

SOME REMARKS ON SPECIAL DISCONJUGACY CRITERIA FOR DIFFERENTIAL SYSTEMS

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The present note is devoted to the discussion of a special class of transformations that preserves the oscillatory or disconjugacy nature of solutions for a type of first order differential system in a $2n$ -dimensional vector function, and which includes as a particular instance a well-known transformation of a real scalar second order homogeneous ordinary differential equation to the canonical form $y''(t) + \gamma(t)y(t) = 0$.

It is well-known, (see, for example, Reid [4, 6, 7]), that for self-adjoint differential systems conditions of oscillation may be characterized by variational criteria, and, in particular, the property of disconjugacy is equivalent to the positive definiteness of an associated hermitian integral functional on a suitable class of vector functions. Moreover, (see, for example, Reid [4; § 5]; Hartman and Wintner [2]), results for self-adjoint differential systems may be applied to yield sufficient conditions for disconjugacy in the case of nonself-adjoint systems. These criteria and procedures are of basic significance for the utilization of the considered transformations in the study of oscillation and comparison phenomena.

Matrix notation is used throughout; in particular, matrices of one column are called vectors, and for a vector $u = (u_\alpha)$, ($\alpha = 1, \dots, n$), the norm $|u|$ is given by $(|u_1|^2 + \dots + |u_n|^2)^{1/2}$; the linear vector space of ordered n -tuples of complex numbers, with complex scalars, is denoted by C_n . The $n \times n$ identity matrix is denoted by E_n , or by merely E when there is no ambiguity, while 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is denoted by M^* . If M is an $n \times n$ matrix the symbol $\nu[M]$ is used for the maximum of $|My|$ on the unit ball $\{y: |y| \leq 1\}$ in C_n . The notation $M \geq N$, $\{M > N\}$, is used to signify that M and N are hermitian matrices of the same dimensions, and $M - N$ is a nonnegative, {positive} definite hermitian matrix. In general, if M is an $n \times n$ matrix, let $\Re M$ and $\Im M$ denote the hermitian matrices $\Re M = \frac{1}{2}(M + M^*)$, $\Im M = i/2(M^* - M)$ so that $M = \Re M + i \Im M$. If the elements of a matrix function $M(t)$ are a.c. (absolutely continuous) on arbitrary compact subintervals of a given interval I , then $M(t)$ is said to be locally a.c. on I ; moreover, $M'(t)$ signifies the matrix of derivatives at values where these derivatives exist and the zero matrix elsewhere. Correspondingly, if the elements of $M(t)$ are

(Lebesgue) integrable on a compact interval $[a, b]$, then $\int_a^b M(t)dt$ denotes the matrix of integrals of respective elements of $M(t)$. If $M(t)$ and $N(t)$ are equal a.e. (almost everywhere) on their domain of definition we write simply $M(t) = N(t)$. A matrix function is called continuous, integrable, etc., when each of its elements possesses the specified property.

For a given compact interval $[a, b]$ on the real line the symbols $\mathfrak{L}_{nr}[a, b]$, $\mathfrak{L}_{nr}^\infty[a, b]$ are used to denote the classes of $n \times r$ matrix functions $M(t) = [M_{\alpha\beta}(t)]$, ($\alpha = 1, \dots, n; \beta = 1, \dots, r$), which on $[a, b]$ are respectively integrable, measurable and essentially bounded, where in each case measure and integral is understood to be in the sense of Lebesgue. For brevity, $\mathfrak{L}_n[a, b]$ and $\mathfrak{L}_n^\infty[a, b]$ are written for $\mathfrak{L}_{n1}[a, b]$ and $\mathfrak{L}_{n1}^\infty[a, b]$, respectively.

2. Transformations for differential systems. We shall be concerned with a vector differential system of the form

$$(2.1) \quad \begin{aligned} L_1[u, v](t) &\equiv -v'(t) + C(t)u(t) - D(t)v(t) = 0, \\ L_2[u, v](t) &\equiv u'(t) - A(t)u(t) - B(t)v(t) = 0, \end{aligned}$$

in n -dimensional vector functions

$$u(t) \equiv (u_i(t)), v(t) \equiv (v_i(t)),$$

($i = 1, \dots, n$). In terms of the $2n$ -dimensional vector function $y(t) = (y_\sigma(t))$, ($\sigma = 1, \dots, 2n$), with $y_i(t) = u_i(t)$, $y_{n+i}(t) = v_i(t)$, ($i = 1, \dots, n$), this system may be written also as

$$(2.1') \quad \mathcal{L}[y] \equiv \mathcal{J}y'(t) + \mathcal{A}(t)y(t) = 0,$$

where

$$(2.2) \quad \mathcal{J} = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix}, \mathcal{A}(t) = \begin{bmatrix} C(t) & -D(t) \\ -A(t) & -B(t) \end{bmatrix}.$$

Throughout the discussion it will be assumed that the coefficient matrix functions satisfy the following hypothesis.

