $p$-ideal in a* GOML $G$, *and let $I$ be a $p$-ideal in $J$. Then $I$ is a $p$-ideal in $G$.*

*Proof*. We shall show for $e \in I$, $f \in G$ that $e\Psi_f \in I$. Since $e \in J$, a $p$-ideal in $G$, then $e\Psi_f \in J$. Therefore $e\Psi_{e\Psi_f} \in I$. A computation shows that $e\Psi_{e\Psi_f} = e\Psi_f$.

THEOREM 9. *Let $G$ be a* GOML. *For $G$ to be solvable it is a necessary and sufficient condition that $G$ be distributive.*

*Proof*. Theorem 7 proves the sufficiency. We shall prove the necessity by showing that $G_2 = G_1$ and hence that $G_n = G_1$ for all positive integers $n$.

Let $G \leqq L$ be the Janowitz embedding, and let $'$ be the orthocomplementation of $L$. For elements $e, f \in G$, set $c = (e' \vee f') \wedge (e' \vee f) \wedge e$ and $d = (f' \vee e') \wedge (f' \vee e) \wedge f$. Then $c \vee d = [e, f]$ by the computation of Lemma 2. Moreover,

$$\begin{aligned} & c \vee d' \\ &= [(e' \vee f') \wedge (e' \vee f) \wedge e] \vee (e \wedge f) \vee (f \wedge e') \vee f' \\ &\qquad (e \wedge f)c(e' \vee f'),\ (e' \vee f),\ e \\ &= [(e' \vee f) \wedge e] \vee (f \wedge e') \vee f' \\ &\qquad (e' \vee f)ce,\ f' \\ &= (e \vee f') \vee (f \wedge e') = 1\ . \end{aligned}$$

Similarly $c' \vee d = 1$. Also $c' \vee d' \geqq (e \wedge f) \vee e' \vee f' = 1$.

We have shown for any $e, f \in G$ and for $c, d$ as above that $[e, f] = [c, d] = c \vee d$. Here $c, d \leqq [c, d]$ imply that $c, d \in G_1$, and thus $[e, f] = [c, d] \in G_2$. This completes the proof that $G_1 = G_2$.

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