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CONE RELATIONSHIPS OF BIORTHOGONAL SYSTEMS

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It is shown in this paper that total biorthogonal systems have the same cone if and only if they differ at most by rearrangement and by positive scalar multiplication. A connection is demonstrated between this result and work done by R. E. Fullerton in which he characterized the existence of an unconditional basis in terms of the existence of certain type cones. The paper is concluded by generalizing the first result to the situation in which two biorthogonal systems have cones which induce order isomorphic orderings.

1. Definitions and notations. In this paper we will assume that all vector spaces considered are real and that all topological vector spaces are Hausdorff. E' will denote the topological dual of E, and the letter N will denote the set of natural numbers.

An ordered vector space is a vector space E equipped with a transitive, reflexive relation $\leqq$ satisfying the following conditions:

(a) If x, y, z are elements in E and $x \leqq y$, then $x + z \leqq y + z$.

(b) If x, y are elements in E such that $x \leqq y$ and if a is a nonnegative scalar, then $ax \leqq ay$.

An ordered topological vector space is a topological vector space which is also an ordered vector space. The positive cone K of an ordered vector space E is defined by $K = \{x \in E: x \geqq 0\}$. It has the properties that $K + K \subset K$ and $aK \subset K$ for each nonnegative scalar a. A subset K of any vector space E with these two properties is called a cone. If K is a cone in the vector space E, then $(E, \leqq)$ is an ordered vector space where the relation $\leqq$ is defined on E by $x \leqq y$ whenever $y - x \in K$.

If x and y are elements of E, then the order interval between x and y is the set $I(x, y) = \{z \in E: x \leqq z \leqq y\}$. The positive cone K of E is said to be generating if $E = K - K$ and proper if $K \cap - K = \{0\}$. If E is an ordered topological vector space, its positive cone is said to be normal if there exists a local base of neighborhoods of zero for the given topology with the property that $V = \cup \{I(x, y): x, y \in V\}$ for each basic neighborhood V.

In a topological vector space (E, T), a pair of indexed sets (x_α, f_α), $\alpha \in A$, with $\{x_\alpha\} \subset E$ and $\{f_\alpha\} \subset E'$ is called a biorthogonal system in E if $f_\alpha(x_\beta) = 0$ for $\alpha \neq \beta$ and $f_\alpha(x_\alpha) = 1$. The set $K = \{x \in E: f_\alpha(x) \geqq 0, \alpha \in A\}$ is a cone and is called the cone of the biorthogonal system (x_α, f_α).

We will call a biorthogonal system (x_α, f_α) total provided it has the following two properties:

(1) The closed linear span of the x_α's is E.

(2) The f_α's are total, i.e., $f_\alpha(x) = 0$ for each $\alpha \in A$ implies that $x = 0$.

We will pay special attention to biorthogonal systems (x_n, f_n), $n \in N$, which are Schauder bases and will use the term basis to mean a Schauder basis.

2. Biorthogonal systems having the same cone. We will approach the proof of Theorem 2.6 by characterizing the extreme subsets of the positive cone. Propositions 2.4 and 2.5 although not used in the proof of Theorem 2.6, are of interest in themselves. An extreme subset of a cone K is a convex set A such that the following holds: if $u, v \in K$, $0 < t < 1$, and $tu + (1-t)v \in A$, then u and v are elements of A.

If x is a nonzero element of K, we will denote by $R(0, x)$ the ray $R(0, x) = \{tx : 0 \leqq t\}$. If $R(0, x)$ is an extreme subset of K, we will call it an extreme ray.

PROPOSITION 2.1. *Let E be a vector space ordered by the cone K. Let A be a convex subset of K. Then A is an extreme subset of K if and only if the following two conditions are satisfied:*

(a) *$I(0, x) \subset A$ for each $x \in A$.*

(b) *$R(0, x) \subset A$ for each $x \in A$.*

Proof. Let A be an extreme subset of K. Suppose that $A \neq \phi$ and that $A \neq \{0\}$. Let x be a nonzero element of A, and let $\alpha > 1$. Then $(\alpha - 1)/\alpha \cdot 0 + (1/\alpha) \cdot (\alpha x) = x \in A$. Therefore, $0, \alpha x \in A$. Since $\alpha > 1$ was arbitrary and A is convex, $R(0, x) \subset A$. Now let $y \in I(0, x)$. Then $x = (1/2) \cdot 2(x - y) + (1/2) \cdot (2y)$ and thus $2(x - y), 2y \in A$. Therefore, $y \in A$ and hence $I(0, x) \subset A$.

