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SOME TOPOLOGICAL PROPERTIES WEAKER THAN COMPACTNESS

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Many topological properties may be described by covering relations which may also generally be easily described in terms of filter relations. For example, a space is compact if and only if each open cover of the space contains a finite subcover, or equivalently, if each filter has an adherent point. In this paper, characterizations are given of some topological properties weaker than compactness, both in terms of filters and coverings. In the final section a question posed by Viglino and by Dickman and Zame is answered.

2. Definitions and notations. (a) A space for which distinct points may be separated by disjoint closed neighborhoods (i.e., a Urysohn space) will be labeled a $T_{2(1/2)}$-*space*. Let $\nu = 2, 2\frac{1}{2}$, or 3. A T_ν-space is said to be T_ν-*closed* if it is closed in each T_ν-extension. A T_ν-space (X, τ) is said to be T_ν-*minimal* if there exists no T_ν topology on X strictly weaker than τ.

(b) A Hausdorff space (X, τ) is *C-compact* if given a closed set Q of X and a τ-open cover $\mathcal{O}$ of Q, then there exists a finite number of elements of $\mathcal{O}$, say 0_i, $1 \leqq i \leqq n$, with $Q \subset \mathrm{cl}_X \bigcup_{i=1}^n 0_i$.

(c) A Hausdorff space X is *functionally compact* if for every open filter $\mathcal{U}$ in X such that the intersection A of the elements of $\mathcal{U}$ equals the intersection of the closures of the elements of $\mathcal{U}$, then $\mathcal{U}$ is the neighborhood filter of A.

(d) A filter is *open* (closed) if it has a base of open (closed) sets. A *regular filter* is a filter which is both open and closed.

(e) Let A be a subset of a space X. An open cover, $\mathcal{U}$, of A will be said to be a *Urysohn cover* if for each $x \in A$ there exist elements $0_1, 0_2$ of $\mathcal{U}$ with $x \in 0_1 \subset \mathrm{cl}\, 0_1 \subset 0_2$.

(f) Let A be a subset of a space X. An open cover, $\mathcal{S}$, of A will be said to be a *strong cover* if for each $x \in A$ there exist $\{0_n\}_{n=1}^\infty \subset \mathcal{S}$ with $x \in 0_1$ and $\mathrm{cl}\, 0_i \subset 0_{i+1}$ for each i.

(g) A closed subset Y of a space X is *regular closed* if given $x \in X \setminus Y$, then there exists an open set 0 with $x \in 0 \subset \mathrm{cl}\, 0 \subset Y^c$.

3. Covering theorems. Filter characterizations for T_ν-closed and T_ν-minimal spaces are listed below. The proof of (a) may be found in [2]; (b) in [1]; (c), (d), and (e) in [5]; (f) in [4].

THEOREM I. (a) *A T_2-space is T_2-closed if and only if every* *open filter has an adherent point.*

(b) *A T_2-space is T_2-minimal if and only if every open filter with unique adherent point converges.*

(c) *A $T_{2(1/2)}$-space is $T_{2(1/2)}$-closed if and only if for each two open filters $\mathscr{F}_1$, $\mathscr{F}_3$ and each closed filter $\mathscr{F}_2$ such that $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3$, then $\mathscr{F}_1$ has an adherent point.*

(d) *A $T_{2(1/2)}$-space is $T_{2(1/2)}$-minimal if and only if for each two open filters $\mathscr{F}_1$, $\mathscr{F}_3$ and each closed filter $\mathscr{F}_2$ such that $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3$ with $\mathscr{F}_1$ having a unique adherent point, then $\mathscr{F}_3$ converges.*

(e) *A T_3-space is T_3-closed if and only if every regular filter has an adherent point.*

(f) *A T_3-space is T_3-minimal if and only if every regular filter with unique adherent point converges.*

Of the six properties listed in the above theorem, only the first has a well known covering characterization. (See Theorem II (a′), below). Herrlick has listed in [5] covering characterizations for $T_{2(1/2)}$-closed and T_3-closed spaces. We list in the following theorem covering characterizations for each of the six properties. These characterizations emphasize the relationship between T_ν-closed and T_ν-minimal spaces. The proof of each part of Theorem II follows from its counterpart in Theorem I. We offer proofs for parts (d′) and (f′).

