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AN ELEMENTARY PROOF OF THE UNIQUENESS OF THE FIXED POINT INDEX

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In 1953, Barrett O'Neill stated axioms for a "fixed point index" and obtained existence and uniqueness theorems for the index on finite polyhedra. The proof of uniqueness consisted of showing that any function which satisfied the axioms must agree with the index he had already defined. This paper presents a proof of the uniqueness of the fixed point index on finite polyhedra which depends only on the axioms and therefore is "elementary" in the sense that it is independent of the existence of an index. The proof is "elementary" also in that all the techniques used are taken from geometric topology or calculus so that, in particular, no algebraic topology is required. An elementary proof of the uniqueness of the fixed point index on compact metric absolute neighborhood retracts is an immediate consequence of the material in this paper.

1. The axioms. Let $\mathscr{C}$ be a collection of topological spaces. Denote by $\mathscr{C}'$ the collection of all triples (X, f, U) where X is in $\mathscr{C}$, $f: X \to X$ is a map, and U is an open subset of X such that there are no fixed points of f on the boundary of U.

A *fixed point index on* $\mathscr{C}$ is a function $i: \mathscr{C}' \to Z$ (the integers) such that

I. (*Localization*). If $(X, f, U) \in \mathscr{C}'$ and $g: X \to X$ is a map such that $g(x) = f(x)$ for all x in the closure of U, then $i(X, f, U) = i(X, g, U)$.

II. (*Homotopy*). Given a map $H: X \times I \to X$, define $h_t: X \to X$, for $t \in I = [0, 1]$, by $f_t(x) = H(x, t)$. If $(X, h_t, U) \in \mathscr{C}'$ for all $t \in I$, then $i(X, h_0, U) = i(X, h_1, U)$.

III. (*Additivity*). If $(X, f, U) \in \mathscr{C}'$ and $U_1, \cdots, U_s$ is a set of mutually disjoint open subsets of U such that $f(x) \neq x$ for all $x \in [U - \bigcup_{j=1}^{s} U_j]$, then $i(X, f, U) = \sum_{j=1}^{s} i(X, f, U_j)$.

IV. (*Weak Normalization*). If $(X, f, U) \in \mathscr{C}'$ where f is the constant map such that $f(X) = x_0 \in U$, then $i(X, f, U) = 1$.

V. (*Commutativity*). If X and X' are in $\mathscr{C}$ and $f: X \to X'$, $g: X' \to X$ are maps such that $(X, gf, U) \in \mathscr{C}'$ then $i(X, gf, U) = i(X', fg, g^{-1}(U))$.

The axiom list above is a modified version of the one used by Browder in [1]. It differs somewhat from O'Neill's original list [5].

2. Proof of uniqueness. Given a space X and a function $f: X \to R^n$ (euclidean n-dimensional space), we will always denote by the corresponding capital letter $F: X \to R^n$ the function such that $F(x) = x - f(x)$ for all $x \in X$. Let O be the origin in R^n.

LEMMA 1. *If U is an open subset of R^n containing O and f: $U \to R^n$ is a linear function whose only fixed point is O, then $\partial F(O)/\partial(x_1, \cdots, x_n)$ (the Jacobian of F at O) is not zero.*

Proof. Suppose that $\partial F(O)/\partial(x_1, \cdots, x_n) = 0$. Since f is linear on U and $f(O) = O$, there exists a linear transformation $g: R^n \to R^n$ such that $g(x) = f(x)$ for all $x \in U$. We observe that $G: R^n \to R^n$ is also a linear transformation so, letting dG denote the differential of G, we have $dG(x) = G$ for all $x \in R^n$. Since G is identical with F in a neighborhood of O, the determinant of G is

$$\det(G) = \frac{\partial G(O)}{\partial(x_1, \cdots, x_n)} = \frac{\partial F(O)}{\partial(x_1, \cdots, x_n)} = 0$$

and G is a singular linear transformation. Therefore, there exists a vector subspace S of R^n, of dimension at least one, such that $G(S) = O$. Thus, for $x \in S \cap U$, we have

$$0 = G(x) = x - g(x) = x - f(x)$$

which means that the points of $S \cap U$ are fixed points of f, in contradiction to the hypothesis that O is the only fixed point.