Conversely, let $u, v \in K$ and $0 < t < 1$ such that $tu + (1 - t)v = x \in A$. Then $x - tu = (1 - t)v \in K$. Therefore, $tu \in I(0, x)$ which implies that $tu \in A$ by (a) and hence that $u \in A$ by (b). Similarly v is also an element of A. Therefore, A is an extreme subset of K.

COROLLARY 2.2. *Let E be a vector space ordered by a cone K, and let x be a nonzero element of K. Then the following statements are equivalent.*

(a) *$R(0, x)$ is an extreme ray*

(b) *$I(0, x) \subset R(0, x)$*

(c) *$I(0, x) = \{tx : 0 \in t \leqq 1\}$.*

Proof. The implications $a \rightarrow b$ and $c \rightarrow a$ follow trivially from Proposition 2.1. To show that $b \rightarrow c$, we need only show that $I(0, x) \subset$

$\{tx: 0 \leqq t \leqq 1\}$. The reverse inclusion is clear. Thus suppose $z \in I(0, x)$. By (b) there exists $t \geqq 0$ such that $z = tx$. But $tx \in I(0, x)$ implies that $x - tx = (1 - t)x \in I(0, x) \subset R(0, x)$. Then $1 - t \geqq 0$ and $t \leqq 1$. Thus $z \in \{tx: 0 \leqq t \leqq 1\}$.

We remark that (c) of Corollary 2.2 is well known and is sometimes used as the definition of an extreme ray of a cone [5, p. 10].

PROPOSITION 2.3. *Let E be a topological vector space ordered by the cone K of a biorthogonal system (x_α, f_α), $\alpha \in A$, for which the f_α's are total, and let $x \in E$. Then $R(0, x)$ is an extreme ray of K if and only if there exists $\beta \in A$ and $b > 0$ such that $x = bx_\beta$.*

Proof. Suppose that $R(0, x)$ is an extreme ray for K. Since $x \neq 0$ and since the f_α's are total, there exists $\beta \in A$ such that $f_\beta(x) > 0$. Then $f_\beta(x)x_\beta \in I(0, x)$ which is contained in $R(0, x)$ by Corollary 2.2. Thus $R(0, x_\beta) = R(0, x)$ and there exists $b > 0$ such that $x = bx_\beta$.

Conversely, suppose that $\alpha \in A$. Consider the ray $R(0, x_\alpha)$. If $z \in I(0, x_\alpha)$, then $f_\beta(z) = 0$ for $\beta \neq \alpha$. Hence $f_\beta(z - f_\alpha(z)x_\alpha) = 0$ for each $\beta \in A$. Since the f_α's are total, we conclude that $z = f_\alpha(z)x_\alpha$. Thus $I(0, x_\alpha) \subset R(0, x_\alpha)$ and $R(0, x_\alpha)$ is an extreme ray, Corollary 2.2.

PROPOSITION 2.4. *If E is a topological vector space ordered by the cone K of biorthogonal system (x_α, f_α), then K has extreme rays if and only if the f_α's are total.*

Proof. If the f_α's are total, then by Proposition 2.3, each ray $R(0, x_\alpha)$ is an extreme ray of K.

Suppose, however, that the f_α's are not total. Then there exists a nonzero x in E such that $f_\alpha(x) = 0$ for each $\alpha \in A$. Thus x and $-x \in K$. If $y \neq 0$ is an element of K, then $(1/2) \cdot x + (1/2) \cdot -x = 0 \in R(0, y)$. However, either x or $-x$ is not an element of $R(0, y)$, and thus $R(0, y)$ cannot be an extreme ray of K. Since y was an arbitrary element of K, we conclude that K has no extreme rays.

