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ASCOLI'S THEOREM FOR SPACES OF MULTIFUNCTIONS

Y.-F. LIN and DAVID A. ROSE

The purpose of this paper is to prove that if X and Y are two arbitrary topological spaces and if $M(X, Y; c)$ denotes the space of all multi-valued functions on X to Y with the compact-open topology, then a closed set $\mathscr{F} \subset M(X, Y; c)$ is compact if at each point $x \in X, \mathscr{F}(x) = \cup \{F(x) \mid F \in \mathscr{F}\}$ has a compact closure in Y, and $\mathscr{F}$ is evenly continuous.

The classical theorem of Ascoli [1] asserts that a uniformly bounded, equicontinuous family of functions has a compact closure in the space of continuous functions with the topology of uniform convergence. This theorem has been the center of many papers, notably the works of Gale [2], Myers [5], Weston [6], and Morse-Kelley [3, pp. 233-237]. The purpose of this paper is to establish a similar version of the Ascoli theorem in the space of multi-valued functions.

Let X and Y be two nonvoid topological spaces. Then $F \subset X \times Y$ is said to be a *multifunction* on X to Y, denoted by $F: X \rightarrow Y$, if and only if for each $x \in X$,

1.1. $$\pi_2((\{x\} \times Y) \cap F) \neq \square ,$$

where π_2 is the second projection of $X \times Y$ onto Y, and $\square$ denotes the empty set. We shall write $F(x)$ for the set defined in 1.1. Thus, loosely speaking, a multifunction is a point-to-set correspondence. If A and B are subsets of X and Y respectively, then we denote $F(A) = \cup \{F(x) \mid x \in A\}$ and $F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \square\}$.

DEFINITION 1.2. A multifunction $F: X \rightarrow Y$ is *continuous* if and only if for each open set V in Y, the set $F^{-1}(V)$ is open and the set $F^{-1}(Y - V)$ is closed in X.

Let Y^X be the set of all (single-valued) functions on X to Y, and let $M(X, Y)$ be the set of all multifunctions on X to Y. Note that Y^X is a subset of $M(X, Y)$. For our later convenience in expressions, let us agree that for any $A \subset X$ and $B \subset Y$:

$$(A, B) = \{F \in M(X, Y) \mid F(A) \subset B\} ,$$
$$)A, B(= \{F \in M(X, Y) \mid A \subset F^{-1}(B)\} .$$

Recall that the compact-open topology [3, p. 221] for Y^X has as a subbase the totality of sets $(K, U) \cap Y^X$ where K is a compact subset of X and U is an open subset of Y. Although the set $(K, U) \cap Y^X$ is always the same as $)K, U(\cap Y^X$, the sets (K, U) and $)K, U($ are in general distinct in $M(X, Y)$. It then seems very natural to make the following definition.

DEFINITION 1.3. The *compact-open topology* for $M(X, Y)$ is the topology defined by taking the totality of (K, U) and $)L, V($ as subbasic open sets, where K and L are any compact subsets of X and U and V are any open sets in Y. The *point-open topology* for $M(X, Y)$ is the topology which has as a subbase the totality of the sets (p, U) and $)p', U'($, where p and p' are points of X and U and U' are open in Y.

We recall that a family $\mathscr{F} \subset Y^X$ is said to be *evenly continuous* [3, p. 235] if and only if for each x in X, each y in Y, and each open neighborhood V of y there exist an open neighborhood U of x and an open neighborhood W of y such that if $f \in \mathscr{F}$ and $f(x) \in W$ then $f(U) \subset V$. The concept of even continuity has played an important role in [3, pp. 235–237] and [6]. We generalize this to any subset of $M(X, Y)$:

DEFINITION 1.4. A family $\mathscr{F} \subset M(X, Y)$ is *evenly continuous* if and only if for each x in X, each y in Y, and each open neighborhood V of y there exist an open neighborhood U of x and an open neighborhood W of y such that:

(i) if $F \in \mathscr{F}$ and $F(x) \cap W \neq \square$ then $U \subset F^{-1}(V)$, and

(ii) if, in addition to (i), $F(x) \subset V$ then $F(U) \subset V$.

