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A v-INTEGRAL REPRESENTATION FOR LINEAR OPERATORS ON SPACES OF CONTINUOUS FUNCTIONS WITH VALUES IN TOPOLOGICAL VECTOR SPACES

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Suppose X and Y are topological vector spaces. This paper gives an analytic representation of continuous linear operators from C into Y, where C denotes the space of continuous functions from the interval $[0, 1]$ into X with the topology of uniform convergence. In order to obtain an integral representation theorem analogous to the ones given by R. K. Goodrich for the locally convex setting in Trans. Amer. Math. Soc. 131 (1968), 246–258, certain strong hypotheses on C must be assumed because of the need to be able to extend the operators to a subset of the double dual of C. However, by using the notion of v-integral, it is possible to avoid this problem and give a representation theorem without additional hypothesis.

Let $\mathscr{I}$ be the collection of intervals in $(0, 1]$ of the form $(a, b]$ and let $L[X, Y]$ denote the space of linear operators from X into Y. Then the set function K from $\mathscr{I}$ into $L[X, Y]$ is said to be convex with respect to length if $K(I) = \sum_{i=1}^{n} [l(I_i)/l(I)]K(I_i)$ whenever $I = \bigcup_{i=1}^{n} I_i$, and where $1(I)$ denotes the length of I. If K is convex with respect to length, then K is said to be v-integrable with respect to f if $\lim_{|\sigma|\to 0} \sum K(I_i)(\Delta_i f) = v\int K df$ exists in $\bar{Y}$, the completion of Y (by $\Delta_i f$ we mean $f(t_{i+1}) - f(t_i)$ where $\{t_i\}$ is the partition σ of $[0, 1]$).

If $I \in \mathscr{I}$, with endpoints a and b, then the function Ψ_I defined by $\Psi_I(t) = 0$ for $t \leqq a$, $\Psi_I(t) = (t-a)/(b-a)$ for $a \leqq t \leqq b$, and $\Psi_I(t) = 1$ for $t \geqq b$, is called the fundamental function associated with I. A set function K whose domain is $\mathscr{I}$ and whose range is in $L[X, Y]$ is said to be quasi-Gowurin if given a neighborhood V of θ_Y, there is a neighborhood U of θ_C such that $\sum \Psi_{I_i} \cdot x_i \in U$ implies $\sum [K(I_i)](x_i) \in V$.

Finally, if $f \in C$ and σ is a partition of $[0, 1]$, then pf_σ denotes the polygonal function determined by σ and f.

2. The representation theorem.

Let C_θ denote the subspace of C such that $f(0) = \theta_X$.

THEOREM 2.1. *Suppose K is a set function on $\mathscr{I}$ with values* *in* $L[X, Y]$ *which is convex with respect to length and which is quasi-Gowurin. Then* $T(f) = v\int Kdf$ *is a continuous linear operator from* C_θ *into* $\bar{Y}$.

Proof. First we show that $v\int Kdf$ exists in $\bar{Y}$ for each $f \in C_\theta$. Suppose V is a neighborhood of θ_Y. Since K is quasi-Gowurin, there is a neighborhood U of θ_C such that $\sum \Psi_{I_i} \cdot x_i \in U$ implies $\sum [K(I_i)](x_i)$ is in V. Since pf_σ converges to f in the topology of uniform convergence, there is a δ such that $|\sigma_1|, |\sigma_2| < \delta$ implies $pf_{\sigma_1} - pf_{\sigma_2} \in U$. Let $\sigma_1 * \sigma_2$ denote the common refinement of σ_1 and σ_2. It follows from 7.2 in [1] that

$$(*) \quad \sum_{\sigma_1} [K(I_i)](\Delta_i f) - \sum_{\sigma_2} [K(I_j)](\Delta_j f) = \sum_{\sigma_1 * \sigma_2} [K(I_k)](\Delta_k (pf_{\sigma_1} - pf_{\sigma_2})) .$$

Since $\sum \Psi_{I_K} \cdot (\Delta_K (pf_{\sigma_1} - pf_{\sigma_2})) = pf_{\sigma_1} - pf_{\sigma_2} \in U$, then it follows that (*) is in V from which we conclude that $\{\sum_\sigma [K(I_i)](\Delta_i f)\}_\sigma$ is Cauchy. Hence, $v\int Kdf$ exists in $\bar{Y}$. Suppose $f_\alpha \to f$ in C_θ. Suppose V is a neighborhood of θ_Y. Then there is a neighborhood V' of θ_Y such that $V' + V' + V' \subset V$. Since K is quasi-Gowurin, there is a neighborhood U of θ_C such that $\sum \Psi_{I_i} \cdot x_i \in U$ then $\sum [K(I_i)](x_i) \in V$. There exists a neighborhood U' of θ_C such that $U' + U' + U' \subset U$. Since f_α converges to f, then, there is a β such that $\alpha > \beta$ implies $f_\alpha - f \in U'$. Suppose $\alpha > \beta$. Then there is a δ such that $|\sigma| < \delta$ implies each of $p(f_\alpha)_\sigma - f_\alpha \in U'$, $f - pf_\sigma \in U'$, $v\int Kdf - \sum_\sigma [K(I_i)](\Delta_i f) \in V'$, and $\sum_\sigma [K(I_i)](\Delta_i f_\alpha) - v\int Kdf \in V'$. Then,