For $r \neq 0$ a real number, define $\sigma(r) = 1$ if $r > 0$ and $\sigma(r) = -1$ if $r < 0$.

If $f: X \to X$ is a map, x is a fixed point of f, and there exists an open set V in X containing x such that the closure of V contains no other fixed point of f, then define the *index of f at x* by $i(X, f, x) = i(X, f, V)$. The additivity axiom implies that the definition is independent of the choice of V.

THEOREM 2. *Let $O \in U \subset X \subset R^n$ where U is open in R^n and X is a finite polyhedron. If $f: X \to X$ is of class C^1 on $U, f(O) = O$, and $\partial F(O)/\partial(x_1, \cdots, x_n) \neq 0$, then $i(X, f, O) = \sigma(\partial F(O)/\partial(x_1, \cdots, x_n))$ for any index i on finite polyhedra.*

The proof of this theorem, using only the axioms, is given in the next section.

Let K be a finite simplicial complex and let $|K|$ denote its geometric realization. A pair $T = (K, \tau)$, where $\tau = |K| \to X$ is a homeomorphism, is a *triangulation* of the polyhedron X. A *simplicial map*

$g: (X, T_1) \to (X, T_2)$ is a map induced by a simplicial function from K_1 to K_2. A *maximal simplex* of $T = (K, \tau)$ is a subset $\tau|s|$ of X where $|s|$ is an open simplex of $|K|$ which is not a face of any other simplex.

THEOREM 3 (*Hopf* [3]). *Let X be a finite polyhedron and $f: X \to X$ a map. Given $\varepsilon > 0$ there exist triangulations T_1 and T_2 of X and a simplicial approximation $g: (X, T_1) \to (X, T_2)$ to f, whose distance from f is less than ε, such that all fixed points of g are in maximal simplices of T_1.*

We are now able to prove the uniqueness of the fixed point index on finite polyhedra.

Let $\mathscr{C}$ be the collection of all finite polyhedra and suppose that $i: \mathscr{C}' \to Z$ is a fixed point index. Given $(X, f, U) \in \mathscr{C}'$, we will show how to compute $i(X, f, U)$ directly from the axioms and that will imply that the index, if it exists, is unique.

Since the boundary of U is compact and contains no fixed point of f, there exists $\varepsilon > 0$ such that, for all x in the boundary of U, the distance from x to $f(x)$ is greater than ε. Applying Theorem 3, we obtain a simplicial approximation $g: (X, T_1) \to (X, T_2)$ to f. Recall that consequently there is a homotopy $h_t: X \to X$, $t \in I$, along line segments in X such that $h_0 = f$ and $h_1 = g$. Since the distance from f to g is less than ε, the distance from f to each h_t is less than ε so $(X, h_t, U) \in \mathscr{C}'$ and, by the homotopy axiom, $i(X, f, U) = i(X, g, U)$.

If a single maximal simplex of T_1 contained two fixed points of g, then since g is linear on each closed simplex of T_1, g would be the identity map on the line segment (in the closure of the maximal simplex) which is determined by the two points. But the segment would intersect the boundary of the simplex, which is a union of nonmaximal simplices. Thus there would be a nonmaximal simplex of T_1 containing a fixed point of g, contrary to Theorem 3. Each maximal simplex of T_1, therefore, contains at most one fixed point of g and g has only a finite number of fixed points. Let $x_1, \cdots, x_r$ be the fixed points of g in U, then, by the additivity axiom, $i(X, g, U) = \sum_{j=1}^{r} i(X, g, x_j)$.