PROPOSITION 2.5. *Let (E, T) be a topological vector space ordered by the cone K of a biorthogonal system (x_n, f_n) which is a basis on K, i.e., $x = \sum_{n=1}^{\infty} f_n(x)x_n$ for each $x \in K$. Then A is a closed extreme subset of K if and only if there exists a subset Δ of N such that $A = \{y \in K: f_n(y) = 0$ for $n \in \Delta\}$.*

Proof. Let $\Delta \subset N$ and let $A = \{y \in K: f_n(y) = 0$ for $n \in \Delta\}$. It is clear that $I(0, x) \subset A$ and $R(0, x) \subset A$ for each $x \in A$. Therefore, by Proposition 2.1, A is an extreme subset of K. Clearly A is closed.

Conversely, suppose that A is a closed extreme subset of K. Let

$\Delta = \{n \in N: f_n(y) = 0$ for each $y \in A\}$. We need only to show that $A = A_\Delta$ where $A_\Delta = \{y \in K: f_n(y) = 0$ for $n \in \Delta\}$. Clearly $A \subset A_\Delta$. Thus suppose $z \in A_\Delta$, then $z = \sum_{n=1}^{\infty} f_n(z)x_n$ and $f_n(z) = 0$ for $n \in \Delta$. Hence z is contained in the closure of the convex hull of the $R(0, x_n)$, $n \in N - \Delta$. However, for each $n \in N - \Delta$ there exists $y \in A$ such that $f_n(y) > 0$. Since $f_n(y)x_n$ is an element of $I(0, y)$ and since A is an extreme subset of K, we have that $R(0, f_n(y)x_n) \subset A$ by Proposition 2.1. Hence $R(0, x_n) \subset A$ for each $n \in N - \Delta$. A is also closed and convex, since it is a closed extreme subset of K. Therefore, $z \in A_\Delta$ and we have that $A = A_\Delta$.

THEOREM 2.6. *Let (x_α, f_α), $\alpha \in A$, be a total biorthogonal system and (y_β, h_β), $\beta \in B$, a biorthogonal system with cones K_1 and K_2 respectively in a topological vector space E. Then $K_1 = K_2$ if and only if there exists a one-to-one mapping F of A onto B and a collection of positive scalars $\{\lambda_\alpha\}$, $\alpha \in A$, such that $y_{F(\alpha)} = \lambda_\alpha x_\alpha$ and $h_{F(\alpha)} = (1/\lambda_\alpha)f_\alpha$.*

Proof. If there exists a mapping F and positive scalars $\{\lambda_\alpha\}$, $\alpha \in A$ as in the theorem statement, it is clear from the definitions of K_1 and K_2 that they must be equal.

Conversely, suppose that $K_1 = K_2$. By Proposition 2.3, the collection of extreme rays of K_1 is the set $\{R(0, x_\alpha): \alpha \in A\}$. However, $K_1 \cap - K_1 = \{0\}$ since the f_α's are total. Thus, because K_1 is also the cone of (y_β, h_β), we must have the h_β's are also total. Applying Proposition 2.3 again, we have that the collection of extreme rays of K_1 is the set $\{R(0, y_\beta): \beta \in B\}$. Thus, $\{R(0, x_\alpha): \alpha \in A\} = \{R(0, y_\beta): \beta \in B\}$. Define F mapping A into B in the following way: α maps to $F(\alpha)$ provided $R(0, x_\alpha) = R(0, y_{F(\alpha)})$. It is a simple matter to verify that F is a well-defined, one-to-one mapping of A onto B. Therefore, for each $\alpha \in A$ there exists $\lambda_\alpha > 0$ such that $y_{F(\alpha)} = \lambda_\alpha x_\alpha$. Furthermore, since the closed linear span of the x_α's is E, $h_{F(\alpha)} = (1/\lambda_\alpha)f_\alpha$, [3, Proposition 1].