(H₁) $A(t), B(t), C(t), D(t)$ are $n \times n$ matrix functions defined on a given interval I on the real line, and are of class $\mathfrak{L}_{nn}[a, b]$ on arbitrary compact subintervals $[a, b]$ of I .

Corresponding to the vector differential system (2.1), one has the matrix differential system

$$(2.1_M) \quad \begin{aligned} L_1[U, V](t) &\equiv -V'(t) + C(t)U(t) - D(t)V(t) = 0, \\ L_2[U, V](t) &\equiv U'(t) - A(t)U(t) - B(t)V(t) = 0, \end{aligned}$$

in $n \times r$ matrix functions $U(t), V(t)$, while the associated $2n \times r$ matrix function $Y(t) = (U(t); V(t))$ with

$$Y_{i\beta}(t) = U_{i\beta}(t), Y_{n+i,\beta}(t) = V_{i\beta}(t),$$

($i = 1, \dots, n; \beta = 1, \dots, r$), is a solution of the equation

$$(2.1'_M) \quad \mathcal{L}[Y](t) \equiv \mathcal{L} Y'(t) + \mathcal{A}(t)Y(t) = 0.$$

In particular, the linear homogeneous differential system (2.1) is self-adjoint, or Hamiltonian, if the following conditions hold.

(\mathbf{H}_2) $B(t)$ and $C(t)$ are hermitian, and $D(t) = A^*(t)$ for $t \in I$.

In general, two distinct points t_1 and t_2 on I are said to be (*mutually*) *conjugate* with respect to (2.1) if there exists a solution $(u(t); v(t))$ of this system with $u(t_1) = 0 = u(t_2)$, while $u(t) \not\equiv 0$ for t on the subinterval with endpoints t_1 and t_2 . The system (2.1) is said to be *disconjugate* on a subinterval I_0 provided no two distinct points of this interval are conjugate.

In the case of a self-adjoint system (2.1), one has the important fact that if $y_\alpha(t) = (u_\alpha(t); v_\alpha(t))$, $\alpha = 1, 2$, are solutions of (2.1), then the function $y_2^*(t) \mathcal{L} y_1(t) = v_2^*(t)u_1(t) - u_2^*(t)v_1(t)$ is constant on I . If the value of this constant is zero, then $y_1(t)$ and $y_2(t)$ are said to be (*mutually*) *conjoined* solutions of (2.1). In particular, if

$$Y(t) = (U(t); V(t))$$

is a $2n \times n$ matrix which is a solution of (2.1'_M), and whose n column vectors are linearly independent solutions of (2.1') which are mutually conjoined, then for brevity $Y(t)$ is called a *conjoined basis* for (2.1). Moreover, if $Y(t) = (U(t); V(t))$ is a conjoined basis for (2.1), and at a point t_0 of I the $n \times n$ matrix $U(t)$ is of rank $n - k$, then in accordance with the terminology of Morse [3] the point t_0 is said to be a *focal point* of $Y(t)$ of order k .

For a compact subinterval $[a, b]$ of I , let $\mathcal{D}_0[a, b]$ denote the class of n -dimensional vector functions $\eta(t)$ which are a.c. on $[a, b]$ and satisfy the differential equation

$$L_2[\eta, \zeta](t) \equiv \eta'(t) - A(t)\eta(t) - B(t)\zeta(t) = 0$$

with $\zeta(t) \in \mathfrak{S}_n^\infty[a, b]$, while $\eta(a) = 0 = \eta(b)$. Moreover, let $\mathbf{H}_+[a, b]$ denote the condition that the hermitian functional

$$(2.3) \quad J[\eta; a, b] = \int_a^b \{\zeta^*(t)B(t)\zeta(t) + \eta^*(t)C(t)\eta(t)\} dt$$

is positive definite on $\mathcal{D}_0[a, b]$. It is to be remarked that the value of the integral in (2.3) is a function of η on $\mathcal{D}_0[a, b]$, since if $\zeta_i(t)$

and $\zeta_2(t)$ are vector functions in $\mathcal{L}_n^\infty[a, b]$ such that $L_2[\eta, \zeta_\alpha](t) = 0$ for $\alpha = 1, 2$, then $\zeta_1^*(t)B(t)\zeta_1(t) = \zeta_2^*(t)B(t)\zeta_2(t)$. As is well-known, (see, for example, Reid [4; Th. 2.1]; Reid [7; Th. 5.1]), for self-adjoint differential systems (2.1) the condition of disconjugacy is characterized by certain variational conditions, that may be phrased as follows.