Therefore, we need only consider the following situation. We have an open subset $\tau_1|s|$ of X homeomorphic to R^n for some $n \geqq 1$ (because $\tau_1|s|$ is a maximal simplex of T_1) and a map $g: X \to X$ which is linear on $\tau_1|s|$ and which has a single fixed point x in $\tau_1|s|$. Let $\bar{V}$ be a closed n-cell in $\tau_1|s|$ such that $x \in V$ and $g(\bar{V}) \subset \tau_1|s|$. Choose a closed n-cell Y in $\tau_1|s|$ containing $\bar{V} \cup g(\bar{V})$. There are retractions $\rho_\alpha: Y \to \bar{V}$ and $\rho_\beta: X \to Y$. Let $\alpha = g\rho_\alpha: Y \to Y$ and $\beta = \alpha\rho_\beta: X \to X$. The localization axiom tells us that

$$i(X, g, x) = i(X, g, V) = i(X, \beta, V) .$$

Furthermore, for $j: Y \rightarrow X$ the inclusion map, the commutativity axiom implies

$$\begin{aligned} i(X, \beta, V) &= i(X, j\alpha\rho_\beta, V) \\ &= i(Y, \alpha\rho_\beta j, j^{-1}(V)) \\ &= i(Y, \alpha, V) . \end{aligned}$$

Observe that $\alpha(x) = g(x)$ for all $x \in V$, so α is linear on Λ.

There is a linear homeomorphism h taking $\tau_1|s|$ onto an open subset of R^n such that $h(x) = \boldsymbol{O}$. Let $a = h\alpha h^{-1}: h(Y) \rightarrow h(Y)$ then, by the commutativity axiom again,

$$i(Y, \alpha, V) = i(h(Y), a, h(V)) = i(h(Y), a, \boldsymbol{O}) .$$

Since a is linear on $h(V)$, $(\partial A(\boldsymbol{O})/(\partial(x_1, \cdots, x_n)) \neq 0$ by Lemma 1 and so, by Theorem 2,

$$i(X, g, x) = i(h(Y), a, \boldsymbol{O}) = \sigma\Big(\frac{\partial A(\boldsymbol{O})}{\partial(x_1, \cdots, x_n)}\Big) .$$

Therefore, we will have completed our elementary proof of the uniqueness of the fixed point index on finite polyhedra once we prove Theorem 2 directly from the axioms in the next section.

We have not only proved uniqueness but also, in the process, constructed a candidate for a fixed point index. Given $(X, f, U) \in \mathscr{C}'$, we have the map g as above with fixed points $x_1, \cdots, x_r$ in U and we could define

$$i(X, f, U) = \sum_{j=1}^{r} \sigma\Big(\frac{\partial A_j(\boldsymbol{O})}{\partial(x_1, \cdots, x_{n(j)})}\Big)$$

where A_j is defined like the map A above and $n(j)$ is the dimension of the maximal simplex of T_1 containing x_j. However, we would still be required to verify that this function i really satisfies the axioms. Since an elegant proof of the existence of a fixed point index on finite polyhedra is given in the first two sections of [2], such a verification would appear to be of little interest.

We note that our uniqueness proof together with the proof of Lemma 0 of [1] constitutes an elementary proof of the uniqueness of the fixed point index on the collection of all compact metric absolute neighborhood retracts.

3. Proof of Theorem 2. The result we are setting out to prove is not new (see [4, p. 214]). What is new, however, is a proof which depends only on the axioms for a fixed point index. The first step is a proof of Theorem 2 in the case $n = 1$.

LEMMA 4. *If* $0 \in (a, b) \subset R^1$, $f: [a, b] \to [a, b]$ *is differentiable at* $0, f(0) = 0$ *and* $dF(0)/dx > 0$, *then there exists* $\varepsilon > 0$ *and a homotopy* $h_t: [a, b] \to [a, b]$, $t \in I$, *such that* $h_0 = f$, h_1 *maps* $[a, b]$ *to* 0 *and* $h_t(\varepsilon) \neq \varepsilon$, $h_t(-\varepsilon) \neq -\varepsilon$ *for all* $t \in I$.

