PACIFIC JOURNAL OF MATHEMATICS
Vol. 34, No. 1, 1970

THE AMBIENT HOMEOMORPHY OF AN INCOMPLETE SUBSPACE OF INFINITE-DIMENSIONAL HILBERT SPACES

JAMES E. WEST

The pair (H, H_f) is studied from a topological point of view (where H is an infinite-dimensional Hilbert space and H_f is the linear span in H of an orthonormal basis), and a complete characterization is obtained of the images of H_f under homeomorphisms of H onto itself. As the characterization is topological and essentially local in nature, it is applicable in the context of Hilbert manifolds and provides a characterization of (H, H_f)-manifold pairs (M, N) (with M an H-manifold and N an H_f-manifold lying in M so that each coordinate chart f of M may be taken to be a homeomorphism of pairs $(U, U \cap N) \xrightarrow{f} (f(U), f(U) \cap H_f))$.

This implies that in the countably infinite Cartesian product of H with itself, the infinite (weak) direct sum of H_f with itself is homeomorphic to H_f (the two form such a pair), and that if K is a locally finite-dimensional simplicial complex equipped with the barycentric metric (inducing the Euclidean metric on each simplex) and if no vertex-star of K contains more than $\dim(H)$ vertices, then $(K \times H, K \times H_f)$ is an (H, H_f)-manifold pair.

These results are used in [10] to study H_f-manifolds much more intensively to obtain results previously available only for H-manifolds or in the case that H_f is separable, i.e., connected H_f-manifolds are homeomorphic to open subsets of H_f, homotopy-equivalent H_f-manifolds are homeomorphic, and there is an essentially unique completion of an H_f-manifold into an H-manifold, yielding an (H, H_f)-pair.

It should be remarked that this characterization has already been achieved for separable Hilbert spaces by R. D. Anderson [1] and by C. Bessaga and A. Pełczynski [5], and that the observations concerning (H, H_f)-manifold pairs have been made by T. A. Chapman [6, 7] in that case. (Chapman then proceeded to obtain most of the results of [10] in the separable case by methods which seem at the moment to be limited to separability.)

Throughout the discussion, X will denote some complete metric space, and $\mathscr{H}(X)$, the group of all homeomorphisms of X onto itself. The term "isotopy" ("isotopic") will be understood as an abbreviation for "invertible, ambient isotopy", that is, a map $F: X \times [0, 1] \to X$ such that the function $G: X \times [0, 1] \to X \times [0, 1]$ defined from F by setting $G(x, t) = (F(x, t), t)$ is a homeomorphism. (When an embedding

f of a subset of X into X is said to be isotopic to the identity, then, there will exist an extension g of f to an element of $\mathscr{H}(X)$ which is invertibly ambient isotopic to the identity.) If $\mathscr{U}$ is a collection of open sets of X, a map f of a subset Y of X into X will be said to be *limited by* $\mathscr{U}$ if for each point y of Y such that $y \neq f(y)$, there is a member of $\mathscr{U}$ containing both. A homotopy $F: Y \times [0, 1] \to X$ will be said to be *limited by* $\mathscr{U}$ if for each point y of Y such that $F(\{y\} \times [0, 1]) \neq \{y\}$, there is an element of $\mathscr{U}$ containing $F(\{y\} \times [0, 1])$. If $\mathscr{S}$ is a collection of subsets of X then $\mathscr{S}^*$ will denote their union, and $\mathscr{S}$ will be termed *normal* whenever there is an open cover $\mathscr{U}$ of $\mathscr{S}^*$ by mutually disjoint sets with the property that for each U in $\mathscr{U}$, $U \cap \mathscr{S}^* \in \mathscr{S}$. The letter N means the positive integers. Finally, if A is a subset of X and $\mathscr{S}$ is a collection of subsets of X, then $\mathrm{st}(A, \mathscr{S})$ denotes the star of A with respect to $\mathscr{S}$, that is, the union of all members of $\mathscr{S}$ meeting A, and $\mathrm{st}(\mathscr{S}) = \{\mathrm{st}(S, \mathscr{S}) \mid S \in \mathscr{S}\}$. Also,

$$\mathrm{st}^n(A, \mathscr{S}) = \mathrm{st}(\mathrm{st}^{n-1}(A, \mathscr{S}), \mathscr{S}),$$

and $\mathrm{st}^n(S) = \mathrm{st}(\mathrm{st}^{n-1}(\mathscr{S}))$. All refinements used will be understood to be composed of open sets, and $\mathscr{T}$ is a st^n-refinement of $\mathscr{S}$ provided that $\mathrm{st}^n(\mathscr{T})$ refines $\mathscr{S}$.

The first lemma is due to Anderson and Bing [2].