THEOREM 2.1. *If conditions (\mathbf{H}_1) and (\mathbf{H}_2) hold, then for $[a, b]$ a compact subinterval of I condition $\mathbf{H}_+[a, b]$ holds if and only if $B(t) \geq 0$ for t a.e. on $[a, b]$, and one of the following conditions holds:*

(i) (2.1) is disconjugate on $[a, b]$;

(ii) there exists a conjoined basis $Y(t) = (U(t); V(t))$ for (2.1) with $U(t)$ nonsingular on $[a, b]$.

In particular, if $B(t) \geq 0$ for t a.e. on $[a, b]$, then the conditions (i), (ii) and $\mathbf{H}_+[a, b]$ are equivalent.

If $T(t)$ and $S(t)$ are nonsingular $n \times n$ matrix functions which are locally a.c. on I , then under the transformation

$$(2.4) \quad u(t) = T(t)u^0(t), \quad v(t) = S(t)v^0(t),$$

we have the identities

$$(2.5) \quad L_1[u, v](t) \equiv S(t)L_1^0[u^0, v^0](t), \quad L_2[u, v](t) = T(t)L_2^0[u^0, v^0](t),$$

where

$$(2.1)^0 \quad \begin{aligned} L_1^0[u^0, v^0](t) &\equiv -v^{0'}(t) + C^0(t)u^0(t) - D^0(t)v^0(t), \\ L_2^0[u^0, v^0](t) &\equiv u^{0'}(t) - A^0(t)u^0(t) - B^0(t)v^0(t), \end{aligned}$$

and the coefficient matrices of (2.1)⁰ are given by

$$(2.6) \quad \begin{aligned} A^0 &= T^{-1}[AT - T'], \quad B^0 = T^{-1}BS, \\ C^0 &= S^{-1}CT, \quad D^0 = S^{-1}[S' + DS]. \end{aligned}$$

Clearly one has the following result.

LEMMA 2.1. *If (\mathbf{H}_1) is satisfied, then distinct values t_1, t_2 on I are conjugate with respect to (2.1) if and only if they are conjugate with respect to (2.1)⁰; in particular, (2.1) is disconjugate on a subinterval I_0 of I if and only if (2.1)⁰ is disconjugate on this subinterval.*

We shall be concerned with systems (2.1) in which the following additional condition holds.

(\mathbf{H}_3) *On I the matrix function $B(t)$ is nonsingular, locally a.c., and possesses a square root matrix $B^{1/2}(t)$ which is locally a.c. on this interval.*

It is to be remarked that if $B(t)$ is nonsingular and locally a.c.

on I , while there exists a nonzero locally a.c. scalar function $\rho(t)$ such that $\nu[E - \rho(t)B(t)] < 1$ for $t \in I$, then the binomial series may be used to provide a locally a.c. square root $B^{1/2}(t)$ of $B(t)$ with series expansion

$$B^{1/2}(t) = [1/\rho^{1/2}(t)] \left\{ E - \sum_{k=1}^{\infty} c_k [E - \rho(t)B(t)]^k \right\},$$

where $c_1 = 1/2$, $c_k = [1 \cdot 3 \cdots (2k - 3)]/[k! 2^k]$, ($k = 1, 2, \dots$), (for example, see Reid [5; § 4]). In particular, this condition holds whenever $B(t)$ is a positive definite hermitian matrix that is locally a.c. on I ; in this case the corresponding square root matrix provided by this binomial series expansion is the unique positive definite hermitian square root of $B(t)$.

3. Special transformations which preserve disconjugacy. The particular type of transformation to be used in the results of this section is introduced in the following theorem.

