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ON MEARLY COMMUTATIVE DEGREE ONE ALGEBRAS

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The main result in this paper establishes that there do not exist nodal algebras A satisfying the conditions:
(I) $x(xy) + (yx)x = 2(xy)x$
(II) $(xy)x - x(yx)$ **is in** N**, the set of nilpotent elements of** A **over any field** F **whose characteristic is not two.**

Recall that a finite dimensional, power-associative algebra A with identity 1 over a field F is called a nodal algebra if every x in A is of the form $x = \alpha 1 + n$ with α in F and n nilpotent, and if the set N of nilpotent elements of A does not form a subalgebra of A. Following the convention laid down in [5] we call any ring satisfying (I) a nearly commutative ring.

In a recent paper [4] one of the authors has established the results given here if the field F has characteristic zero. In that paper the theorem of Albert [1] that there do not exist commutative, power-associative nodal algebras over fields of characteristic zero was used extensively. Recently, Oehmke [3] proved the same result if the field has characteristic $P \neq 2$. This result of Oehmke's will be used throughout this paper.

The known class of nodal algebras over fields of characteristic P are the truncated polynomial algebras of Kokoris [2] which are flexible. Our results show that if nearly commutative nodal algebras exist over fields of characteristic P they will not fall into the class of Kokoris algebras. In [5] one of the authors has shown that there do not exist nearly commutative nodal algebras over fields of characteristic zero.

Let A be a nearly commutative nodal algebra over a field F whose characteristic is $P \neq 2$. Then A^+ is a commutative, power-associative algebra over F. Therefore by [3] N^+ is a subalgebra of A^+. In particular, N is a subspace of A. The nilindex of A is defined to be the least positive integer k such that $n^k = 0$ for every n in N.

LEMMA 1. *There do not exist any nearly commutative nodal algebras whose nilindex is two over any field of characteristic* $P \neq 2$.

Proof. Let A be such an algebra. Then since $z^2 = 0$ for every z in N and N is a subspace of A we may linearize to get $xy = -yx$ for all x, y in N. Let $xy = \alpha 1 + n$, $yx = -\alpha 1 - n$. It suffices to show that $\alpha = 0$. Using (I) we get $\alpha x + xn - \alpha x - nx = 2\alpha x + 2nx$. Since $xn = -nx$ we have $4xn = 2\alpha x$ and since $P \neq 2$ $xn = (\alpha/2)x$ and $nx = (-\alpha/2)x$. Using (I) again we have $n(nx) + (xn)n = 2(nx)n$ or $(\alpha^2/2)x = (-\alpha^2/2)x$. Thus $\alpha = 0$ and A cannot be nodal.

LEMMA 2. *Let A be a nodal algebra satisfying* (II) *over a field F whose characteristic is not two. Then if $N^{\cdot 2} = \{x \cdot y \mid x, y \text{ in } N\}$, then $N^{\cdot 2}N \subseteqq N$ and $NN^{\cdot 2} \subseteqq N$. (Here $x \cdot y$ denotes the multiplication in A^+)*

Proof. Let x and y be elements of N such that $xy = \alpha 1 + n$ with α in F and n in N. Then $yx = 2x \cdot y - \alpha 1 - n$ and $(x, y, x) = 2\alpha x + nx + xn - 2x(x \cdot y)$. But $nx + xn = 2x \cdot n$ is in N, $2\alpha x$ is in N, and by (II) (x, y, x) is in N. Therefore $x(x \cdot y)$ is in N. Linearizing this we have:

$$(1) \qquad x(z \cdot y) + z(x \cdot y) \text{ is in } N \text{ if } x, y, z \text{ in } N\,.$$

Let $z = y$ in (1). Then $xy^2 + y(x \cdot y)$ is in N. But by the previous remark $y(y \cdot x)$ is in N. Thus we conclude that for all x, y in N, xy^2 is in N. Linearizing this we have that $x(z \cdot y)$ is in N. Since N is an ideal of A^+ [3], $x \cdot (z \cdot y)$ and hence $(z \cdot y)x$ is also in N. Thus $N^{\cdot 2}N$ and $NN^{\cdot 2}$ are contained in N.