LEMMA 1. *Let $\{f_n\}_{n \in N}$ be a sequence of homeomorphisms of the complete metric space X onto itself, and let $\mathscr{U}$ be any open cover of X. If $\{U_n\}_{n=0}^{\infty}$ is a collection of open covers of X such that $\mathrm{st}^2(\mathscr{U}_0)$ refines $\mathscr{U}$ and for each n in N $\mathscr{U}_n$ is a star-refinement of $\mathscr{U}_{n-1}$ of mesh less than $1/2^n$, then $\{f_n \circ \cdots \circ f_1\}_{n \in N}$ converges (uniformly) to a member of $\mathscr{H}(X)$ which is limited by $\mathscr{U}$ provided that for each n in N f_{n+1} is limited by $\mathscr{U}_n$ and mesh*

$$(f_1^{-1} \circ \cdots \circ f_n^{-1}(\mathscr{U}_n)) < 1/2^n .$$

Proof. Anderson and Bing proved that $\{f_n \circ \cdots \circ f_1\}_{n \in N}$ converges uniformly to a member f of $\mathscr{H}(X)$. To verify that f is limited by $\mathscr{U}$, it is sufficient to observe that for each x in X and n in N, there is a $U(x, n)$ in $\mathscr{U}_n$ containing both $f_n \circ \cdots \circ f_1(x)$ and $f_{n+1} \circ \cdots \circ f_1(x)$, and there is also a $U(x, 0)$ in $\mathscr{U}_0$ containing both x and $f_1(x)$. If $V(x, n)$ is an element of $\mathscr{U}_{n-1}$ containing $\mathrm{st}(U(x, n), \mathscr{U}_n)$ for each x and n, then x and $f_{n+1} \circ \cdots \circ f_1(x)$ lie in

$$\begin{aligned}\bigcup_{m=0}^{n} U(x, m) &\subset \bigcup_{m=0}^{n-1} U(x, m) \cup V(x, n) \\ &\subset \bigcup_{m=0}^{n-2} U(x, m) \cup V(x, n-1) \subset \cdots \subset U(x, 0) \cup V(x, 1) \\ &\subset \mathrm{st}(U(x, 0), \mathscr{U}_0),\end{aligned}$$

so x and $f(x)$ must lie in the closure of $\mathrm{st}(U(x, 0), \mathscr{U}_0)$, which is contained in $\mathrm{st}^2(U(x, 0), \mathscr{U}_0)$, which lies in some member of $\mathscr{U}$.

LEMMA 2. *If $\mathscr{U}$ is a collection of pairwise disjoint open subsets of X, then there is an open cover $\mathscr{V}$ of $\mathscr{U}^*$, refining $\mathscr{U}$, with the property that if for each $U \in \mathscr{U}$ f_U is a homeomorphism of U onto itself which is limited by $\mathscr{V}$, then the function f defined by $f(x) = f_U(x)$, if $x \in U$, and $f(x) = x$, if $x \notin \mathscr{U}^*$, is a homeomorphism of X onto itself.*

Proof. Let $\mathscr{V} = \{V(x) = \{y \in X \mid d(y, x) < d(z, x)/2$ for each z in $X \backslash U\} \mid x \in U \in \mathscr{U}\}$, where $d(\cdot, \cdot)$ is the metric for X. Then for any points z of $X \backslash U^*$, and y of X, $d(z, f(y)) \leqq 3d(z, y)$, which establishes continuity. As f must be one-to-one and onto, and the same argument establishes the continuity of f^{-1}, f is a homeomorphism.

Let $\mathscr{K}$ be an hereditary collection of closed subsets of X which is invariant under the action of $\mathscr{H}(X)$, that is, each closed subset of a member of $\mathscr{K}$ is in $\mathscr{K}$ and $f(K) \in \mathscr{K}$ if $K \in \mathscr{K}$ and $f \in \mathscr{K}(X)$. A set A in X will be termed *$\mathscr{K}$-absorptive* if for each open cover $\mathscr{U}$ of a member K of $\mathscr{K}$ and each member K' of $\mathscr{K}$ contained in $K \cap A$, there is a homeomorphism f in $\mathscr{H}(X)$ which is limited by $\mathscr{U}$, is the identity on K', and carries K into A. If f may always be chosen so that there is an isotopy from it to the identity which is limited by $\mathscr{U}$, then A will be called *strongly $\mathscr{K}$-absorptive.*

LEMMA 3. *If A is $\mathscr{K}$-absorptive (strongly $\mathscr{K}$-absorptive), L is an open subset of a member of $\mathscr{K}$, and U is an open cover of L in X, then there is a member f of $\mathscr{H}(X)$ carrying L into A which is limited by $\mathscr{U}$ (is isotopic to the identity by an isotopy limited by $\mathscr{U}$).*