Using the notations we have introduced, the conditions (i) and (ii) above may be expressed equivalently as (i') and (ii'), respectively:

(i') $\mathscr{F} \cap)x, W(\subset)U, V($, and

(ii') $\mathscr{F} \cap)x, W(\cap (x, V) \subset (U, V)$.

It follows that if $\mathscr{F} \subset M(X, Y)$ consists entirely of single-valued functions, then $\mathscr{F}$ satisfies Definition (1.4) if and only if $\mathscr{F}$ is evenly continuous in the sense of Kelley [3, p. 235].

2. Properties of evenly continuous families of multifunctions.

PROPOSITION 2.1. *Every finite set of continuous multifunctions on X to Y is evenly continuous.*

Proof. If x is any point in X and $\mathscr{F} = \{F_1, \cdots, F_k\}$ is a finite collection of continuous multifunctions on X to Y, and if y is any element of Y, let V be any open neighborhood of y. Let $U_1 = \bigcap_{i=1}^{n} F_{k_i}^{-1}(V)$, where $\{F_{k_i} | i = 1, \cdots, n\}$ is the set of all F in $\mathscr{F}$ such that $F(x) \cap V \neq \square$. (If $\{F_{k_i} \mid i = 1, \cdots, n\}$ is empty, let $U_1 = X$.) Let

$$U_2 = X - \bigcup_{j=n+1}^{n+m} F_{k_j}^{-1}(Y - V)$$

where $\{F_{k_j} \mid j = n + 1, \cdots, n + m\}$ is the set of all F in $\mathscr{F}$ such that $x \notin F^{-1}(Y - V)$. (If $\{F_{k_j} \mid j = n + 1, \cdots, n + m\}$ is empty, let $U_2 = X$.) Since $\mathscr{F}$ contains only continuous multifunctions and since the union of a finite number of closed sets is closed and the intersection of a finite number of open sets is open, U_1 and U_2 are both open sets in X containing x. Let $U = U_1 \cap U_2$ and let $W = V$. Then U and W are open sets containing x and y respectively and, $F \in \mathscr{F} \cap)x, W(\Rightarrow F \in \mathscr{F}$ and $F(x) \cap W \neq \square$

$\Rightarrow F \in \mathscr{F}$ and $F(x) \cap V \neq \square$

$\Rightarrow F = F_{k_i}$, for some $i = 1, \cdots, n$

$\Rightarrow F^{-1}(V) \supset \bigcap_{i=1}^{n} F_{k_i}^{-1}(V) = U_1 \supset U$

$\Rightarrow F \in)U, V($.

And, $F \in \mathscr{F} \cap)x, W(\cap (x, V) \Rightarrow F \in \mathscr{F} \cap (x, V)$

$\Rightarrow F \in \mathscr{F}$ and $F(x) \subset V$

$\Rightarrow F \in \mathscr{F}$ and $x \notin F^{-1}(Y - V)$

$\Rightarrow F = F_{k_j}$, for some $j = n + 1, \cdots, n + m$.

Now if $z \in X - \bigcup_{j=n+1}^{n+m} F_{k_j}^{-1}(Y - V)$ then $z \in X - F^{-1}(Y - V)$ and hence $F(z) \cap (Y - V) = \square$. Thus, $F(z) \subset V$ and hence $F \in (U_2, V)$. So $F \in (U, V)$, since $U \subset U_2$. This shows that $\mathscr{F} \cap)x, W(\subset)U, V($ and $\mathscr{F} \cap)x, W(\cap (x, V) \subset (U, V)$. Thus, $\mathscr{F}$ is evenly continuous.