THEOREM II. (a′) *A T_2-space X is T_2-closed if and only if given an open cover, $\mathscr{O}$, of X, then there exists $0_i \in \mathscr{O}$, $1 \leqq i \leqq n$, such that $X = \operatorname{cl} \bigcup_{i=1}^n 0_i$.*

(b′) *A T_2-space X is T_2-minimal if and only if given $p \in X$, an open cover, $\mathscr{O}$, of $X \backslash \{p\}$, and an open neighborhood U of p, then there exist $0_i \in \mathscr{O}$, $1 \leqq i \leqq n$, such that $X = U \cup \operatorname{cl} \bigcup_{i=1}^n 0_i$.*

(c′) *A $T_{2(1/2)}$-space X is $T_{2(1/2)}$-closed if and only if given a Urysohn cover, $\mathscr{U}$ of X, then there exist $0_i \in \mathscr{U}$, $1 \leqq i \leqq n$, such that $X = \operatorname{cl} \bigcup_{i=1}^n 0_i$.*

(d′) *A $T_{2(1/2)}$-space X is $T_{2(1/2)}$-minimal if and only if given $p \in X$, a Urysohn cover, $\mathscr{U}$ of $X \backslash \{p\}$ and an open neighborhood U of p, then there exist $0_i \in \mathscr{U}$, $1 \leqq i \leqq n$, such that $X = U \cup \operatorname{cl} \bigcup_{i=1}^n 0_i$.*

(e′) *A T_3-space X is T_3-closed if and only if given a strong cover, $\mathscr{S}$, of X, then there exist $0_i \in \mathscr{S}$, $1 \leqq i \leqq n$, such that $X = \bigcup_{i=1}^n 0_i$.*

(f′) *A T_3-space X is T_3-minimal if and only if given $p \in X$, a strong cover, $\mathscr{S}$, of $X \backslash \{p\}$, and an open neighborhood U of p, then there exist $0_i \in \mathscr{S}$, $1 \leqq i \leqq n$, such that $X = U \cup \bigcup_{i=1}^n 0_i$.*

Proof of (d′). Let $p \in X$, U an open neighborhood of p, and $\mathscr{U}$ a Urysohn cover of $X \backslash \{p\}$ such that the union of the closure of any finite number of elements of $\mathscr{U}$ fails to cover U^c. Since X is $T_{2(1/2)}$, we may assume that for each $x \in X \backslash \{p\}$ there exist $0_x^1, 0_x^3$ in $\mathscr{U}$ with $x \in 0_x^1 \subset \mathrm{cl}\, 0_x^1 \subset 0_x^3$, and $p \neq 0_x^3$. Let $\mathscr{F}_3$ denote the filter generated by $\overline{\{0_x^3\}}_{x \in X/\{p\}}$, $\mathscr{F}_1$ the filter generated by $\overline{\{0_x^{1c}\}}_{x \in X/\{p\}}$, and $\mathscr{F}_2$ the filter generated by $\overline{\overline{\{0_x^{3c}\}}}_{x \in X/\{p\}}$. Then $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3$ with p the only adherent point of $\mathscr{F}_1$ and $\mathscr{F}_3$ not converging.

Conversely, let $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3$ with $\mathscr{F}_1$ having a unique adherent point, p, and $\mathscr{F}_3$ not converging. Let U be an open neighborhood of p which contains no element of $\mathscr{F}_3$. Then, $\mathscr{U} = \{\bar{F}_1^c / F_1 \in \mathscr{F}_1\} \cup \{\bar{F}_3^c / F_3 \in \mathscr{F}_3\}$ is a Urysohn cover of $X \backslash \{p\}$ such that union of the closure of any number of elements of $\mathscr{U}$ fails to cover U^c.

Proof of (f′). Let $p \in X$, U an open neighborhood of p, and $\mathscr{S}$ a strong cover of $X \backslash \{p\}$ such that the union of any finite number of elements of $\mathscr{S}$ fails to cover U^c. Since X is T_3, we may assume that for each $x \in X \backslash \{p\}$ there exist a sequence $\{0_x^n\}_{n=1}^{\infty}$ of elements of $\mathscr{S}$ with $x \in 0_x^1$, $\mathrm{cl}\, 0_x^n \subset 0_x^{n+1}$ and $p \notin 0_x^n$ for any n. Now, the filter generated by $\{\overline{0_x^{nc}} / x \in X \backslash \{p\}; n = 1, 2, 3, \cdots\}$ is regular with unique adherent point, P, and does not converge.

Conversely, let $\mathscr{F}$ be a regular filter with unique adherent point, p. Let U be an open neighborhood of p which contains no element of $\mathscr{F}$. Then $\{\bar{F}^c / F \in \mathscr{F}\}$ is a strong cover of $X \backslash \{p\}$ and no finite union of elements of this cover contains U^c.