Proof. As $\mathscr{U}^*$ is an open subset of the complete metric space X, it may be given an equivalent metric under which it is itself complete, so Lemma 1 holds under the new metric. Let $\{V_n\}_{n \in N}$ be a sequence of open sets in X such that each contains its successor and $\bigcap_{n \in N} V_n = X \backslash U^*$, and let $\mathscr{W}$ be a refinement of $\mathscr{U}$ which covers $\mathscr{U}^*$ and has the property that any member of $\mathscr{H}(\mathscr{U}^*)$ which is limited by $\mathscr{W}$ extends to an element of $\mathscr{H}(X)$ which is also limited by $\mathscr{W}$. If $\mathscr{K}'$ is the collection of all members of $\mathscr{K}$ which lie in $\mathscr{U}^*$, then from the definition of (strong) $\mathscr{K}$-absorptivity it is immediate that as a subset of $\mathscr{U}^*$, $A \cap U^*$ is (strongly) $\mathscr{K}'$-absorptive. Using Lemma 1 and the fact that $L \backslash V_{n+1}$ contains $L \backslash V_n$ for all n in N and that both are in $\mathscr{K}'$, select a sequence $\{f_n\}_{n \in N}$ of members of $\mathscr{H}(\mathscr{U}^*)$ with $\{f_n \circ \cdots \circ f_1\}_{n \in N}$ converging to a member of $\mathscr{H}(\mathscr{U}^*)$ which is limited by $\mathscr{W}$ and such that for each n, f_n

carries $f_{n-1} \circ \cdots \circ f_1(L \backslash V_n)$ into $A \cap \mathscr{U}^*$ and is the identity on

$$f_{n-1} \circ \cdots \circ f_1(L \backslash V_{n-1}) .$$

This may be done because each of the functions $f_n \circ \cdots \circ f_1$ may be kept limited by $\mathscr{W}$, which ensures that they permute the elements of $\mathscr{K}'$. Extending the limit homeomorphism to all of X so that it is the identity off $\mathscr{U}^*$ produces the desired member of $\mathscr{H}(X)$. (In the case that an isotopy is desired, and that A is strongly $\mathscr{K}$-absorptive, consider the cover $\mathscr{W}' = \{W \times [0, 1] \mid W \in \mathscr{W}\}$ of $\mathscr{U}^* \times [0, 1]$ and construct a level-preserving homeomorphism of $\mathscr{U}^*$ which is limited by $\mathscr{W}'$, is the identity on $\mathscr{U}^* \times \{0\}$, and carried $L \times \{1\}$ into $A \times \{1\}$. The associated isotopy extends to X.)

A collection $\mathscr{A}$ of members of K will be called a *$\mathscr{K}$-complex* if it may be expressed as a countable union $\bigcup_{n=0}^{\infty} \mathscr{A}_n$ of subsets of itself such that $\mathscr{A}^n = \bigcup_{m=0}^{n} \mathscr{A}_m^*$ is closed for each n and $\mathscr{A}[n] = \{A \backslash \mathscr{A}^{n-1} \mid A \in \mathscr{A}_n\}$ is normal for all n. (Here, $\mathscr{A}^{-1} = \varnothing$.) The set $\mathscr{A}^*$ will be said to *admit the structure of a $\mathscr{K}$-complex.* If $\mathscr{A}^*$ is (strongly) $\mathscr{K}$-absorptive, then it will be referred to as a *(strong) $\mathscr{K}$-absorption base.*

THEOREM 1. *If $\mathscr{U}$ is an open cover of X and A^* and B^* are two (strong) $\mathscr{K}$-absorption bases in X, there is a homeomorphism f of X onto itself (an isotopy F of X), limited by $\mathscr{U}$, such that $f(A^*) = B^*(F(A^* \times \{1\}) = B^*)$.*

Proof. Let $\mathscr{A} = \bigcup_{n=0}^{\infty} \mathscr{A}_n$ and $\mathscr{B} = \bigcup_{n=0}^{\infty} \mathscr{B}_n$ be $\mathscr{K}$-complex structures for A^* and B^* respectively. As the construction of an isotopy in the strong case may be handled from the construction of a homeomorphism in the other case as was done in the previous proof, only the latter construction will be made here. It is quite simple. Since $\mathscr{K}$ is invariant under the action of $\mathscr{H}(X)$, so is the collection of (strong) $\mathscr{K}$-absorption bases. A sequence $f_1, g_1, f_2, g_2, \cdots$ of members of $\mathscr{H}(X)$ is to be chosen with $\{g_n^{-1} \circ f_n \circ \cdots \circ g_1^{-1} \circ f_1\}_{n \in N}$ converging to an element f of $\mathscr{H}(X)$ which is limited by $\mathscr{U}$. Furthermore, $f_n(g_{n-1}^{-1} \circ \cdots \circ f(\mathscr{A}^n))$ is to be a subset of $\mathscr{B}^*$, $g_n(\mathscr{B}^n)$ is to be a subset of $f_n \circ g_{n-1}^{-1} \circ \cdots \circ g_1^{-1} \circ f_1(\mathscr{A}^*)$, f_n is to be the identity on $g_{n-1}^{-1} \circ f_{n-1} \circ \cdots \circ g_1^{-1} \circ f_1(\mathscr{A}^{n-1}) \cup \mathscr{B}^{n-1}$, and g_n is to be the identity on $f_n \circ g_{n-1}^{-1} \circ \cdots \circ g_1^{-1} \circ f_1(\mathscr{A}^n) \cup \mathscr{B}^{n-1}$. Then the limit homeomorphism f is limited by $\mathscr{U}$ and $f(\mathscr{A}^*) = \mathscr{B}^*$. The selection of these homeomorphisms may be made inductively so as to satisfy the convergence criterion of Lemma 1 because for each n, $\mathscr{A}[n]$ and $\mathscr{B}[n]$ are normal and $\mathscr{A}^{n-1}$ and $\mathscr{B}^{n-1}$ are closed, so Lemmas 2 and 3 may be applied and the homeomorphisms constructed piecemeal on collections of pairwise disjoint open sets in X.