PROPOSITION 2.2. *If X has the discrete topology then $M(X, Y)$ is evenly continuous.*

Proof. Let x be any point of X, y be any point in Y and V be any open neighborhood of y. Let $U = \{x\}$ and $W = V$ be open neighborhoods of x and y respectively. Then

$$F \in M(X, Y) \cap)x, W(\Rightarrow F \in)x, V(\Rightarrow F \in)U, V(.$$

And

$$F \in M(X, Y) \cap)x, W(\cap (x, V) \Rightarrow F \in (x, V) \Rightarrow F \in (U, V) .$$

PROPOSITION 2.3. *If $\mathscr{F} \subset M(X, Y)$ is evenly continuous and F is any member of $\mathscr{F}$ then F is a continuons multifunction on X to Y.*

Proof. Let V be any open set in Y and let F be any member of $\mathscr{F}$ where $\mathscr{F}$ is an evenly continuous subset of $M(X, Y)$. If $F^{-1}(V) = \square$, then $F^{-1}(V)$ is open in X. If $F^{-1}(V) \neq \square$, then for all x in $F^{-1}(V)$, $F(x) \cap V \neq \square$, and for any y_x in $F(x) \cap V$, by the even continuity of $\mathscr{F}$ there exist open neighborhoods U_x and W_x of x and y_x respectively such that $F \in)U_x, V($ since $F \in \mathscr{F} \cap)x, W_x($. To see that $F \in)x, W_x($, note that $y_x \in F(x) \cap W_x$ and hence $F(x) \cap W_x \neq \square$. Thus, $U_x \subset F^{-1}(V)$ and hence $F^{-1}(V)$ is open. Now let $C = Y - V$. If $F^{-1}(C) = X$ then $F^{-1}(C)$ is closed in X. If $F^{-1}(C) \neq X$, then for all x in $X - F^{-1}(C)$, $F(x) \cap C = \square$. Thus, if x is in $X - F^{-1}(C)$, then $F(x) \subset Y - C = V$ which is open in Y. By the even continuity of $\mathscr{F}$, for any y_x in $F(x)$ there exist neighborhoods U_x and W_x of x and y_x respectively such that $F \in (U_x, V)$ since $F \in \mathscr{F} \cap)x, W_x(\cap (x, V)$.

$$F \in (U_x, V) \Rightarrow F(U_x) \subset V \Rightarrow F(U_x) \cap C = \square \Rightarrow U_x \cap F^{-1}(C) \\ = \square \Rightarrow U_x \subset X - F^{-1}(C) \,.$$

Thus $X - F^{-}(C)$ is open and $F^{-1}(C)$ is closed. This completes the proof.

Propositions (2.1) and (2.3) together say that a finite subset $\mathscr{F}$ of $M(X, Y)$ is evenly continuous if and only if $\mathscr{F}$ consists entirely of continuous multifunctions.

In the sequel, we denote by $M(X, Y; c)$ the space $M(X, Y)$ equipped with the compact-open topology [see Definition (1.3)], and by $M(X, Y; p)$ the space $M(X, Y)$ equipped with the point-open topology.

3. Ascoli's theorem for the space $M(X, Y; c)$ of multifunctions.

From Proposition (2.2), if X is a discrete space then $M(X, Y)$ is evenly continuous; and here the point-open topology and the compact-open topology coincide on $M(X, Y)$. The following lemma generalizes this occurrence.

LEMMA 3.1. *Let $\mathscr{F}$ be an evenly continuous subset of $M(X, Y)$. Then $\mathscr{F}$ has the same closure with respect to both the compact-open topology and the point-open topology; and these topologies induce the same topology in $\mathscr{F}$.*

Proof. Let $\mathscr{G}$ be any subset of $\mathscr{F}$. Let $\mathscr{G}^{*c}$ and $\mathscr{G}^{*p}$ be the closures of $\mathscr{G}$ in $M(X, Y; c)$ and $M(X, Y; p)$ respectively. Since the point-open topology in $M(X, Y)$ is always contained in (=coarser than) the compact-open topology for $M(X, Y)$, $\mathscr{G}^{*c} \subset \mathscr{G}^{*p}$. To show that $\mathscr{G}^{*c} = \mathscr{G}^{*p}$, it remains to show that $\mathscr{G}^{*p} \subset \mathscr{G}^{*c}$. To this end, let F_0 be any multifunction in $\mathscr{G}^{*p}$ and let $K_1, K_2, \cdots, K_n, L_{n+1}, \cdots, L_{n+m}$ be compact subsets of X and $U_1, \cdots, U_n; V_{n+1}, \cdots, V_{n+m}$ be open sets in Y such that for all $i = 1, \cdots, n$, $F_0(K_i) \subset U_i$ and for all