Proof. Since $dF(0)/dx > 0$ then $df(0)/dx < 1$ so there exists $\varepsilon > 0$ such that $0 < |x| \leqq \varepsilon$ implies $(f(x) - f(0))/x < 1$. Therefore, $f(x) < x$ if $0 < x \leqq \varepsilon$ and $f(x) > x$ if $-\varepsilon \leqq x < 0$. Define $h_t(x) = (1 - t)f(x)$ for $x \in [a, b]$.

LEMMA 5. *Let* $0 \in (a, b) \subset R^1$, $f: [a, b] \to [a, b]$ *be a map differentiable at* $0, f(0) = 0$ *and* $dF(0)/dx \neq 0$. *If* $i: \mathscr{C}' \to Z$ *is any index on finite polyhedra, then* $i([a, b], f, 0) = \sigma(dF(0)/dx)$.

Proof. If $dF(0)/dx > 0$, then using Lemma 4 and the homotopy axiom,

$$\begin{aligned} i([a, b], f, 0) &= i([a, b], f, (-\varepsilon, \varepsilon)) \\ &= i([a, b], h_1, (-\varepsilon, \varepsilon)) \end{aligned}$$

where h_1 is the constant map at 0 so $i([a, b], h_1, (-\varepsilon, \varepsilon)) = 1$ by the weak normalization axiom. If $dF(0)/dx < 0$ then there exists $\varepsilon > 0$ such that $f(-\varepsilon) < -\varepsilon$ and $f(\varepsilon) > \varepsilon$. Define a map $g: [a, b] \to [a, b]$ as follows. If $x \in [-\varepsilon, \varepsilon]$, let $g(x) = f(x)$. Choose $c \in (\varepsilon, f(\varepsilon))$, define $g(f(\varepsilon)) = c$ and extend g linearly over $[\varepsilon, f(\varepsilon)]$. Finally, extend g over $[a, b]$. By the additivity axiom

$$i([a, b], g, (-\varepsilon, f(\varepsilon))) = i([a, b], g, (-\varepsilon, \varepsilon)) + i([a, b], g, (\varepsilon, f(\varepsilon)) .$$

Since $g(\varepsilon) > \varepsilon$ and $g(f(\varepsilon)) < f(\varepsilon)$, an argument like the one in Lemma 4 and the first part of this proof shows that $i([a, b], g, (\varepsilon, f(\varepsilon))) = 1$. Next observe that $g(-\varepsilon) < -\varepsilon$ and $g(f(\varepsilon)) < f(\varepsilon)$ so, choosing $e \in (g(-\varepsilon), -\varepsilon)$, the homotopy $h_t(x) = (1 - t)g(x) + te$ for $x \in [a, b]$ has the properties $h_0 = g$, h_1 is the constant map at e, and $h_t(-\varepsilon) \neq -\varepsilon$, $h_t(f(\varepsilon)) \neq f(\varepsilon)$ for all $t \in I$. Therefore, by the homotopy axiom,

$$i([a, b], g, (-\varepsilon, f(\varepsilon))) = i([a, b], h_1, (-\varepsilon, f(\varepsilon))) .$$

However, h_1 has no fixed points on $[-\varepsilon, f(\varepsilon)]$ so, by the additivity axiom, $i([a, b], h_1, (-\varepsilon, f(\varepsilon))) = 0$. Consequently, $i([a, b], g, (-\varepsilon, \varepsilon)) = -1$ and, by the localization axiom, $i([a, b], f, (-\varepsilon, \varepsilon)) = -1$.