By replacing in Theorem II (b′) the point, p, by a closed (regular closed) set, we obtain the covering characterization for C-compact (functionally compact) spaces listed below. The proof follows easily from definition b (c).

THEOREM III. *A T_2-space X is C-compact (functionally compact) if and only if given a closed (regular closed) subset C of X, an open cover $\mathscr{O}$, of $X \backslash C$, and an open neighborhood U of C, then there exist $0_i \in \mathscr{O}$, $1 \leq i \leq n$, such that $X = U \cup \mathrm{cl} \bigcup_{i=1}^n 0_i$.*

4. A counterexample. One can easily show that every continuous function from a C-compact space into a Hausdorff space is closed [6]. The question as to whether or not the converse is valid was posed in [6]. Dickman and Zame [4] have since then shown that a necessary and sufficient condition that a Hausdorff space be functionally compact is that each continuous function of the space into a Hausdorff space be closed. We resolve the question posed in [6] by constructing, in the following example, a space which is functionally compact but not C-compact. The example is a modification of that given in [4] showing that a functionally compact space need not be compact.

EXAMPLE. Let $I = [0, 1]$. For each integer $n \geqq 2$, let $\{a_n^j\}_{j=1}^{\infty}$ be a strictly decreasing sequence in $(1/n, 1/n-1)$ converging to $1/n$. Let $X = I \setminus \bigcup_{n \geqq 2}^{j \geqq 1} \{a_n^j\}$. Topologize X as follows: Let $X \setminus (\{1/n\}_{n=1}^{\infty} \cup \{0\})$ retain the usual topology. Let a neighborhood system of the point 0 be composed of all sets of the form $\{x \in X / |x| < 1/m\} \setminus \{1/n\}_{n=1}^{\infty}$, m an integer. Let a neighborhood system of the point $1/n$ be composed of all sets of the form $0 \cap X$ where 0 is an open set in I with $\{1/n, a_{n-1}^1, a_{n-3}^2, \cdots, a_2^{n-1/2}\} \subset 0$ in the case that n is odd, and with $\{1/n, a_{n-1}^1, a_{n-3}^2, \cdots, a_3^{n/2-1}\} \subset 0$ in the case that n is even (where for $n = 2$ we simply have $\{1/2\}$). Clearly X is Hausdorff. Let $0_{2n} = \{x \in X / |x - 1/2n| < 1/3n\} \cup \bigcup_{i=1}^{n-1} \{x \in X / |x - a_{2n-2i+1}^i| < 1/3n\}$ for each $n > 1$. Then $\{0_{2n}\}_{n>1}$ is an open cover of the closed set $\{1/2n\}_{n>1}$ and the closure of any finite union of elements in $\{0_{2n}\}_{n>1}$ fails to contain $\{1/2n\}_{n>1}$. Hence, X is not C-compact. We apply Theorem III and show that X is functionally compact.

Let C be a regular closed subset of X, $\mathcal{O}$ an open cover of $X \setminus C$, U an open neighborhood of C. Suppose first that C contains infinitely many elements of $\{1/n\}_{n=1}^{\infty}$. Then by the regularity of C, $\{1/n\}_{n=1}^{\infty} \cup \{0\} \subset C$ so that $X \setminus U$ is compact. Suppose that $0 \in C$ and that only finitely many elements of $\{1/n\}_{n=1}^{\infty}$ are contained in C. Choose $1/2n$ and $1/2n+1$ such that neither is in C. Let $0_{2n}, 0_{2n+1}$ be elements of $\mathcal{O}$ containing $1/2n$ and $1/2n+1$ respectively. Then, $\{1/n / n \geqq k\} \subset \mathrm{cl}\,(0_{2n} \cup 0_{2n+1})$ for some k. It is easy to see that the closure of a finite number of elements of $\mathcal{O}$ contains $X \setminus (\mathrm{cl}\,(0_{2n} \cup 0_{2n+1}) \cup U)$. Suppose that $0 \notin C$ and that only finitely many elements of $\{1/n\}_{n=1}^{\infty}$ are contained in C. Once more let $0_{2n}, 0_{2n+1}$ be elements of $\mathcal{O}$ containing $1/2n$ and $1/2n+1$ respectively, with neither $1/2n$ nor $1/2n+1$ in C. Let 0_i be an element of $\mathcal{O}$ containing 0. It is easy to see that the closure of a finite number of elements of $\mathcal{O}$ contains $X \setminus (\mathrm{cl}\,(0_i \cup 0_{2n} \cup 0_{2n+1}) \cup U)$. Hence, X is functionally compact.

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