$j = n+1, \cdots, n+m$, $L_j \subset F_0^{-1}(V_j)$.
Thus,

$$F_0 \in (K_1, U_1) \cap \cdots \cap (K_n, U_n) \cap)L_{n+1}, V_{n+1}(\cap \cdots \cap)L_{n+m}, V_{n+m}(\ .$$

If $1 \leqq i \leqq n$, then since $\mathscr{G}$ is evenly continuous ($\mathscr{G} \subset \mathscr{F}$ and $\mathscr{F}$ is evenly continuous), for any $x \in K_i$ and for any y_x in $F_0(x) \subset U_i$, there exist open neighborhoods U_x and W_x of x and y_x respectively such that $\mathscr{G} \cap)x, W_x(\cap (x, U_i) \subset (U_x, U_i)$. Since K_i is compact, K_i is covered by a finite number of the open sets U_x. Let $\{U_{x_{(k,i)}} \mid k = 1, \cdots, k(i)\}$ be such a finite open cover of K_i for each $i = 1, \cdots, n$; and for each $i = 1, \cdots, n$ let $\{W_{x_{(k,i)}} \mid k = 1, \cdots, k(i)\}$ be the set of corresponding open neighborhoods of points in U_i existing by the even continuity of $\mathscr{G}$ such that for each $i = 1, \cdots, n$ and for all $k = 1, \cdots, k(i)$,

$$\mathscr{G} \cap)x_{(k,i)}, W_{x_{(k,i)}}(\cap (x_{(k,i)}, U_i) \subset (U_{x_{(k,i)}}, U_i) \ .$$

If $n+1 \leqq j \leqq n+m$ and $x \in L_j$, then again by the even continuity of $\mathscr{G}$, for any y_x in $F_0(x) \cap V_j$, there exist open neighborhoods U_x and W_x of x and y_x respectively such that $\mathscr{G} \cap)x, W_x(\subset)U_x, V_j($. Since L_j is compact, L_j is covered by a finite number of the open sets U_x. Let $\{U_{x_{(q,j)}} \mid q = 1, \cdots, q(j)\}$ be such a finite open covering of L_j for each $j = n+1, \cdots, n+m$; and for each $j = n+1, \cdots, n+m$, let $\{W_{x_{(q,j)}} \mid q = 1, \cdots, q(j)\}$ be the corresponding open neighborhoods of points in $F_0(x_{(q,j)}) \cap V_j$ existing by the even continuity of $\mathscr{G}$ such that $\mathscr{G} \cap)x_{(q,j)}, W_{x_{(q,j)}}(\subset)U_{x_{(q,j)}}, V_j($. Thus

$$N_1 = \bigcap_{(k,i)=(1,1)}^{(k(i),n)} [)x_{(k,i)}, W_{x_{(k,i)}}, (\cap (x_{(k,i)} U_i)]$$

and

$$N_2 = \bigcap_{(q,j)=(1,n+1)}^{(q(j),n+m)})x_{(q,j)}, W_{x_{(q,j)}}($$

are both basic open neighborhoods of F_0 in $M(X, Y; p)$. And hence $N_1 \cap N_2$ is an open neighborhood of F_0 in $M(X, Y; p)$. $F_0 \in \mathscr{G}^{*p}$ implies that there exists a multifunction F in $N_1 \cap N_2 \cap \mathscr{G}$. Since $\mathscr{G}$ is evenly continuous, $F \in \mathscr{G} \cap N_1$ implies that for each $i = 1, \cdots, n$ and for each $k = 1, \cdots, k(i)$, $F \in (U_{x_{(k,i)}}, U_i)$; and $F \in \mathscr{G} \cap N_2$ implies that for each $j = n+1, \cdots, n+m$ and for each $q = 1, \cdots, q(j)$, $F \in)U_{x_{(q,j)}}, V_j($. Since for each $i = 1, \cdots, n$, $\{U_{x_{(k,i)}} \mid k = 1, \cdots, k(i)\}$ is a cover of K_i, and for each $j = n+1, \cdots, n+m$, $\{U_{x_{(q,j)}} \mid q = 1, \cdots, q(j)\}$ is a cover of L_j, we have $F \in (K_i, U_i)$ for each $i = 1, \cdots, n$ and $F \in)L_j, V_j($ for each $j = n+1, \cdots, n+m$. Thus,