COROLLARY 2.7. *Let (E, T) be a topological vector space ordered by the cone K of basis (x_n, f_n). If (y_α, h_α), $\alpha \in A$, is any biorthogonal system whose cone is also K, then $A = N$ and there exists a permutation τ of N such that $(y_{\tau(n)}, h_{\tau(n)})$ is also a basis.*

Proof. By Theorem 2.6, there exists a one-to-one mapping of N onto A and positive scalars $\{b_n: n \in N\}$ such that $y_{\tau(n)} = b_n x_n$ and $h_{\tau(n)} = (1/b_n)f_n$. Since (x_n, f_n) is a basis, $(b_n x_n, (1/b_n)f_n) = (y_{\tau(n)}, h_{\tau(n)})$ is also a basis.

COROLLARY 2.8. *If (E, T) is a topological vector space ordered by the cone K of an unconditional basis (x_n, f_n) and if (y_α, h_α), $\alpha \in A$, is* *any biorthogonal system having the same cone K, then $A = N$ and (y_n, h_n) is also an unconditional basis (E, T).*

3. **Remarks.** We next point out a connection between the above work and some work done by Fullerton [4]. To do this let us notice that if (x_n, f_n) is a basis in a locally convex topological vector space (E, T), then the rays $\{R(0, x_n): n \in N\}$ satisfy the following conditions.

(a) $\bigcup \{R(0, x_n): n \in N\}$ generates a linear space dense in E.

(b) For each $n \in N$, the closed linear subspace L_n generated by the set $\{R(0, x_j): j \neq n\}$ is a hyperplane.

(c) $\bigcap \{L_n: n \in N\} = \{0\}$.

Furthermore, if H_n is the closed half-space bounded by L_n and containing $R(0, x_n)$, then $K = \bigcap \{H_n: n \in N\}$ is a cone and in fact is the cone of the basis (x_n, f_n).

If (x_n, f_n) is an unconditional basis and (E, T) is sequentially complete, it is true that the cone K generated by the rays $\{R(0, x_n): n \in N\}$ satisfies two additional conditions.

(d) $K \cap x - K$ is compact for each $x \in K$.

(e) $K - K = E$.

Statement (d) is equivalent to unconditional convergence of the series $\sum_{n=1}^{\infty} f_n(x) x_n$ for each $x \in K$ [9, p. 16]. Statement (e) is well known and follows from the fact that in a sequentially complete space unconditional convergence is equivalent to sub-series convergence [8, p. 17].

REMARK. In a complete locally convex topological vector space, the existence of an unconditional basis is equivalent to the existence of a cone K defined by a collection of rays $\{R(0, x_n): n \in N\}$ satisfying conditions (a) through (e). Furthermore, it can be shown that K is the cone of that basis.

The above remark is essentially the work of Fullerton [4]; however, it can also be obtained using a theorem of McArthur [9, p. 16].

We would like to point out here that even though Fullerton did not include condition (e) when claiming the above remark to be true, it is necessary. This can be seen by the following example. Let $(E, T) = C[0, 1]$ with the sup-norm topology. Let (x_n, f_n) be the usual Schauder basis given for $C[0, 1]$, [2, p. 69]. It is well known that the cone K of this basis is normal. Thus by theorems of McArthur [10, pp. 6 and 16], we have that $K \cap x - K$ is compact for each $x \in K$. Consequently, the rays $R(0, x_n)$ of this basis satisfy conditions (a) through (d) and the cone K would be called an absolute basis cone by Fullerton [4]. Hence, by Fullerton's statement of the above remark, the existence of the cone K is equivalent to the existence of an unconditional basis system (y_α, h_α), $\alpha \in A$, having K as its cone. It is well known that the basis (x_n, f_n) above is a conditional basis [7]. Furthermore by Theorem 2.6, if (y_α, h_α), $\alpha \in A$, is any arbitrary biorthogonal system having K for a cone, then $A = N$ and there exists a permutation τ of N and a collection $\{\lambda_n : n \in N\}$ of positive scalars such that $y_n = \lambda_n x_{\tau(n)}$. Since (x_n, f_n) is a conditional basis, $(\lambda_n x_{\tau(n)}, (1/\lambda_n) f_{\tau(n)})$ must also be a conditional basis if indeed it is a basis at all. Thus we have that K satisfied conditions (a) through (d), but no unconditional basis system exists which has K for its cone. We note that K is not generating since it is the cone of a conditional basis and is normal [9, p. 20]. Hence, condition (e) is indeed necessary in order that the above remark be true.