THEOREM 3.1. *If hypotheses (H_1) and (H_3) hold for (2.1), and $B^{1/2}(t)$ is a square root of $B(t)$ as in (H_3) , with inverse $B^{-1/2}(t)$, then under the transformation*

$$(3.1) \quad u(t) = B^{1/2}(t)u^0(t), \quad v(t) = B^{-1/2}(t)v^0(t),$$

the system (2.1) is transformed into the system

$$(3.2) \quad \begin{aligned} L_1^0[u^0, v^0](t) &\equiv -v^{0'}(t) + C^0(t)u^0(t) - D^0(t)v^0(t) = 0, \\ L_2^0[u^0, v^0](t) &\equiv u^{0'}(t) - A^0(t)u^0(t) - v^0(t) = 0, \end{aligned}$$

where

$$(3.3) \quad \begin{aligned} A^0 &= B^{-1/2}\{AB^{1/2} - [B^{1/2}]'\}, \quad C^0 = B^{1/2}CB^{1/2}, \\ D^0 &= [-(B^{1/2})' + B^{1/2}D]B^{-1/2}. \end{aligned}$$

Now let the matrix functions $A_1(t)$ and $A_2(t)$ be defined as

$$(3.4) \quad \begin{aligned} A_1(t) &= \frac{1}{2}[A^0(t) + A^{0*}(t)] = \Re(A^0(t)), \\ A_2(t) &= \frac{1}{2}[A^0(t) - A^{0*}(t)] = i \Im A^0(t), \end{aligned}$$

so that $A^0(t) = A_1(t) + A_2(t)$ and $A^{0*}(t) = A_1(t) - A_2(t)$. We shall also assume that we have satisfied the further differentiability condition

(H_4) $A_1(t) = \Re A^0(t)$ is locally a.c. on I .

In particular, (H_4) holds when $A^0(t)$ is locally a.c. on I , and when in addition to the previous condition on $B(t)$ we have that $[B^{1/2}(t)]'$ is locally a.c. on I . In turn, this latter condition holds when $B(t)$ and the associated nonzero scalar function $\rho(t)$ appearing in the discussion

following (\mathbf{H}_3) are individually continuously differentiable on I , and the respective derivative matrix and scalar function are locally a.c. on I .

The following result may be verified directly.

LEMMA 3.1. *If hypotheses (\mathbf{H}_1) , (\mathbf{H}_3) and (\mathbf{H}_4) hold for the differential system (2.1), then under the transformation*

$$(3.5) \quad u^0(t) = u_1(t), \quad v^0(t) = v_1(t) - A_1(t)u_1(t),$$

the system (3.2) becomes

$$(3.6) \quad \begin{aligned} -v_1'(t) + C_1(t)u_1(t) - D_1(t)v_1(t) &= 0, \\ u_1'(t) - A_2(t)u_1(t) - v_1(t) &= 0, \end{aligned}$$

where

$$(3.7) \quad \begin{aligned} C_1(t) &= C^0(t) + A_1(t)A_2(t) + D^0(t)A_1(t) + A_1'(t), \\ D_1(t) &= D^0(t) - A_1(t). \end{aligned}$$

Now let $M(t)$ be the solution of the first order differential system

$$(3.8) \quad M'(t) - A_2(t)M(t) = 0, \quad M(t_0) = E,$$

where t_0 is some fixed point of I . Since $A_2^*(t) = -A_2(t)$, the solution $M(t)$ of (3.8) is unitary for $t \in I$, so that

$$M(t)M^*(t) \equiv E \equiv M^*(t)M(t).$$

Moreover, under the transformation

$$(3.9) \quad u_1(t) = M(t)u_2(t), \quad v_1(t) = M(t)v_2(t),$$

the system (3.6) becomes

$$(3.10) \quad \begin{aligned} -v_2'(t) + C_2(t)u_2(t) - D_2(t)v_2(t) &= 0, \\ u_2'(t) - v_2(t) &= 0, \end{aligned}$$

where the matrix functions $C_2(t)$ and $D_2(t)$ are given by

$$(3.11) \quad C_2 = M^*C_1M, \quad D_2 = M^*[A_2 + D_1]M = M^*[D^0 - A^{0*}]M.$$

In particular, $(u_2(t); v_2(t))$ is a solution of (3.10) if and only if $x(t) = u_2(t)$ is a solution of the linear homogeneous second order vector differential equation

$$(3.12) \quad x''(t) + D_2(t)x'(t) - C_2(t)x(t) = 0,$$

and $v_2(t) = x'(t)$.

Combining these results, we have the following theorem.