THEOREM 2. *If U is an open subset of X, A^* is a (strong) $\mathscr{K}$-absorption base for X, and $\mathscr{K}'$ is the set of all members of $\mathscr{K}$ contained in U, then $A^* \cap U$ is a (strong) $\mathscr{K}'$-absorption base for U.*

Proof. It has already been remarked that $A^* \cap U$ is (strongly) $\mathscr{K}'$-absorptive, so all that is necessary is to demonstrate that it admits the structure of a $\mathscr{K}'$-complex. If $A^* \cap U = \emptyset$, then $\mathscr{K}' = \{\emptyset\}$, and $A^* \cap U$ is a strong $\mathscr{K}'$-absorption base for U. Otherwise, let $\{V_n\}_{n \in N}$ be a collection of open sets with $X \backslash U \subset V_{n+1} \subset \bar{V}_{n+1} \subset V_n$ for each n, and with $\bigcap_{n \in N} V_n = X \backslash U$. Now, let

$$\mathscr{B}_{2n} = \bigcup_{m=0}^{n} \{A \backslash V_{2(n-m+1)} \mid A \in \mathscr{A}_m\}$$

and $\mathscr{B}_{2n+1} = \bigcup_{m=0}^{n+1} \{A \backslash V_{2(n-m+1)+1} \mid A \in \mathscr{A}_m\}$. If $\bigcup_{n=0}^{\infty} \mathscr{B}_n$ is denoted by $\mathscr{B}$, it is apparent that $\mathscr{B}^n$ is closed for each n. To see that $\mathscr{B}[n]$ is normal for each n, let $\{\mathscr{U}_n\}_{n \in N}$ be a collection of sets of mutually disjoint open sets of X with the property that $\mathscr{U}_n{}^*$ contains $\mathscr{A}[n]^*$ and that for each U in $\mathscr{U}_n$, $U \cap \mathscr{A}[n]^* \in \mathscr{A}[n]$. Then define $\mathscr{W}_{2n} = \bigcup_{m=0}^{n} \{U \cap V_{2(n-m)+1} \backslash \bar{V}_{2(n-m+1)+1} \mid U \in \mathscr{U}_m\}$ and

$$\mathscr{W}_{2n+1} = \bigcup_{m=0}^{n+1} \{U \cap V_{2(n-m+1)} \backslash \bar{V}_{2(n-m+2)} \mid U \in \mathscr{U}_m\} ,$$

for each $n = 0, 1, \cdots$ The collections $\mathscr{W}_n$ are composed of pairwise disjoint open sets separating members of $\mathscr{B}[n]$, so $\mathscr{B}$ is a $\mathscr{K}'$-complex. Since $\mathscr{B}^* = \mathscr{A}^* \cap U$, the proof is complete.

If $\{Y_n\}_{n \in N}$ is a collection of spaces, then $\prod_{n \in N} Y_n$ will denote their Cartesian product. If, for each n, $y_n \in Y_n$, then $\prod_{n \in N} (Y_n, y_n)$ will denote that subset of $\prod_{n \in N} Y_n$ composed of those points with n-th coordinate differing from y_n for at most finitely many n. Also, let $\mathscr{C}$ be a class of spaces which is closed under the operations of taking closed subsets and of taking finite products, and for each space Y, let $\mathscr{C}(Y)$ denote the collection of images of members of $\mathscr{C}$ under closed embeddings in Y.