$$F \in (K_1, U_1) \cap \cdots \cap (K_n, U_n) \cap)L_{n+1}, V_{n+1}(\cap \cdots \cap)L_{n+m}, V_{n+m}(\ .$$

Since $(K_1, U_1) \cap \cdots \cap (K_n, U_n) \cap)L_{n+1}, V_{n+1}(\cap \cdots \cap)L_{n+m}, V_{n+m}($ is an arbitrary basic open neighborhood of F_0 in $M(X, Y; c)$ and this neighborhood always contains a member F of $\mathscr{G}$, F_0 belongs to $\mathscr{G}^{*c}$. Since F_0 was an arbitrary element of $\mathscr{G}^{*p}$, we have shown $\mathscr{G}^{*p} \subset \mathscr{G}^{*c}$, and hence $\mathscr{G}^{*c} = \mathscr{G}^{*p}$. Since $\mathscr{G}$ was an arbitrary subset of $\mathscr{F}$, in particular $\mathscr{F}^{*c} = \mathscr{F}^{*p}$ and the subtopology for $\mathscr{F}$ induced by the compact-open topology for $M(X, Y)$ coincides with the subtopology for $\mathscr{F}$ induced by the point-open topology for $M(X, Y)$. This completes the proof of the lemma.

In [4], the *cartesian M-product* $P_M\{X_\lambda \mid \lambda \in \Lambda\}$ of the family $\{X_\lambda \mid \lambda \in \Lambda\}$ of spaces is defined to be the set of all multifunctions $F: \Lambda \to \cup\{X_\lambda \mid \lambda \in \Lambda\}$ such that $F(\lambda) \subset X_\lambda$ for each $\lambda \in \Lambda$; and the *M-product topology* for $P_M\{X_\lambda \mid \lambda \in \Lambda\}$ is the topology having as a subbase the totality of subsets (λ, U_λ) and $)\mu, V_\mu($, where U_λ and V_μ are any arbitrary open sets in X_λ and X_μ respectively, and λ and μ are any members of Λ. It is proved in [4, p. 400] that if each X_λ is compact then $P_M\{X_\lambda \mid \lambda \in \Lambda\}$ with the M-product topology is compact.

We now prove the main theorem.

THEOREM 3.2. *Let X and Y be arbitrary topological spaces and let $M(X, Y; c)$ be the space of all multifunctions on X to Y with the compact-open topology. Then a closed set $\mathscr{F}$ in $M(X, Y; c)$ is compact if, at each point x in X, $\mathscr{F}(x) = \cup\{F(x) \mid F \in \mathscr{F}\}$ has a compact closure in Y, and $\mathscr{F}$ is evenly continuous.*

Proof. Suppose that $\mathscr{F}$ is a closed evenly continuous subset of $M(X, Y; c)$ such that for each x in X, $\mathscr{F}(x)$ has a compact closure in Y. Let $\mathscr{F}(x)^*$ denote the closure of $\mathscr{F}(x)$ for all x in X, and let $Y_x = Y$ for all x in X. Since $\mathscr{F}(x)^* \subset Y = Y_x$ for all x in X, $P_M\{\mathscr{F}(x)^* \mid x \in X\} \subset P_M\{Y_x \mid x \in X\} = M(X, Y; p)$. Since $\mathscr{F}(x)^*$ is compact for each x in X, by Tychonoff's theorem for the space of multifunctions [4], $P_M\{\mathscr{F}(x)^* \mid x \in X\}$ is compact with the M-product topology. The M-product topology for $P_M\{\mathscr{F}(x)^* \mid x \in X\}$ coincides with the M-product topology for $P_M\{Y_x \mid x \in X\}$ relativized on the subset $P_M\{\mathscr{F}(x)^* \mid x \in X\}$, and the M-product space $P_M\{Y_x \mid x \in X\}$ is the same as the space $M(X, Y; p)$. Consequently, the M-product space $P_M\{\mathscr{F}(x)^* \mid x \in X\}$ is a compact subspace of $M(X, Y; p)$. Since for all $F \in \mathscr{F}$, $F(x) \subset \mathscr{F}(x)^*$ for every $x \in X$, we have