LEMMA 3. *Let A be a nodal algebra satisfying* (I) *and* (II) *over a field F whose characteristic is not two. Then $S = N^{\cdot 2} + N^{\cdot 2}N$ is an ideal of A which is contained in N.*

Proof. Linearizing (I) we have

$$(\mathrm{I}') \qquad x(zy) + z(xy) + (yx)z + (yz)x = 2(xy)z + 2(zy)x\,.$$

Let $z = u \cdot v$ with u, v in N. Then we have

$$(2) \qquad \begin{aligned} &x((u \cdot v)y) + (y(u \cdot v))x - 2((u \cdot v)y)x \\ &= 2(xy)(u \cdot v) - (u \cdot v)(xy) - (yx)(u \cdot v)\,. \end{aligned}$$

Clearly the right hand side is in S. Therefore

$$(3) \quad x((u \cdot v)y) + (y(u \cdot v))x - 2((u \cdot v)y)x \text{ is in } S \text{ if } x, y, u, v, \text{ are in } N\,.$$

Adding and subtracting $2((u \cdot v)y)x$ to (3) we have: $2x \cdot ((u \cdot v)y) + 2(y \cdot (u \cdot v))x - 4(((u \cdot))y)x)$ is in S. But $((u \cdot v) \cdot y)x \in N^{\cdot 3}x \subseteqq N^{\cdot 2}x \subseteqq S$. Also by Lemma 2

$$(u \cdot v)y \in N,\ x \cdot ((u \cdot v)y) \in N^{\cdot 2} \subseteqq S.$$

Thus, $((u \cdot v)y)x \in S$ and combining this with $2x \cdot ((u \cdot v)y) \in S$ we have

$x((u\cdot v)y)\in S$. This shows that $(N^{\cdot 2}N)N\subseteqq S$ and $N(N^{\cdot 2}N)\subseteqq S$ which proves that S is an ideal of A. The fact that $S\subseteqq N$ follows directly from Lemma 2.

THEOREM 1. *There do not exist any simple nodal algebras satisfying* (I) *and* (II) *over any field F whose characteristic is not two.*

Proof. We show that if A is a simple nodal algebra satisfying (I) and (II) then the nilindex of A is two contradicting Lemma 1. By Lemma 3, S is an ideal of A contained in N. Then by the simplicity of A, $S=0$. Let y be any element of N. Clearly $y^2\in S$. Therefore $y^2=0$ and the nilindex of A is two.

THEOREM 2. *There do not exist any nodal algebras satisfying* (I) *and* (II) *over any field whose characteristic is not two.*

Proof. For if B is such an algebra it would have a homomorphic image which is a simple nodal algebra contradicting Theorem 1.

COROLLARY 1. *There are no nearly commutative nodal algebras satisfying $(x, x, z)=(z, x, x)$ over any field F whose characteristic is not two.*

Proof. Let A be such an algebra with x, z in N. From $(x, x, z)=(z, x, x)$ we obtain: $(zx)x+x(xz)=zx^2+x^2z$. The right hand side is in N by [3] and the left hand side is just $2(xz)x$ by (I). Therefore $(xz)x$ is in N. Using (I) it is an easy matter to show that $x(zx)$ is also in N. Thus (x, z, x) is in N if x and z are in N. Therefore A satisfies condition (II) and by Theorem 2, A cannot be nodal.

An algebra satisfying the identity $(x, x, z)=(z, x, x)$ is called an anti-flexible algebra [6].

COROLLARY 2. *If A is a nearly commutative algebra over a field F of characteristic not two and if A has an anti-automorphism then A cannot be nodal.*

Proof. Let ϕ be the anti-automorphism and let $x\phi=x'$ for every x in A. Applying ϕ to the identity (I) we get:

$$x'(x'y')+(y'x')x'=2x'(y'x')\ . \tag{4}$$

But by (I) $x'(x'y')+(y'x')x'=2(x'y')x'$. Therefore we have $(x'y')x'=x'(y'x')$ for all x', y' in A. But ϕ is onto. Therefore $(xy)x=x(yx)$ for all x, y in A and A is flexible. By Theorem 2, A cannot be nodal.

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