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WILSON ANGLES IN LINEAR NORMED SPACES

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The purpose of this note is to give a complete answer to the question: which linear normed spaces over the field of reals have the property that an angle (determined by two metric rays) can be defined in terms of the euclidean law of cosines?

Menger [7] introduced a system of axioms for "angle spaces" and related problems. Wilson [11] has shown that a theory of angles analogous to that of euclidean space is possible for complete, convex metric spaces any four points of which are congruent with four points of euclidean space. However, he also proved [10] that a complete, convex, externally convex metric space with the property that each four points of the space are congruent to four points of euclidean space is an inner-product space. In [12] Wilson extended his definition of angle to general metric spaces in the following way (for definitions of metric concepts used in this paper see [1]). If a, b, c, are points of a metric space, with distance between pairs denoted by ab, ac, bc, the symbol bac is called an angle with vertex a and its value is defined by the formula

$$bac = \text{Arc cos}\,[(ab^2 + ac^2 - bc^2)/2ab \cdot ac]\ .$$

This definition is possible by virtue of the triangle inequality. If R, R' are two metric rays, (congruent images of half-lines), with common initial point a and if b, c are points on R, R', respectively, $b \neq a \neq c$, then R, R' make an angle $[R; R']$ if lim bac exists as b and c tend to a on the metric rays R, R' respectively.

Wilson notes that in general metric spaces angles defined in this way lack many important properties usually associated with angles and suggests that a further investigation of the types of spaces admitting these properties and of conditions for the existence of angles between rays is needed. In this paper we restrict the class of metric spaces to the class of linear normed spaces over the field of reals. We show that if such a space admits an angle as defined above, then the linear normed space is an inner-product space. Thus, a linear normed space over the reals which admits an angle for each pair of rays with a common point is an inner-product space and consequently has the euclidean four-point property postulated by Wilson in [11]. In light of [10], this then is a partial converse of [11].

It should be noted that in this paper a local property is given which characterizes inner-product spaces among the class of linear normed spaces over the reals. So far as the authors know, this is the only local characterization that has been given. We will show that the criteria of Blumenthal [1] are satisfied and thus obtain our result.

2. Angles in linear normed spaces. In the discussion that follows B will denote a linear normed space with the property that for each point a and each pair of rays R, R' with common initial point a, $\lim bac$ exists as b and c tend to a on the rays R, R', respectively. For convenience we will denote "lim bac as b and c tend to a on R, R', respectively" by $\lim_{b, c \to a} bac$.

We note that if a, b are distinct points of B, then the algebraic line determined by a, b, denoted by

$$L(a, b) = \{x \in B \mid x = \lambda a + (1 - \lambda) b\}$$

is a metric line; since the mapping $\lambda a + (1 - \lambda) b \to (1 - \lambda) \mid a - b \mid$ is a congruence between $L(a, b)$ and the real line.

THEOREM 1. *If $R(a, b)$ and $R(a, c)$ are algebraic rays in B, (i.e., rays which are contained in algebraic lines) with common initial point a, then the angle $[R(a, b); R(a, c)]$ is equal to*

$$\operatorname{Arc\,cos} [(ab^2 + ac^2 - bc^2)/2ab \cdot ac] \ .$$

Proof. Since $\lim_{b, c \to a} [ab^2 + ac^2 - bc^2)/2ab \cdot ac]$ exists and $ab = |a - b|$, this limit is independent of the way in which b and c tend to a on the rays $R(a, b)$ and $R(a, c)$, respectively. Thus,