LEMMA 6. *If, for* $j = 1, 2, 0 \in (a_j, b_j) \subset R^1$, $f_j: [a_j, b_j] \to [a_j, b_j]$ *is differentiable at* $0, f_j(0) = 0$ *and* $dF_j(0)/dx < 0$, *then there exists* $\varepsilon > 0$

and a homotopy $h_t\colon [a_1, b_1] \times [a_2, b_2] \to [a_1, b_1] \times [a_2, b_2]$ *such that* $h_0(x, y) = (f_1 \times f_2)(x, y) = (f_1(x), f_2(y))$, h_1 *is the constant map at* $\boldsymbol{O}$, *and* $h_t(p) \neq p$ *for all* $t \in I$ *and all* p *on the boundary of* $V = [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$.

Proof. The hypothesis $dF_j(0)/dx < 0$ implies that there exists $\varepsilon > 0$ such that $f_j(x) > x$ for $0 < x \leqq \varepsilon$ and $f_j(x) < x$ when $-\varepsilon \leqq x < 0$. Let B_ε denote the closed ball in R^2 of radius ε centered at $\boldsymbol{O}$ and let S_ε be the boundary of B_ε. Define $r\colon V \to B_\varepsilon$ to be the retraction such that if $p \notin B_\varepsilon$ then $r(p)$ is the point where the ray from $\boldsymbol{O}$ through p intersects S_ε. Define $s\colon B_\varepsilon \to V$ to be radial projection. If $p \in B_\varepsilon$ then $p = \alpha e^{i\theta}$ for some $\alpha \in [0, \varepsilon]$ and $\theta \in [0, 2\pi)$. For a real number t, define $\rho_t\colon B_\varepsilon \to B_\varepsilon$ to be the rotation $\rho_t(\alpha e^{i\theta}) = \alpha e^{i(\theta + t\pi)}$. Finally, define $h_t\colon V \to [a_1, b_1] \times [a_2, b_2]$ as follows

$$h_t(p) = \begin{cases} (f_1 \times f_2) s \rho_t r(p) & 0 \leqq t \leqq 1/2 \\ 2(1-t) h_{1/2}(p) & 1/2 \leqq t \leqq 1 \end{cases}$$

and extend h_t to $h_t\colon [a_1, b_1] \times [a_2, b_2] \to [a_1, b_1] \times [a_2, b_2]$ so that $h_0 = f_1 \times f_2$ and h_1 is constant.

LEMMA 7. *Let* X, Y, *and* $X \times Y$ *be spaces in a collection* $\mathscr{C}$ *and let* $i\colon \mathscr{C}' \to Z$ *be an index. If* $f\colon X \to X$ *is a map with an isolated fixed point at* x_0 *and* $k\colon Y \to Y$ *is the constant map such that* $k(Y) = y_0$, *then* $i(X \times Y, f \times k, (x_0, y_0)) = i(X, f, x_0)$.

Proof. Define $\pi\colon X \times Y \to X$ by $\pi(x, y) = x$ and $j\colon X \to X \times Y$ by $j(x) = (x, y_0)$. Let U be an open subset of X containing x_0 whose closure contains no other fixed point of f. By the commutativity axiom,

$$\begin{aligned} i(X \times Y, f \times k, (x_0, y_0)) &= i(X \times Y, f \times k, U \times Y) \\ &= i(X \times Y, j\pi(f \times k), U \times Y) \\ &= i(X, \pi(f \times k)j, j^{-1}(U \times Y)) \\ &= i(X, f, U) = i(X, f, x_0)\ . \end{aligned}$$

Next we prove Theorem 2 for a special kind of map.