THEOREM 3. *If $\{X_n\}_{n \in N}$ is a sequence of complete metric spaces and if, for each n, $\mathscr{A}(n)$ is a $\mathscr{C}(X_n)$-complex, and x_n is a point of $\mathscr{A}(n)^*$, then $\prod_{n \in N}(\mathscr{A}(n)^*, x_n)$ admits the structure of a $\mathscr{C}(\prod_{n \in N} X_n)$-complex.*

Proof. For each finite subset S of N, let f denote the natural injection of $\prod_{n \in S} X_n$ into $\prod_{n \in N} (X_n, x_n)$. Now, for each ordered n-tuple $(m_1, \cdots, m_n)$ of nonnegative integers, each of which is no greater than n, let $\mathscr{B}(n; m_1, \cdots, m_n) = \{f(\prod_{i=1}^{n} A_i) \mid A_i \in \mathscr{A}(i)_{m_i}\}$. Order the set of all these collections in such a manner that

$$\mathscr{B}(n; m_1, \cdots, m_n) \geqq \mathscr{B}(n'; m'_1, \cdots, m'_{n'})$$

if $n \geqq n'$ or if $n = n'$ and $m_j \geqq m'_j$ for all j. The order selected will be isomorphic to the nonnegative integers, so index the $\mathscr{B}$'s by them in a manner consistent with the above requirements. Let $\mathscr{B} = \bigcup_{n=0}^{\infty} \mathscr{B}_n$. For each n, $\mathscr{B}_n^*$ is closed, so $\mathscr{B}^n$ is, also. Thus, in order to check that $\mathscr{B}$ is a $\mathscr{C}(\prod_{n \in N} X_n)$-complex, it is only necessary to verify that $\mathscr{B}[i]$ is normal for each i. However, for n and $(m_1, \cdots, m_n)$ such that $\mathscr{B}_i = \mathscr{B}(n; m_1, \cdots, m_n)$, and for B in $\mathscr{B}_i$, $B \backslash \mathscr{B}^{i-1} \subset B \backslash f(\prod_{j=1}^{n} \mathscr{A}(j)^{m_j - 1})$, so if for each n in N and each nonnegative integer m, $\mathscr{U}_m^n$ is an open cover of $\mathscr{A}(n)[m]^*$ in X_n by pairwise disjoint open sets U with the property that

$$U \cap \mathscr{A}(n)[m]^* \in \mathscr{A}(n)[m] ,$$

then $\mathscr{V}_i = \{\prod_{j=1}^{n} U_j \times \prod_{j=n+1}^{\infty} X_j \mid U_j \in \mathscr{U}_{m_j}^j \text{ for } j = 1, \cdots, n\}$ is a cover of $\mathscr{B}[i]$ by mutually disjoint open sets of $\prod_{n \in N} X_n$ with the property that the intersection of each with $\mathscr{B}[i]^*$ is a member of $\mathscr{B}[i]$. Thus, each $\mathscr{B}[i]$ is normal and $\mathscr{B}$ is a $\mathscr{C}(\prod_{n \in N} X_n)$-complex. As it is immediate that $\mathscr{B}^* = \prod_{n \in N}(\mathscr{A}(n)^*, x_n)$, the theorem has been proved.

REMARK. It was tacitly assumed above that there were infinitely many X_n's. Of course, the same proof works for a finite collection.

COROLLARY 1. *If, in the above,* $\prod_{n \in N} (\mathscr{A}(n)^*, x_n)$ *is (strongly)* $\mathscr{C}(\prod_{n \in N} X_n)$*-absorptive, then it is a (strong)* $\mathscr{C}(\prod_{n \in N} X_n)$*-absorption base.*

REMARK. It is clear from the definitions that if X and Y are homeomorphic, then any homeomorphism between them carries the $\mathscr{C}(X)$-complexes to the $\mathscr{C}(Y)$-complexes and the (strong) $\mathscr{C}(X)$-absorption bases to the (strong) $\mathscr{C}(Y)$-absorption bases.

From now on, $\mathscr{C}$ will denote the class of all finite-dimensional compact metric spaces. The next lemma is an extension of Proposition 4.5 of [5] to the nonseparable case and to isotopies. It consists of combining Theorem 4.2 of [3] with the Bartle-Graves Theorem.

LEMMA 4. *If X is an infinite-dimensional Fréchet space and K is a compact subset of X, then for each open cover $\mathscr{U}$ of K there is a second, $\mathscr{V}$, such that any embedding of K in X which is limited by $\mathscr{V}$ is (invertibly ambient) isotopic to the identity by an isotopy which is limited by $\mathscr{U}$.*

Proof. For a real number (positive) r and a point x in a metric space, $N(x, r)$ will denote the open ball centered at x with radius r.