$$\begin{aligned}
&\lim_{b, c \to a} bac \\
&= \operatorname{Arc\,cos} \lim_{\lambda \to 1} \frac{| a - (1-\lambda)b - \lambda a |^2 + | a - (1-\lambda)c - \lambda a |^2}{2 \mid a - (1-\lambda)b - \lambda a \mid} \\
&\qquad\qquad\qquad \frac{- |(1-\lambda)b + \lambda a - (1-\lambda)c - \lambda a |^2}{\times \mid a - (1-\lambda)c - \lambda a \mid} \\
&= \operatorname{Arc\,cos} \lim_{\lambda \to 1} \frac{(1-\lambda)^2 \mid a - b \mid^2 + (1-\lambda)^2 | a - c |^2 - (1-\lambda)^2 \mid b - c \mid^2}{2(1-\lambda)^2 \mid a - b \mid \mid a - c \mid} \\
&= \operatorname{Arc\,cos} [(\mid a - b \mid^2 + \mid a - c \mid^2 - \mid b - c \mid^2)/2 \mid a - b \mid \ \mid a - c \mid] \ .
\end{aligned}$$

THEOREM 2. *If a, b, c, d is a quadruple of points of B with b, c, d on an algebraic line, then points a', b', c', d' of the euclidean plane E_2 exist which are congruent to a, b, c, d.*

Proof. Since b, c, d lie on an algebraic line, one of them, say c is between the other two. Now points a', b', d' of E_2 exist which are congruent to a, b, d. Let c' be the point in E_2 between b' and d' such that $b'c' = bc$ and $c'd' = cd$. Now, $ab = a'b'$, $ad = a'd'$, $bc = b'c'$, and $cd = c'd'$, and it suffices to show that $ac = a'c'$. By Theorem 1,

$$\begin{aligned}\cos [R(b, a); R(b, c)] &= (ab^2 + bc^2 - ac^2)/2ab \cdot bc \\ &= (ab^2 + bd^2 - ad^2)/2ab \cdot bd \\ &= (a'b'^2 + b'd'^2 - a'd'^2/2a'b' \cdot b'd' \\ &= (a'b'^2 + b'c'^2 - a'c'^2)/2a'b' \cdot b'c'\end{aligned}$$

and it follows that $ac = a'c'$ which completes the proof.

COROLLARY. *Let a, b, c, d be a quadruple of points of B with b, c, d on an algebraic line with c between b and d. If $R(c, b), R(c, d)$, and $R(c, a)$ are algebraic rays, then $[R(c, b); R(c, a)] + R(c, a); R(c, d)] = \pi$.*

Proof. By Theorem 2, points $a', b', c, 'd'$ of E_2 exist which are congruent to a, b, c, d. Moreover, by Theorem 1, the angle between two algebraic rays is given by the euclidean law of cosines, which is also true for triangles in E_2. The corollary now follows.

THEOREM 3. *If a, b, c are linear points of B with b between a and c, for any rays $R(b, a)$, $R(b, c)$, $[R(b, c); R(b, c)] = \pi$.*

Proof. Let $\{a_n\}$, $\{c_n\}$ be sequences of points on $R(b, a)$ and $R(b, c)$, respectively, such that $a_n \neq b \neq c_n$. Then $a_nc_n = a_nb + bc_n$ and $(a_nb^2 + c_nb^2 - a_nc_n^2)/2a_nb \cdot c_nb = -1$, and consequently, $\lim_{a, b \to b} abc = \pi$.

THEOREM 4. *If a, b are any two distinct points of B, then a, b determine a unique metric line; viz. the algebraic line.*

Proof. We first show that a, b are endpoints of exactly one metric segment. It is known that a, b are endpoints of an algebraic segment $S(a, b)$, which is also a metric segment. Suppose a, b are endpoints of another metric segment $S_1(a, b)$. Let d be a point of $S_1(a, b) - S(a, b)$, choose a point e on $S(a, b)$ such that $be = bd$, and let c be a point such that b is between a and c. It follows from the transitive property of betweeness that b is between d and c. If $R(b, d)$ and $R(b, c)$ are the algebraic rays through b, d and b, c, respectively, then by Theorem 3, $[R(b, d); R(b, c)] = \pi$. Moreover, if $R(b, e)$ is the algebraic ray of b, e it follows from the corollary of Theorem 2 that $[R(b, e); R(b, d)] + [R(b, d); R(b, c)] = \pi$. Consequently,

$[R(b, e); R(b, d)] = 0$. Thus, $(bd^2 + be^2 - de^2)/2be \cdot bd = 1$ or $(bd - be)^2 = de^2$ and $bd = be + de$ or $bd + de = be$. But this implies that $de = 0$ or $d = e$, contrary to fact. Therefore, each two distinct points are endpoints of exactly one segment.