Note that Corollary 2.8 states that unconditional basis cones are unique among biorthogonal system cones in any topological vector space. Therefore, it seems likely that the above remark could be generalized to general topological vector spaces. Corollary 2.7 indicates that some type of a similar remark should be true for conditional bases.

4. **Biorthogonal systems giving order isomorphic orderings.** In this section we will prove a type of analogue to Theorem 2.6. For this work one might think of two cones as being the same if they give order isomorphic orderings.

If (E_1, T_1, K_1) and (E_2, T_2, K_2) are ordered topological vector spaces with positive cones K_1 and K_2 respectively, then E_1 and E_2 are said to be order isomorphic to each other if there exist a linear isomorphism T mapping E_1 onto E_2 such that $T(K_1) = K_2$. If T and T^{-1} are also continuous, E_1 and E_2 are topologically order isomorphic.

Let (x_α, f_α) be a biorthogonal system in a topological vector space E_1 and (y_α, h_α) a biorthogonal system in a topological vector space E_2. We say that (x_α, f_α) and (y_α, h_α) are equivalent systems if for each $x \in E_1$ and $y \in E_2$ there exists a corresponding $y' \in E_2$ and $x' \in E_1$ such that $f_\alpha(x) = h_\alpha(y')$ and $f_\alpha(x') = h_\alpha(y)$ for each α. If (x_n, f_n) and (y_n, h_n) are Schauder bases for E_1 and E_2 respectively, we say that (x_n, f_n) and (y_n, h_n) are equivalent bases provided $\{(a_n): \sum_{n=1}^\infty a_n x_n \text{ converges in } E_1\} = \{(b_n): \sum_{n=1}^\infty b_n y_n \text{ converges in } E_2\}$.

THEOREM 4.1. *Let (E_1, K_1) and (E_2, K_2) be topological vector spaces ordered respectively by the cones K_1 and K_2 of total biorthogonal systems (x_α, f_α) and (y_α, h_α), $\alpha \in A$. The following statements can be proved.*

(a) *If (E_1, K_1) and (E_2, K_2) are topologically order isomorphic, there exists a permutation τ of A and positive scalars $\{\lambda_\alpha : \alpha \in A\}$ such that (x_α, f_α) and $(\lambda_\alpha y_{\tau(\alpha)}, (1/\lambda_\alpha) h_{\tau(\alpha)})$ are equivalent systems.*

(b) *If (x_α, f_α) and (y_α, h_α) are equivalent, then (E_1, K_1) and (E_2, K_2)*

are order isomorphic vector spaces.

Proof. Let $T\colon E_1 \to E_2$ be a topological order isomorphism mapping E_1 onto E_2. Then $(T(x_\alpha), f_\alpha T^{-1})$ is a biorthogonal system in E_2. Let K_3 denote the cone of $(T(x_\alpha), f_\alpha T^{-1})$. Then $K_2 = K_3$. Therefore by Theorem 2.6, there exists a permutation of A and positive scalars $\{\lambda_\alpha\colon \alpha \in A\}$ such that $T(x_\alpha) = \lambda_\alpha y_{\tau(\alpha)}$ and $f_\alpha T^{-1} = (1/\lambda_\alpha)h_{\tau(\alpha)}$. Furthermore (x_α, f_α) and $(\lambda_\alpha y_{\tau(\alpha)}, (1/\lambda_\alpha)h_{\tau(\alpha)})$ are equivalent systems.

Suppose now that (x_α, f_α) and (y_α, h_α) are equivalent. If $x \in E_1$, let $T(x)$ denote that element y of E_2 for which $f_\alpha(x) = h_\alpha(y)$ for each $\alpha \in A$. Then T is a linear isomorphism of E_1 onto E_2 and $T(K_1) = K_2$.