Let λ be a Lebesgue number of $\mathscr{U}$ with respect to K, let $\mathscr{V}_1 = \{N(x, \lambda/3^6) \mid x \in K\}$, and, inductively, for $n > 1$, let

$$\mathscr{V}_n = \{N(x, \lambda/3^{n+5}) \mid x \in \mathscr{V}_{n-1}^*\} .$$

Now, let $\mathscr{V} = \bigcup_{n \in N} \mathscr{V}_n$. If f embeds K in X and is limited by $\mathscr{V}$, let Y be the closed linear span in X of the image of $F: K \times [0, 1] \rightarrow X$ defined by $F(x, t) = (1 - t)x + tf(x)$. Let $p_Y: X \rightarrow X/Y$ be the canonical projection, and let $q_Y: X/Y \rightarrow X$ be a right inverse for p_Y sending 0 to 0. (This is by the Bartle-Graves Theorem. For a proof see [11].) Now, the function $h_f: X/Y \times Y \rightarrow X$ defined by $h_f = q_Y p_1 + p_2$ is a homeomorphism, where p_1 and p_2 denote the projections onto the first and second factors, respectively.

From the definition of $\mathscr{V}$, it follows that for each element V of $\mathrm{st}^4(\mathscr{V})$, $V + N(0, \lambda/3)$ is contained in some member of $\mathscr{U}$, where here "+" denotes the set of all sums of pairs of elements, one from the first set and one from the second. Letting W be a neighborhood of the origin in X/Y which q_Y carries into $N(0, \lambda/3)$, one sees that $h_f(W \times (\mathscr{V}^* \cap Y))$ lies in $\mathscr{U}^*$ and, indeed, that $\{h_f(W \times V) \mid V \in \mathrm{st}^4(\mathscr{V} \mid Y)\}$ refines $\mathscr{U}$. (Here, $\mathscr{V} \mid Y = \{V \cap Y \mid V \in \mathscr{V}\}$.)

Select a map $g: X/Y \rightarrow [0, 1]$ such that $g^{-1}(0) \supset (X/Y)\backslash W$ and $0 \in g^{-1}(1)$. Since Y is separable and $\mathscr{V}^* \cap Y$ is open in Y, [3] yields an isotopy $G: (\mathscr{V}^* \cap Y) \times [0, 1] \rightarrow \mathscr{V}^* \cap Y$ from the identity homeomorphism at $t = 0$ to an extension to $\mathscr{V}^* \cap Y$ of f at $t = 1$ which is limited by $\mathrm{st}^4(\mathscr{V} \mid Y)$. Then $H: X \times [0, 1] \rightarrow X$ given by

$$H(x, t) = \begin{cases} h_f(p_Y(x), G(p_2 \circ h_f^{-1}(x), t \cdot g \circ p_Y(x))), & \text{if } x \in h_f(W \times (\mathscr{V}^* \cap Y)) \\ x & \text{, if } x \notin h_f(W \times (\mathscr{V}^* \cap Y)) \end{cases}$$

is the desired isotopy.

Let H be an infinite-dimensional (real) Hilbert space, let E be a complete, orthonormal basis for H, and denote by H_f the collection of all (finite) linear combinations of members of E.

THEOREM 4. *H_f is a strong $\mathscr{C}(H)$-absorption base.*

Proof. Two things must be shown, namely, that H_f admits the structure of a $\mathscr{C}(H)$-complex and that it is strongly $\mathscr{C}(H)$-adsorptive. To see the first, let $\mathscr{A}_0$ be the set of all integral linear combinations of members of E. For $n > 0$, let

$$\mathscr{Q}_n = \left\{Q_n = \left\{\sum_{m=1}^{n} t_m e_m \mid t_m \in [0, 1], m = 1, \cdots, n\right\} \middle| e_1, \cdots, e_n \text{ are } n \text{ distinct elements of } E\right\},$$

and let $\mathscr{A}_n = \{A = Q_n + x \mid Q_n \in \mathscr{Q}_n, x \in \mathscr{A}_0\}$. It is readily seen that $\mathscr{A} = \bigcup_{n=0}^{\infty} \mathscr{A}_n$ is a $\mathscr{C}(H)$-complex with $\mathscr{A}^* = H_f$.

By Lemma 4, in order to demonstrate that H_f is strongly $\mathscr{C}(H)$-absorptive one must only show that for each member K of $\mathscr{C}(H)$, each open cover $\mathscr{U}$ of K, and for each closed subset K' of $K \cap H_f$, there is an embedding f of K in H_f, limited by $\mathscr{U}$, which is the identity on K'. Since K is compact, there exists a Lebesgue number λ for $\mathscr{U}$ with respect to K, so one must only find an embedding f of K in H_f which moves no point as much as λ and is the identity on K'. However, the total boundedness of K and the denseness in H of H_f lead to the existence of a sequence $\{e_i\}_{i \in N}$ in E and a sequence $\{n(i)\}_{i \in N}$ in N such that if p_i is the orthogonal projection of H onto the span of $\{e_j\}_{j=n(i-1)+1}^{n(i)}$, then $\| \sum_{i=1}^{m} p_i(x) - x \| < 2^{-m-2}\lambda$ for each $m \in N$ and $x \in K$. Also, since K is finite-dimensional, for each set S of $2\dim(K) + 2$ distinct elements of E, there is an embedding of K in the unit sphere (=elements of norm one) of the subspace spanned by S. Assume that for each i, $n(i) - n(i-1) \geqq 2\dim(K) + 2$, and let f_i be an embedding of K in the unit sphere of the span of $\{e_j\}_{j=n(i-1)+1}^{n(i)}$. Now, let g map K into $[0, 1]$ such that $K' = g^{-1}(0)$, and for each i let h_i map $[0, 1]$ into $[0, 1]$ such that $h_1^{-1}(0) = [0, 1/n(3)]$ and $h_1^{-1}(1) = [1/n(2), 1]$ and for $i > 1$,