THEOREM 3.2. *If hypotheses (\mathbf{H}_1) , (\mathbf{H}_3) and (\mathbf{H}_4) hold, and $M(t)$ is the solution of the differential system (3.8), then $(u(t); v(t))$ is a solution of (2.1) if and only if*

$$(3.13) \quad x(t) = M^*(t)B^{-1/2}(t)u(t)$$

is a solution of (3.12), with $x'(t) = M^(t)[B^{1/2}v(t) + A_1(t)M(t)x(t)]$.*

From the form of the transformations occurring in (3.1), (3.5) and (3.9), it follows readily that distinct points t_1, t_2 on I are conjugate with respect to (2.1) if and only if these points are conjugate with respect to the system (3.6) and (3.9). Also, (2.1) is disconjugate on a subinterval I_0 if and only if the systems (3.6) and (3.9) are disconjugate on I_0 . In particular, from § 5 of Reid [4] we have the result of the following theorem. For systems (2.1) with real coefficient matrices, the conclusion of this theorem with $\lambda = 0$ is also a consequence of results of Hartman and Wintner [2].

THEOREM 3.3. *Suppose that hypotheses (\mathbf{H}_1) , (\mathbf{H}_3) and (\mathbf{H}_4) hold. Then (2.1) is disconjugate on I if for each compact subinterval $[a, b]$ of I there exists a corresponding real constant λ such that the self-adjoint equation*

$$(3.14) \quad [x'(t) + Q(t, \lambda)x(t)]' - [Q^*(t, \lambda)x'(t) + P(t, \lambda)x(t)] = 0$$

is disconjugate on this interval, where

$$(3.15) \quad \begin{aligned} P(t, \lambda) &= \Re C_2(t) + \lambda \Im C_2(t), \\ Q(t, \lambda) &= -\frac{1}{2}(1 + \lambda i)D_2^*(t). \end{aligned}$$

4. Self-adjoint differential systems. If in the original differential system (2.1) we have $D(t) \equiv A^*(t)$, then whenever hypotheses (\mathbf{H}_1) holds and $B(t)$ is positive definite and locally a.c. we have

$$D^0(t) \equiv A^{0*}(t)$$

in (3.3). Correspondingly, when also (\mathbf{H}_4) holds, in (3.7) we have $D_1 = D^0 - A_1 = A^{0*} - A_1 = -A_2$, and consequently $D_2(t) \equiv 0$ in (3.11), so that in this case (3.10) reduces to the system

$$(4.1) \quad \begin{aligned} -v_2'(t) + C_2(t)u_2(t) &= 0, \\ u_2'(t) - v_2(t) &= 0, \end{aligned}$$

and (3.11) becomes

$$(4.2) \quad x''(t) - C_2(t)x(t) = 0.$$

Finally, if we have the further condition that $C(t)$ is hermitian for

t on I , then one may verify readily that $C^0(t)$, $C_1(t)$ and $C_2(t)$ are also hermitian matrix functions on I . Consequently, for self-adjoint systems (2.1) we have the following result.

THEOREM 4.1. *Suppose that hypotheses (\mathbf{H}_1) and (\mathbf{H}_2) hold for (2.1), and that $B(t)$ is a continuously differentiable positive definite matrix function on I , such that $A(t)$ and $B'(t)$ are locally a.c. on I . If $M(t)$ is the solution of the corresponding system (3.8), then under the transformation*

$$(4.3) \quad \begin{aligned} u_2(t) &= M^*(t)B^{-1/2}(t)u(t), \\ v_2(t) &= M^*(t)[B^{1/2}(t)v(t) + A_1(t)B^{-1/2}(t)u(t)], \end{aligned}$$

where $A_1(t)$ is defined by (3.3), (3.4), the system (2.1) is reduced to the system (4.1), with $C_2(t)$ defined by (3.3), (3.7), (3.11).

Under the hypotheses of Theorem 4.1 it may be verified readily that $(u_\alpha(t); v_\alpha(t))$, $(\alpha = 1, 2)$, are conjoined solutions of (2.1) if and only if the associated $(u_{2\alpha}(t); v_{2\alpha}(t))$ defined for (4.3) are conjoined solutions of (4.1). In particular, if $(U(t); V(t))$ is a conjoined basis for (2.1) then the associated $(U_2(t); V_2(t))$ defined by (4.3) is a conjoined basis for (4.1), and the order of a value $t = t_0$ as a focal point of $(U(t); V(t))$ for (2.1) is the same as the order of that value as a focal point for the associated conjoined basis $(U_2(t); V_2(t))$ for (4.1). Consequently, results on disconjugacy and focal points for one of these differential systems are immediately translatable into results for the associated system. Under the assumption that the hypotheses of Theorem 4.1 hold, let $\Gamma(t; A, B, C)$ denote the corresponding matrix function $C_2(t)$ defined by (3.3), (3.7), (3.11). As examples of such results, we state the following, which are consequences of well-known oscillation and comparison theorems for general self-adjoint systems, (see, for example, Morse [3] and Reid [4, 6, 7]).

