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A GENERALIZATION OF MARTINGALES AND TWO CONSEQUENT CONVERGENCE THEOREMS

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Loeve has observed that a discrete stochastic process can be interpreted as a game and that a martingale can be interpreted as a "fair" game. In this context, the notion of a martingale is enlarged to a game which becomes "fairer with time" and then this concept is utilized to establish two convergence theorems.

Let $(\Omega, \mathfrak{A}, p)$ be a probability space with $\{\mathfrak{A}_n\}_{n\geqq 1}$ an increasing family of sub σ-algebras of $\mathfrak{A}$ to which the process $\{X_n\}_{n\geqq 1}$ is adapted, (see [3, p. 65]). Henceforth, the process $\{X_n\}_{n\geqq 1}$ will be referred to as a game.

DEFINITION. The game $\{X_n\}_{n\geqq 1}$ will be said to become *fairer with time* if for every $\varepsilon > 0$.

$$p[|E(X_n \mid \mathfrak{A}_m) - X_m| > \varepsilon] \to 0$$

as $n, m \to \infty$ with $n \geqq m$.

It should be noted that any martingale is a game which becomes fairer with time. An easy example of a game which is not a martingale or a sub or a super martingale but does become fairer with time is constructed by considering a game which consists of tossing a die. Here, let

$$\mathfrak{A}_n = \mathfrak{A}, \text{ all } n$$

and

$$X_n(\{i\}) \equiv i + (-1)^n/n \ .$$

The main results. Let $\{\alpha_n : n \geqq 1\}$ be a monotonic sequence decreasing to zero with finite sum. The game $\{X_n\}_{n\geqq 1}$ may be decomposed with respect to $\{\alpha_n : n \geqq 1\}$ as

$$X_n = Y_n - Z_n, \text{ where } \{Y_n\}_{n\geqq 1} \text{ and } \{Z_n\}_{n\geqq 1} \tag{1.1}$$

are defined inductively by:

$$\begin{aligned} Y_1 &= X_1 \\ &\vdots \\ Y_n &= Y_{n-1} + [X_n - E(X_n \mid \mathfrak{A}_{n-1})] + \alpha_{n-1} \end{aligned} \tag{1.2}$$

$$Z_n = Z_{n-1} + [X_{n-1} - E(X_n \mid \mathfrak{A}_{n-1})] + \alpha_{n-1} . \tag{1.3}$$

We note that $\{Y_n\}_{n\geqq 1}$ is adapted to the sequence of σ-algebras $\{\mathfrak{A}_n\}_{n\geqq 1}$ and forms a submartingale with respect to it.

We will call the decomposition of the game $\{X_n\}_{n\geqq 1}$ according to (1.1) – (1.3) a Doob-like decomposition. (See [3, p. 104–105].)

Also, we define the collection of sets $\{B^{\alpha}_{n,m}\}$ for $m = 1, 2, \cdots$ and $n \geqq m$ by

$$B^{\alpha}_{n,m} \equiv \{w: \mid E(X_n \mid \mathfrak{A}^m) - X_m \mid > \alpha_m\} .$$

THEOREM 1. *Let $\{X_n\}_{n\geqq 1}$ be a uniformly integrable game and $\{Y_n\}_{n\geqq 1}$, the submartingale associated with its Doob-like decomposition, be uniformly dominated in absolute value by an element of $L_1(\Omega, \mathfrak{A}, p)$. Suppose for every $\delta > 0$ there exists an integer $N(\delta)$, such that*

$$P[B^{\alpha}_{n,m}] < \delta \text{ whenever } n \geqq m \geqq N(\delta) , \tag{1.4}$$

and

$$\sim B^{\alpha}_{n,m} \subset \sim B^{\alpha}_{k,k-1} \text{ whenever } n \geqq k \geqq k-1 \geqq m \geqq N(\delta) . \tag{1.5}$$

Then, there exists a function X in $L_1(\Omega, \mathfrak{A}, p)$ such that

$$\lim_{n\to\infty} \int_\Omega \mid X_n - X \mid dp = 0 .$$

Proof. It will be sufficient to show the game $\{X_n\}_{n\geqq 1}$ is Cauchy in the L_1 norm. For every pair (n, m) of positive integers write:

$$\int_\Omega \mid X_n - X_m \mid dp = \int_{B^{\alpha}_{n,m}} \mid X_n - X_m \mid dp + \int_{\sim B^{\alpha}_{n,m}} \mid X_n - X_m \mid dp .$$

Since $p[B^{\alpha}_{n,m}] \to 0$ as $n, m \to \infty$ and since the game $\{X_n\}_{n\geqq 1}$ is uniformly integrable (see [1, p. 89]), it is immediate that $\int_{B^{\alpha}_{n,m}} \mid X_n - X_m \mid dp$ can be made arbitrarily small for sufficiently large n and m.

By utilizing the Doob-like decomposition of $\{X_n\}_{n\geqq 1}$, we can write

$$\int_{\sim B^{\alpha}_{n,m}} \mid X_n - X_m \mid dp \leqq \int_{\sim B^{\alpha}_{n,m}} \mid Y_n - Y_m \mid dp + \int_{\sim B^{\alpha}_{n,m}} \mid Z_n - Z_m \mid dp .$$

Since there exists an integrable function which uniformly dominates the process $\{Y_n\}_{n\geqq 1}$ in absolute value, it is immediate that $\{Y_n\}_{n\geqq 1}$ is a convergent submartingale. Moreover, the dominated convergence theorem can be used to show that $\int_{\sim B^{\alpha}_{n,m}} \mid Y_n - Y_m \mid dp$ can be made arbitrarily small for sufficiently large n and m.