$$h_i^{-1}(0) = [1/n(i-1), 1] \cup [0, 1/n(i+2)]$$

and $h_i^{-1}(1) = [1/n(i+1), 1/n(i)]$. Finally, set

$$f(x) = \sum_{i \in N} (\max_{j \geq i} \{h_j \circ g(x)\}) p_i(x) + \sum_{i \in N} 2^{-i-1}\lambda \cdot h_i \circ g(x) f_{i+3}(x) .$$

This is the desired embedding.

COROLLARY 2. *If $\mathscr{U}$ is any collection of open sets of H and Y is any $\mathscr{C}(\mathscr{U}^*)$-absorption base in $\mathscr{U}^*$, then there is an ambient, invertible isotopy of H onto itself which is limited by $\mathscr{U}$, is the identity at $t = 0$, and at $t = 1$ is a homeomorphism h_1 such that $h_1(Y) = \mathscr{U}^* \cap H_f$.*

Proof. Lemma 4 shows the equivalence of the concepts of $\mathscr{C}(\mathscr{U}^*)$-absorption base and strong $\mathscr{C}(\mathscr{U}^*)$-absorption base, Theorem 4 combined with Theorem 2 gives that $\mathscr{U}^* \cap H_f$ is also a strong $\mathscr{C}(\mathscr{U}^*)$-absorption base, and Theorem 1 supplies the isotopy on $\mathscr{U}^*$ limited by an open cover given by Lemma 2 which refines $\mathscr{U}$ and has the property that any isotopy limited by it may be extended trivially to one on H.

COROLLARY 3. *Let $\{H_n\}_{n\in N}$ be an indexed, countably infinite collection of copies of H, and let Y be the subspace of $\prod_{n\in N} H_n$ consisting of all points with at most finitely many nonzero coordinates, each of which lies in the appropriate copy of H_f. Then Y is homeomorphic to H_f.*

Proof. It is easy to modify the proof of Theorem 4 to show that Y is $\mathscr{C}(\prod_{n\in N} H_n)$-absorptive. If the copy of H_f in H_n is denoted by $(H_f)_n$, then $Y = \prod_{n\in N}((H_f)_n, 0)$, so Corollary 1 applies to show that Y is a $\mathscr{C}(\prod_{n\in N} H_n)$-absorption base. However, $\prod_{n\in N} H_n$ is homeomorphic to H by a theorem of Bessaga and Pełczynski [4], so by the remark following Corollary 1, Y may be embedded in H as a $\mathscr{C}(H)$-absorption base. Corollary 2 now applies to finish the proof.

The above result is crucial to [10]. The next two results identify some simplicial complexes whose products with H_f are H_f-manifolds.

THEOREM 5. *If K is a metric simplicial complex and $K \times H$ is an H-manifold, then $K \times H_f$ is an H_f-manifold.*

Proof. By Theorem 3 (the remark after Theorem 3), $K \times H_f$ is a $\mathscr{C}(K \times H)$-complex, since K is by definition a $\mathscr{C}(K)$-complex. The strategy of the proof is to show that $K \times H_f$ is a $\mathscr{C}(K \times H)$-absorption base, to embed $K \times H$ component-wise in H as open subsets (using a theorem of Henderson [8]) and then to use Corollary 2 to find a homeomorphism of the open subsets in question onto themselves throwing the images of $K \times H_f$ onto $H_f \cap$ (the open subsets). Thus, all that is necessary is to establish the $\mathscr{C}(K \times H)$-absorptivity of $K \times H_f$. In fact, since for each vertex v of K, $\mathrm{st}^0(v, K)$— the open star of v in K— is a contractible open set, $\mathrm{st}^0(v, K) \times H$ will be homeomorphic to H by [9], so all that is needed is to show that $\mathrm{st}^0(v, K) \times H_f$ is $\mathscr{C}(\mathrm{st}^0(v, K) \times H)$-absorptive. Therefore, let X be a finite-dimensional compactum of $\mathrm{st}^0(v, K) \times H$, let $\mathscr{U}$ be an open cover of X in $\mathrm{st}^0(v, K) \times H$ and let X' be a closed subset of $X \cap (\mathrm{st}^0(v, K) \times H_f)$. Lemma 4 together with the fact that $\mathrm{st}^0(v, K) \times H$ is homeomorphic to H establishes that it is sufficient to find an embedding of X in $\mathrm{st}^0(v, K) \times H_f$ which is limited by $\mathscr{U}$, and is the identity on X'. Let λ be a Lebesgue number for $\mathscr{U}$ with respect to X, and let p_H denote the projection of $K \times H$ onto H. As noted in the proof of Theorem 4, there exists a sequence $\{e_i\}_{i\in N}$ in E and another sequence $\{n(i)\}_{i\in N}$ in N such that $n(i) - n(i-1) \geqq 2\dim(X) + 2$ for each i and $\|\sum_{i=1}^{m} p_i \circ p_H(x) - p_H(x)\| < 2^{-m-2}\lambda$ for each $m \in N$ and $x \in X$, the rest of the notation being as in the proof of Theorem 4. Constructing $f_0: X \to H_f$ by the same method as used in Theorem 4, except for the substitution of $p_i \circ p_H$ for p_i, and setting $f = (p_K, f_0)$ produces the desired embedding, if p_K denotes the projection of $K \times H$ onto K.