THEOREM 4.2. *Suppose that the hypotheses of Theorem 4.1 hold, and $[a, b]$ is a compact subinterval of I . Then the number of points $t_0 \in (a, b)$ which are conjugate to a , where each conjugate point is counted a number of times equal to its order, is equal to the (negative) index of the hermitian functional*

$$(4.4) \quad \int_a^b \{|\eta'(t)|^2 + \eta^*(t)\Gamma(t; A, B, C)\eta(t)\} dt$$

on the class $\mathcal{D}_0[a, b]$ of vector functions which are a.c. on $[a, b]$, with $\eta'(t) \in \mathfrak{R}_n^\infty[a, b]$, and $\eta(a) = \mathbf{0} = \eta(b)$. In particular, (2.1) is disconjugate on $[a, b]$ if and only if the functional (4.4) is positive definite on $\mathcal{D}_0[a, b]$.

THEOREM 4.3. *Suppose that $A^{(\alpha)}(t), B^{(\alpha)}(t), C^{(\alpha)}(t)$, $\alpha = 1, 2$, are triples of matrix functions which individually satisfy the conditions of Theorem 4.2, and for $\alpha = 1, 2$ let $(2.1)_\alpha$ denote the system (2.1) with $(A, B, C) = (A^{(\alpha)}, B^{(\alpha)}, C^{(\alpha)})$. If*

$$\Gamma(t; A^{(2)}, B^{(2)}, C^{(2)}) \geq \Gamma(t; A^{(1)}, B^{(1)}, C^{(1)})$$

for $t \in I$, and $(2.1)_1$ is disconjugate on a subinterval I_0 of I , then $(2.1)_2$ is also disconjugate on I_0 . Also, if $(U_\alpha(t); V_\alpha(t))$ is a conjoined basis for $(2.1)_\alpha$, and for $[a, b]$ a compact subinterval of I we have

$$(U_1(a); V_1(a)) = (U_2(a); V_2(a)),$$

then the number of focal points of $(U_1(t); V_1(t))$ on $[a, b]$ is not less than the number of focal points of $(U_2(t); V_2(t))$ on $[a, b]$.

It is to be noted that the above reduction of (2.1) to (4.1) is a generalization of a well-known reduction of the scalar differential equation

$$(4.5) \quad [r(t)u'(t)]' + p(t)u(t) = 0,$$

to the canonical form

$$(4.6) \quad y''(t) + \gamma(t)y(t) = 0,$$

where

$$(4.7) \quad \gamma(t) = \{[r'(t)]^2 - 2r(t)r''(t) + 4r(t)p(t)\} / \{4r^2(t)\}.$$

In connection with this particular result, the reader is referred to Birkhoff and Rota [1; p. 38, Exercises 4,6]. In particular, the comparison theorem on the distance between zeros of solutions of a pair of ordinary differential equations which Sather [8; § 5] deduced from properties of solutions of certain boundary problems for hyperbolic partial differential equations may be established readily with the aid of a preliminary translation of the involved intervals of existence, and a subsequent application of the most elementary comparison theorem of the classical Sturmian theory to the associated canonical equations (4.6).

REFERENCES

1. G. Birkhoff, and G.-C. Rota, *Ordinary Differential Equations*, Ginn and Company, 1959.
2. P. Hartman, and A. Wintner, *On disconjugate differential systems*, *Canad. J. Math.* **8** (1956), 72-81.
3. M. Morse, *A generalization of the Sturm separation and comparison theorems in n -space*, *Math. Ann.* **103** (1930), 52-69.

4. W. T. Reid, *Oscillation criteria for linear differential systems with complex coefficients*, Pacific J. Math. **6**(1956), 733-751.
5. ———, *A Prüfer transformation for differential systems*, Pacific J. Math. **8** (1958), 575-584.
6. ———, *Oscillation criteria for self-adjoint differential systems*, Trans. Amer. Math. Soc. **101** (1961), 91-106.
7. ———, *Principal solutions of nonoscillatory linear differential systems*, J. Math. Anal. Appl. **9** (1964), 397-423.
8. D. Sather, *Maximum and monotoneity properties of initial-boundary value problems for hyperbolic equations*, Pacific J. Math. **19** (1966), 141-157.

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