The algebraic line through distinct points a, b is a metric line. That the segment $S(a, b)$ cannot be prolonged to another line, follows as above, except that the point d is chosen so that d is between a and b and c is chosen on the algebraic line through a, b such that b is between a and c.

The proof of Theorem 4 shows that if three points are linear then the three points lie on an algebraic line.

THEOREM 5. *Any set of four points of B which contains a linear triple is congruently imbeddable in E_2.*

Proof. Theorem 4 and Theorem 2.

It now becomes possible to complete the proof of the final result. This depends on a property known as the weak euclidean four point property which is defined in the following way [1, p. 123].

DEFINITION. A metric space M has the weak euclidean four-point property provided that each quadruple of pairwise distinct points of M containing a linear triple is congruently imbeddable E_2.

The importance of the weak euclidean four-point property lies in its usefulness as a means of characterizing inner-product spaces. Blumenthal (loc. cit.) has shown that a complete, convex, externally convex metric space with the weak euclidean four-point property is an inner-product space. Moreover, he points out [2] that completeness is not essential in the setting of a linear normed space.

Since each two-dimensional subspace of a real inner-product space is congruent to the euclidean plane, and since each two intersecting lines of such a space lie in a two-dimensional subspace, a real inner-product space satisfies our criteria. This observation together with an application of the above result of Blumenthal yields the following theorem, which characterizes inner-product spaces among the class of linear normed spaces over the field of reals.

THEOREM 6. *The linear normed space B over the field of reals is an inner-product space if and only if* lim bac *exists as b, c tend to a on the rays ρ and σ, respectively for each triple of points a, b, c, and each pair ρ and σ of metric rays through a, b and a, c, respectively.*

It should be noted that Blumenthal's example of a convexly metrized tripod shows that the hypothesis that B is a linear normed space in Theorem 6 can not be deleted.

REFERENCES

1. L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon, Oxford, 1953.

2. ———, *An extension of a theorem of Jordan and von Neuman*, Pacific J .Math. **5** (1955), 161-167.

3. M. M. Day, *Some characterizations of inner product spaces*, Trans. Amer. Math. Soc. **62** (1947), 320-337.

4. ———, *On criteria of Kasahara and Blumenthal for inner-product spaces*, Proc. Amer, Math. Soc. **10** (1959), 92-100.

5. R. W. Freese, *Criteria for inner-product spaces*, Proc, Amer. Math. Soc. **19**(1968), 953-958.

6. P. Jordan and J. von Neumann, *On inner products in linear metric spaces*, Ann. of Math. **36** (1935), 719-723.

7. K. Menger, *Some applications of point-set methods*, Ann. of Math. **32** (1931), 739-760.

8. I. J. Schoenberg, *A remark on M. M. Day's characterization of inner product spaces and a conjecture of L. M. Blumenthal*, Proc. Amer. Math. Soc. **3** (1952), 961-964.

9. W. L. Stamey, *On generalized euclidean and non-euclidean spaces*, Pacific J. Math. **7** (1957), 1505-1511.

10. W. A. Wilson, *A relation between metric spaces and euclidean spaces*, Amer. J. Math. **54** (1932), 505-517.

11. ———, *On angles in certain metric spaces*, Bull. Amer Math. Soc. **38** (1932), 580-588.

12. ———, *On certain types of continuous transformations of metric spaces*, Amer. J. Math. **57** (1935), 62-68.

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