LEMMA 8. *For* $j = 1, \cdots, n$, *let* $0 \in (a_j, b_j) \subset R^1$, $f_j\colon [a_j, b_j] \to [a_j, b_j]$ *be maps differentiable at* 0, $f_j(0) = 0$, *and* $dF_j(0)/dx \neq 0$. *Let* $W = \prod_{j=1}^{n} [a_j, b_j] \subset R^n$ *and* $f = f_1 \times \cdots \times f_n$, *then*

$$i(W, f, \boldsymbol{O}) = \sigma\left(\frac{\partial F(\boldsymbol{O})}{\partial(x_1, \cdots, x_n)}\right)$$

where $i\colon \mathscr{C}' \to Z$ *is any index on finite polyhedra.*

Proof. Certainly,

$$\frac{\partial F(\boldsymbol{O})}{\partial(x_1, \cdots, x_n)} = \frac{dF_1(0)}{dx} \cdots \frac{dF_n(0)}{dx} .$$

By the commutativity axiom, $i(W, f, \boldsymbol{O})$ is independent of the ordering of the f_j's, so assume $dF_j(0)/dx$ is negative for $j = 1, \cdots, m$ and positive for $j = m + 1, \cdots, n$ (where it may be that $m = 0$ or $m = n$). Then we see that $\sigma(\partial F(\boldsymbol{O})/\partial(x_1, \cdots, x_n)) = (-1)^m$. Write $m = 2r + \delta$ where $\delta = 0$ or $\delta = 1$. Applying Lemma 4 $n - m$ times and Lemma 6 r times, we obtain $\varepsilon > 0$ and a homotopy $h_t: W \to W, t \in I$, such that $h_0 = f, h_1$ is the constant map taking W to $\boldsymbol{O} \in R^n$ if $\delta = 0$, $h_1 = f_1 \times k$ where k maps $\prod_{j=2}^n [a_j, b_j]$ to $\boldsymbol{O} \in R^{n-1}$ if $\delta = 1$, and $h_t(p) \neq p$ p on the boundary of $[-\varepsilon, \varepsilon] \times \cdots \times [-\varepsilon, \varepsilon] \subset R^n$. Thus, if m is even,

$$i(W, f, \boldsymbol{O}) = i(W, h_1, \boldsymbol{O}) = 1$$

by the weak normalization axiom and, if m is odd,

$$\begin{aligned} i(W, f, \boldsymbol{O}) &= i(W, f_1 \times k, \boldsymbol{O}) \\ &= i([a_1, b_1], f_1, 0) \\ &= -1 \end{aligned}$$

by Lemmas 5 and 7. In either case, $i(W, f, \boldsymbol{O}) = (-1)^m$ which completes the proof.

LEMMA 9. *Suppose $0 \in (a_j, b_j) \subset R^1$ for $j = 1, \cdots, n$ and let $W = \prod_{j=1}^n [a_j, b_j]$. Let $f: W \to W$ be a map such that $f(\boldsymbol{O}) = \boldsymbol{O}, f$ is of class C^1 on a neighborhood of $\boldsymbol{O}$ and $\partial F(\boldsymbol{O})/\partial(x_1, \cdots, x_n) \neq 0$. There exists a homotopy $h_t: W \to W$ and an open subset V of W containing $\boldsymbol{O}$ such that $h_0 = f$, $h_1 = g_1 \times \cdots \times g_n$ where $g_j: [a_j, b_j] \to [a_j, b_j]$, $h_t(\boldsymbol{O}) = \boldsymbol{O}$, $J(t): I \to R^1$ defined by $J(t) = \partial H_t(\boldsymbol{O})/\partial(x_1, \cdots, x_n)$ is a continuous non-vanishing function and H_t is one-to-one on V, for all $t \in I$.*

Since this result is stated without proof in [4, p. 215] one must assume that it is a well-known fact. However, for the convenience of the reader, a proof is given in the appendix below.