$$\mathscr{F} \subset P_M\{\mathscr{F}(x)^* \mid x \in X\} \subset M(X, Y) .$$

It then follows from Lemma (3.1) that $\mathscr{F}$ is closed in $M(X, Y; p)$, and consequently it is closed in the compact space $P_M\{\mathscr{F}(x)^* \mid x \in X\}$. Thus, $\mathscr{F}$ is compact in $P_M\{\mathscr{F}(x)^* \mid x \in X\}$ and hence it is compact in $M(X, Y; p)$. By Lemma (3.1), the compact-open subtopology and the point-open subtopology for $\mathscr{F}$ coincide. Therefore, $\mathscr{F}$ is compact in the space $M(X, Y; c)$. This completes the proof of the theorem.

4. A remark. In the proof of Ascoli's theorem for multifunctions we make use of the Tychonoff theorem for multifunctions. It is interesting to observe that the Tychonoff theorem for multifunctions can be deduced directly from the Ascoli theorem for multifunctions, and hence these two theorems are, in a sense, equivalent. We now indicate a proof of this fact:

Let $\{Y_x \mid x \in X\}$ be a family of compact topological spaces, and let $Y^x = Y_x \times \{x\}$ for each $x \in X$. Then, $\{Y^x \mid x \in X\}$ is a family of disjoint spaces such that each Y^x is homeomorphic with Y_x. It follows that the two M-product spaces $P_M\{Y_x \mid x \in X\}$ and $P_M\{Y^x \mid x \in X\}$ are homeomorphic. Now, let X have the discrete topology and let $Y = \cup\{Y^x \mid x \in X\}$ be the free union of the spaces $Y^x, x \in X$. That is, a set $U \subset Y$ is *open* in Y if and only if $U \cap Y^x$ is open in Y^x for each x in X. Let $\mathscr{F} = P_M\{Y^x \mid x \in X\} \subset M(X, Y)$. It can be shown that $\mathscr{F}$ with M-product topology is closed in $M(X, Y; c) = M(X, Y; p)$, because X is discrete. Also, we have each $\mathscr{F}(x) = Y^x$ closed in Y, and hence each $\mathscr{F}(x)^* = Y^x$ which is compact. By Proposition (2.2), $M(X, Y)$ is evenly continuous and hence so is $\mathscr{F}$. Now Theorem (3.2) applies here to conclude that $\mathscr{F}$ is compact, and hence so is $P_M\{Y_x \mid x \in X\}$.

BIBLIOGRAPHY

1. G. Ascoli, *Le curve limite di una varieta datà di curve*, Mem. Accad. Lincei (3) **18** (1883), 521–586.

2. D. Gale, *Compact sets of functions and function rings*, Proc. Amer. Math. Soc. **1** (1950), 303–308.

3. J. L. Kelley, *General topology*, Van Nostrand, Princeton, 1955.

4. Y.-F. Lin, *Tychonoff's theorem for the space of multifunctions*, Amer. Math. Monthly **74** (1967), 399–400.

5. S. B. Myers, *Equicontinuous sets of mappings*, Ann. of Math. (2) **47** (1946), 496–502.

6. J. D. Weston, *A generalization of Ascoli's theorem*, Mathematika **6** (1959), 19–24.

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THE UNIVERSITY OF SOUTH FLORIDA
TAMPA, FLORIDA