COROLLARY 4.2. *Let (E, T) be a sequentially complete topological vector space ordered by the cone K_1 of an unconditional basis (x_n, f_n). Let $\{b_n\colon n \in N\}$ be any sequence of nonzero scalars, $z_n = b_n x_n$, and $h_n = (1/b_n)f_n$. Then if K_2 is the cone of (z_n, h_n), we have that (E, K_1) and (E, K_2) are order isomorphic as vector spaces.*

Proof. We begin by showing that (x_n, f_n) and $(a_n x_n, a_n f_n)$ are equivalent bases where $a_n = \operatorname{sgn} b_n$. In a sequentially complete space, $\sum_{n=1}^\infty f_n(x)x_n$ converges unconditionally to x if and only if $\sum_{n=1}^\infty d_n f_n(x)x_n$ converges to x for every sequence $\{d_n\colon d_n = \pm 1\}$, [8, p. 17]. Using this fact again we have that $\sum_{n=1}^\infty d_n f_n(x)x_n$ converges unconditionally to x for each such sequence and for each $x \in E$. Thus $\sum_{n=1}^\infty b_n x_n$ converges if and only if $\sum_{n=1}^\infty b_n(a_n x_n)$ converges, i.e., (x_n, f_n) and $(a_n x_n, a_n f_n)$ are equivalent bases. Thus by (b) of Theorem 4.1, we have that (E, K_1) and (E, K_3) are order isomorphic vector spaces where K_3 is the cone of the basis $(a_n x_n, a_n f_n)$. However, $K_2 = K_3$, thus (E, K_1) and (E, K_2) are order isomorphic vector spaces.

PROPOSITION 4.3. *Let E_1 and E_2 be complete metric linear spaces ordered by the cones K_1 and K_2 respectively of the total biorthogonal systems (x_α, f_α) and (y_α, h_α), $\alpha \in A$. Then (E_1, K_1) and (E_2, K_2) are topologically order isomorphic if and only if there exists positive scalars $\{\lambda_\alpha\colon \alpha \in A\}$ and a permutation τ of A such that the systems (x_α, f_α), $\alpha \in A$, and $(\lambda_\alpha y_{\tau(\alpha)}, (1/\lambda_\alpha)h_{\tau(\alpha)})$ are equivalent.*

Proof. If (E_1, K_1) and (E_2, K_2) are order isomorphic topological vector spaces, we get the desired result by applying (a) of Theorem 4.1.

Conversely, we can assume without loss of generality that (x_α, f_α) and (y_α, h_α) are equivalent. Then by the isomorphism theorem of Arsove and Edwards [1], there exists an isomorphism T of E_1 onto E_2 such that $T(x_\alpha) = y_\alpha$. Furthermore T is defined as in the proof of (b) of Theorem 2.6. Hence $T(K_1) = K_2$, i.e., T is a topological order isomorphism.

PROPOSITION 4.4. *Let* (E_1, T_1) *and* (E_2, T_2) *be barrelled spaces ordered by the cones* K_1 *and* K_2 *of Schauder bases* (x_n, f_n) *and* (y_n, h_n) *respectively. Then* (E_1, T_1, K_1) *and* (E_2, T_2, K_2) *are topologically order isomorphic if and only if there exists a permutation* τ *of* N *and positive scalars* $\{\lambda_n: n \in N\}$ *such that* (x_n, f_n) *and* $(\lambda_n y_{\tau(n)}, (1/\lambda_n)h_{\tau(n)})$ *are equivalent bases.*

Proof. Suppose that (E_1, T_1, K_1) and (E_2, T_2, K_2) are topologically order isomorphic. Let $T: E_1 \rightarrow E_2$ be the defining order homeomorphism. As in the proof of Theorem 4.1, there exists a permutation τ of N and positive scalars $\{\lambda_n: n \in N\}$ such that $T(x_n) = \lambda_n y_{\tau(n)}$. Thus (x_n, f_n) and $(\lambda_n y_{\tau(n)}, (1/\lambda_n)h_{\tau(n)})$ are equivalent bases [6, p. 678].

Conversely, we can assume without loss of generality that (x_n, f_n) and (y_n, h_n) are equivalent bases. Then $F: E_1 \rightarrow E_2$ defined by $F(x) = \sum_{n=1}^{\infty} f_n(x)y_n$ is a linear homeomorphism [6, p. 678] and clearly $F(K_1) = K_2$.

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