We can now prove Theorem 2. Choose $0 \in (a_j, b_j) \subset R^1$ so that $W = \prod_{j=1}^n [a_j, b_j]$ is contained in the open set U. Let $Y = f^{-1}(W) \cap W$, then there is an open subset of R^n containing $\boldsymbol{O}$ in Y. Extend $f|Y$, the restriction of f to Y, to a map $k: W \to W$. By Lemma 9, there is a homotopy $h_t: W \to W$ such that $h_0 = k$, $h_1 = g_1 \times \cdots \times g_n$ where $g_j: [a_j, b_j] \to [a_j, b_j]$, $h_t(\boldsymbol{O}) = \boldsymbol{O}$ and H_t is one-to-one on a neighborhood V of $\boldsymbol{O}$, for all $t \in I$. Therefore $\boldsymbol{O}$ is the only fixed point of each map h_t on V and, by the homotopy axiom, $i(W, k, \boldsymbol{O}) = i(W, h_1, \boldsymbol{O})$.

Applying Lemma 8, $i(W, h_1, O) = \sigma(\partial H_1(O)/\partial(x_1, \cdots, x_n))$. Lemma 9 also states that $J(t) = \partial H_t(O)/\partial(x_1, \cdots, x_n)$ is a continuous nonvanishing function of t so $\sigma(\partial H_1(O)/\partial(x_1, \cdots, x_n)) = \sigma(\partial H_0(O)/\partial(x_1, \cdots, x_n))$. But $h_0 = k$ and k agrees with f near O, therefore $\sigma(\partial H_0(O)/\partial(x_1, \cdots, x_n)) = \sigma(\partial F(O)/\partial(x_1, \cdots, x_n))$. On the other hand, if we extend k to a map $k: X \rightarrow X$, then the commutativity axiom implies that $i(X, k, O) = i(W, k, O)$. Applying the localization axiom, $i(X, f, O) = i(X, k, O)$. We have proved that $i(X, f, O) = \sigma(\partial F(O)/\partial(x_1, \cdots, x_n))$.

Appendix. Proof of Lemma 9. Write

$$f(x_1, \cdots, x_n) = (f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n))$$

where $f_j: W \rightarrow [a_j, b_j]$. Let $S(n)$ denote the symmetric group on n symbols. For $\varphi \in S(n)$ define $h_t^\varphi: W \rightarrow W$ by

$$h_{1-t}^\varphi(x_1, \cdots, x_n) = (f_1(tx^{\varphi(1)}), \cdots, f_n(tx^{\varphi(n)}))$$

where

$$tx^{\varphi(j)} = (tx_1, \cdots, tx_{\varphi(j)-1}, x_{\varphi(j)}, tx_{\varphi(j)+1}, \cdots, tx_n) \in W .$$

Let $F_{jk} = -\partial f_j(O)/\partial x_k$ if $j \neq k$ and let $F_{jj} = 1 - \partial f_j(O)/\partial x_j$ then

$$\frac{\partial H_{1-t}^\varphi(O)}{\partial(x_1, \cdots, x_n)} = \pi(\varphi) F_{1\varphi(1)} \cdots F_{n\varphi(n)} + \sum_{\psi \neq \varphi} \pi(\psi) t^n F_{1\psi(1)} \cdots F_{n\psi(n)}$$

where $\pi(\varphi)$ is 1 if φ is an even permutation and -1 if φ is odd. It is clear that $h_0^\varphi = f$ and that $h_1^\varphi = g_1 \times \cdots \times g_n$ where

$$g_j(x_j) = f_k(0, \cdots, 0, x_j, 0, \cdots, 0)$$

and $\varphi(k) = j$. Also $J^\varphi(t) = \partial H_t^\varphi(O)/\partial(x_1, \cdots, x_n)$ is certainly a continuous function of t.

However, there is a problem because there may exist $t_\varphi \in I$ such that $J^\varphi(t_\varphi) = 0$. Observe first that the hypothesis $\partial F(O)/\partial(x_1, \cdots, x_n) \neq 0$ implies that $t_\varphi < 1$. Next note that we have in fact defined $n!$ different homotopies h_t^φ, one for each element of $S(n)$. The question is whether there exist $n!$ numbers t_φ, $0 \leqq t_\varphi < 1$, such that $J^\varphi(t_\varphi) = 0$. We claim that there do not.