COROLLARY 4. *If K is a metric, locally finite-dimensional, simplicial complex such that no vertex-star contains more vertices than* $\dim(H)$, *then $K \times H_f$ is an H_f-manifold.*

Proof. By Theorem 4 of [12], $K \times H$ is an H-manifold, so Theorem 5 applies. (This metric is assumed that in the abstract.)

Actually, if a pair (X, Y) of spaces, $Y \subset X$, is called a (H, H_f)-*manifold pair* provided that X is a paracompact H-manifold and there is an open cover $\mathscr{U}$ of X by sets U for which there are open embeddings $f_U: U \to H$ such that $f_U(U \cup Y) = f_U(U) \cap H_f$, then the following have been established.

THEOREM 6. *The pair (X, Y) is a (H, H_f)-manifold pair if and only if Y is a $\mathscr{C}(X)$-complex, X is an H-manifold, and the following weak $\mathscr{C}(X)$-absorptivity condition is satisfied: For each finite-dimensional compactum C of X, each open cover $\mathscr{U}$ of C, and each compact subset C' of $C \cap Y$, there is an embedding of C in Y which is limited by $\mathscr{U}$ and extends the inclusion of C'. If (X, Z) is another (H, H_f)-manifold pair and $\mathscr{V}$ is an open cover of X, then there is an isotopy of X, limited by $\mathscr{V}$, from the identity to a pair homeomorphism of (X, Y) onto (X, Z).*

COROLLARY 5. *If (X, Y) and (X', Y') are (H, H_f)-manifold pairs, then $(X \times X', Y \times Y')$ is an (H, H_f)-manifold pair.*

COROLLARY 6. *If (X, Y) is an (H, H_f)-manifold pair and K is a metric, locally finite-dimensional, simplicial complex such that no vertex-star contains more than* $\dim(H)$ *vertices, then $(X \times K, Y \times K)$ is an (H, H_f)-manifold pair.*

REFERENCES

1. R. D. Anderson, *On sigma-compact subsets of infinite-dimensional spaces* (to appear, Trans. Amer. Math. Soc.)

2. R. D. Anderson and R. H. Bing, *A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **74** (1968), 771–792.

3. R. D. Anderson and J. D. McCharen, *On extending homeomorphisms to Fréchet manifolds* (to appear, Proc. Amer. Math. Soc.)

4. Cz. Bessaga and A. Pełczynski, *Some remarks on homeomorphisms of F-spaces*, Bull. Polon. Acad. Sci. **10** (1962), 265–270.

5. Cz. Bessaga and A. Pełczynski, *Estimated extension theorem, homogeneous collec-*

tions and skeletons, and their applications to topological classification of linear metric spaces and convex sets (to appear, Fund. Math.)

6. T. A. Chapman, *Four classes of separable, metric, infinite-dimensional manifolds*, Bull. Amer. Math. Soc. **76** (1970), 399-403.

7. ———, *Dense sigma compact subsets of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. (to appear).

8. D. W. Henderson, *Stable classification of infinite-dimensional manifolds by homotopy type* (to appear. Ann. of Math.)

9. D. W. Henderson and R. M. Schori, *Topological classification of infinite-dimensional manifolds by homotopy type*, Bull. Amer. Math. Soc. **76** (1970), 121-124.

10. D. W. Henderson and J. E. West *Triangulated infinite-dimensional manifolds*, Bull. Amer. Math. Soc. **76** (1970), 655-660.

11. E. A. Michael, *Convex structures and continuous selections*, Canad. J. Math. **11** (1959), 556-576.

12. J. E. West, *Products of complexes and Fréchet spaces which are manifolds.*

Received December 10, 1969. The author was entirely supported by NSF Grant GP-9397 while this research was done.

CORNELL UNIVERSITY
ITHACA, NEW YORK