$$\begin{aligned} v\int Kdf - v\int Kdf_\alpha &= v\int Kdf - \textstyle\sum_\sigma [K(I_i)](\Delta_i f) \\ &\quad + \textstyle\sum_\sigma [K(I_i)](\Delta_i f) - \textstyle\sum_\sigma [K(I_i)](\Delta_i f_\alpha) \\ &\quad + \textstyle\sum_\sigma [K(I_i)](\Delta_i f_\alpha) - v\int Kfd_\alpha \\ &\in \textstyle\sum_\sigma [K(I_i)](\Delta_i (f - f_\alpha)) + V' + V' . \end{aligned}$$

However,

$$\sum_\sigma \Psi_{I_i} \cdot (\Delta_i (f - f_\alpha)) = pf_\sigma - p(f_\alpha)_\sigma = (pf_\sigma - f) + (f - f_\alpha) + (f_\alpha - p(f_\alpha)_\sigma)$$

which is in $U' + U' + U' \subset U$. Hence $\sum_\sigma [K(I_i)](\Delta_i (f - f_\alpha)) \in V'$, from which it follows that $v\int Kdf - v\int Kdf_\alpha \in V' + V' + V' \subset V$. Therefore, $v\int Kdf_\alpha$ converges to $v\int Kdf$, and hence T is continuous.

THEOREM 2.2. *Suppose* T *is a continuous linear operator from* C_θ *into* Y *.Then there is a set function* $\mathscr{F}$ *with values in* $L[X, Y]$

which is convex with respect to length and quasi-Gowurin such that $T(f) = v\int K df$ *for each* $f \in C_0$.

Proof. Define K from $\mathscr{I}$ into $L[X, Y]$ by $[K(I)](x) = T(\Psi_I \cdot x)$, $x \in X$. Then K is convex with respect to length because T is linear and because of the manner in which fundamental functions combine. Suppose V is a neighborhood of θ_Y. Since T is continuous, there is a neighborhood U of θ_C such that $T(U) \subset V$. Therefore $\sum \Psi_{I_i} \cdot x_i \in U$ implies $\sum[K(I_i)](x_i) = T(\sum \Psi_{I_i} \cdot x_i) \in V$, which implies K is quasi-Gowurin. Suppose $f \in C_\theta$. Since pf_σ converses to f in C_θ, then

$$T(f) = \lim_{|\sigma|} T(pf_\sigma) = \lim_{|\sigma|} T(\textstyle\sum_\sigma \Psi_{I_i}(\Delta_i f))$$
$$= \lim_{|\sigma|} \textstyle\sum_\sigma [K(I_i)](\Delta_i f) = v\int K df \, .$$

The last equality follows from 2.1. The theorem is established.

COROLLARY 2.3. *Suppose Y is complete. Then, T is a continuous linear operator from C into Y if and only if there is an element $\mu \in L[X, Y]$ and a set function on $\mathscr{I}$ with values in $L[x, y]$ which is convex with respect to length and quasi-Gowurin such that* $T(f) = \mu(f(0)) + v\int K df$.

3. The locally convex setting. In this section, for the purpose of comparison, we consider the special case when $H = [0, 1]$ of the setting in which Goodrich gives his representation theorem [3], that is, we assume additionally that X and Y are locally convex spaces. The condition of quasi-Gowurin becomes: given a neighborhood V of θ_Y there is a neighborhood U of θ_X such that if $\{\sum_{i=1}^j x_i : j = 1, \cdots, n\} \subset U$, then $\sum_{i=1}^n [K(I_i)](x_i) \in V$. This condition stated in terms of the semi-norms becomes, using Swongs notation [4], there is a pairing (p, q) and a constant W_{p-q} for each pair of semi-norms p and q in the pairing such that $q(\sum_\sigma [K(I_i)](x_i)) \leqq W_{p-q} \max_j p(\sum_{i=1}^j x_i)$ for each partition of $(0, 1]$ and each corresponding collection $\{x_i\}$ in X. This property is the analogy of Goodrich's bounded (p, q) variation. A set function which satisfies this property is said to be of bounded (p, q) convex variation. In order to be able to state an optimal result in the following theorem we shall assume that Y is quasi-complete, i.e., each closed and bounded set in Y is complete.

THEOREM 3.1. *Suppose T is a linear operator from C into Y (which is quasi-complete). Then T is continuous if and only if there exists a $\mu \in L[X, Y]$ and a set function on $\mathscr{I}$ with values in $L[X, Y]$ which is convex with respect to length and of bounded*

(p, q) convex variation such that $T(f) = \mu(f(0)) + v\int K df$. Furthermore if T' denotes the restriction of T to C_θ, then $|T'|_{p-q} = W_{p-q}$.

The theorem follows from 2.3.

REMARK 3.2. It is immediate that the K function of 2.2, 2.3, and 3.1 is unique.

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