The system of equations $J^\varphi(t_\varphi) = 0, \varphi \in S(n)$, can be written in the following form. Order the elements of $S(n)$ and let $\mathfrak{F} = (F_1, \cdots, F_{n!})$ where $F_j = F_{1\varphi(1)} \cdots F_{n\varphi(n)}$ and φ is the j-th element of $S(n)$. Also write $t_\varphi = t_j$ in this case. The system can be written $T\mathfrak{F} = O$ where

$$T = T(t_1, \cdots, t_n) = \begin{bmatrix} 1 & t_1 & \cdots & t_1 \\ t_2 & 1 & \cdots & t_2 \\ \vdots & \vdots & & \vdots \\ t_{n!} & t_{n!} & \cdots & 1 \end{bmatrix} .$$

Since $\partial F(\boldsymbol{O})/\partial(x_1, \cdots, x_n) \neq 0$, it must be that $\mathfrak{F} \neq \boldsymbol{O}$ so there is a solution to $T\mathfrak{F} = \boldsymbol{O}$ if and only if $\det(T) = 0$.

Observe that since all $t_j < 1$ then

$$\det T(1, t_2, \cdots, t_{n!}) = \prod_{j=2}^{n!} (1 - t_j) > 0 .$$

If we write

$$\det T(1, t_2, \cdots, t_{n!}) = \det T(t_2, \cdots, t_{n!}) - B_{n!}$$

then

$$B_{n!} = t_2(\det T(1, t_3, \cdots, t_{n!})) + \cdots + t_{n!}(\det T(1, t_2, \cdots, t_{n!-1}))$$

so $B_{n!} \geqq 0$. Now

$$\det(T) = \det T(t_2, \cdots, t_{n!}) - t_1 B_{n!}$$

so since $0 \leqq t_j < 1$ for $j = 2, \cdots, n!$ then $\det(T) = 0$ implies $B_{n!} \neq 0$ and

$$t_1 = \frac{\det T(t_2, \cdots, t_{n!})}{B_{n!}} > 1$$

which establishes a contradiction and verifies the claim.

Thus there exists the required homotopy h_t so that $\partial H_t(\boldsymbol{O})/\partial(x_1, \cdots, x_n) \neq 0$ for all $t \in I$. It remains to find a single neighborhood V of $\boldsymbol{O}$ on which each H_t is one-to-one. By hypothesis there exists $\delta > 0$ such that if $p \in R^n$ and $|p| \leqq \delta$ then f is of class C^1 at p. Let $B_\delta = \{p \in R^n \,|\, |p| \leqq \delta\}$ and define $D: B_\delta \times \cdots \times B_\delta \times I \to R^1$ by

$$D(p_1, \cdots, p_n, t) = \det \begin{bmatrix} \dfrac{\partial H_t(p_1)}{\partial x_1} & \cdots & \dfrac{\partial H_t(p_1)}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial H_t(p_n)}{\partial x_1} & \cdots & \dfrac{\partial H_t(p_n)}{\partial x_n} \end{bmatrix} .$$

Note that $D(\boldsymbol{O} \times \cdots \times \boldsymbol{O} \times I) \subset R^1 - 0$. Let C be the component of $D^{-1}(R^1 - 0)$ containing $\boldsymbol{O} \times \cdots \times \boldsymbol{O} \times I$, then there exists $\varepsilon > 0$ such that $|p_j| < \varepsilon$ for $j = 1, \cdots, n$ implies $(p_1, \cdots, p_n, t) \in C$ for all $t \in I$. Let $V = \{p \in R^n \,|\, |p| < \varepsilon\}$ then the Inverse Function Theorem tells us that each H_t is one-to-one on V.

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