Thus, it remains to show that $\int_{\sim B^{\alpha}_{n,m}} |Z_n - Z_m| \, dp$ can be made arbitrarily small for sufficiently large n and m and the proof will be complete.

On $\sim B^{\alpha}_{n,m}$ it follows that

$$X_m \geqq E(X_n \mid \mathfrak{A}_m) - \alpha_m .$$

In particular, on $\sim B^{\alpha}_{n,n-1}$

$$X_{n-1} \geqq E(X_n \mid \mathfrak{A}_{n-1}) - \alpha_{n-1}$$

and so where

$$Z_n - Z_{n-1} = X_{n-1} - E(X_n \mid \mathfrak{A}_{n-1}) + \alpha_{n-1} \tag{1.6}$$

we can say

$$Z_n - Z_{n-1} \geqq 0 \text{ on } \sim B^{\alpha}_{n,n-1} . \tag{1.7}$$

Thus, choose any $\delta > 0$ and there exists $N(\delta)$ such that

$$\sum_{k=m}^{\infty} \alpha_k < \delta/2 \text{ for } m \geqq N(\delta) \tag{1.8}$$

and such that (1.5) holds. Hence, with $n \geqq m \geqq N(\delta)$, (1.5) and (1.7), write

$$Z_n - Z_{n-1} \geqq 0 \text{ on } \sim B^{\alpha}_{n,m} . \tag{1.9}$$

By observing the fact that $B^{\alpha}_{n,m} \in \mathfrak{A}_m$ for all n and m, we can write that

$$\int_{\sim B^{\alpha}_{n,m}} |Z_n - Z_m| \, dp = \int_{\Omega} E\{| Z_n - Z_m | I_{\sim B^{\alpha}_{n,m}} \mid \mathfrak{A}_m\} dp . \tag{1.10}$$

By (1.9), $|Z_n - Z_m| I_{\sim B^{\alpha}_{n,m}} = \sum_{k=m+1}^{n} (Z_k - Z_{k-1}) I_{\sim B^{\alpha}_{n,m}}$; this together with (1.6) lets us continue the equality in (1.10) to

$$\begin{aligned}\int_{\sim B^{\alpha}_{n,m}} |Z_n - Z_m| \, dp &= \sum_{k=m+1}^{n} \left\{ \int_{\sim B^{\alpha}_{n,m}} E\{(X_{k-1} - E(X_k \mid \mathfrak{A}_{k-1}) + \alpha_{k-1} | \mathfrak{A}_m\} \right\} dp \\ &= \int_{\sim B^{\alpha}_{n,m}} \{(X_m - E(X_n \mid \mathfrak{A}_m)) + \alpha_m + \cdots + \alpha_{n-1}\} dp \\ &\leqq \int_{\sim B^{\alpha}_{n,m}} \left\{ \alpha_m + \sum_{k=m}^{n-1} \alpha_k \right\} dp < \delta .\end{aligned}$$

By not demanding that the submartingale $\{Y_n\}_{n \geqq 1}$ associated with the Doob-like decomposition of the game $\{X_n\}_{n \geqq 1}$ be uniformly bounded above in absolute value by an element of $L_1(\Omega, \mathfrak{A}, p)$, we get the weaker

THEOREM 2. *Let* $\{X_n\}_{n\geqq 1}$ *be a uniformly integrable game satisfying* (1.4) *and* (1.5). *Then, there exists some constant* c *such that*

$$\lim_{n\to\infty}\int_\Omega X_n dp = c\ .$$

Proof. It will be sufficient to show the sequence $\left\{\int_\Omega X_n dp\right\}_{n\geqq 1}$ is Cauchy. With respect to the Doob-like decomposition of $\{X_n\}_{n\geqq 1}$, we can write

$$\left|\int_\Omega (X_n - X_m)dp\right| \leqq \left|\int_{B^\alpha_{n,m}} (X_n - X_m)dp\right| + \left|\int_{\sim B^\alpha_{n,m}} (X_n - X_m)dp\right|\ . \tag{1.11}$$

Again, $\left|\int_{B^\alpha_{n,m}} (X_n - X_m)dp\right|$ may be made arbitrarily small for sufficiently large m and n by using the uniform integrability of $\{X_n\}_{n\geqq 1}$. In order to deal with the second summand in (1.11), write

$$\left|\int_{\sim B^\alpha_{n,m}} (X_n - X_m)dp\right| \leqq \left|\int_{\sim B^\alpha_{n,m}} (Y_n - Y_m)dp\right| + \int_{\sim B^\alpha_{n,m}} |Z_n - Z_m|dp\ .$$

But $\int_{\sim B^\alpha_{n,m}} |Z_n - Z_m|dp$ can be made arbitrarily small for sufficiently large m and n exactly as in the proof of Theorem 1. Hence, showing that $\left|\int_{\sim B^\alpha_{n,m}} (Y_n - Y_m)dp\right|$ can be made arbitrarily small for sufficiently large m and n will complete the proof. To this end, we use (1.2) and write

$$E((Y_n - Y_m) \mid \mathfrak{A}_m) = \alpha_m + \cdots + \alpha_{n-1}$$

and get

$$\begin{aligned}\int_{\sim B^\alpha_{n,m}} (Y_n - Y_m)dp &= \int_{\sim B^\alpha_{n,m}} E\{(Y_n - Y_m) \mid \mathfrak{A}_m\}dp \\ &= \int_{\sim B^\alpha_{n,m}} (\alpha_m + \cdots + \alpha_{n-1})dp \leqq \sum_{k=m}^{n-1}\alpha_k\ .\end{aligned}$$

But since $\sum_{k=m}^{n-1}\alpha_k$ can be made arbitrarily small for sufficiently large m and n, the result follows.

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