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# LACUNARY SERIES AND PROBABILITY

R. KAUFMAN

**In this note we continue some investigations connecting a lacunary series $\Lambda$ of real numbers**

$$\Lambda: 1 \leqq \lambda_1 < \cdots < \lambda_k < \cdots, q\lambda_k \leqq \lambda_{k+1} \qquad (1 < q)$$

**and a probability measure $\mu$ on $(-\infty, \infty)$ satisfying**

$$\mu([a, a+h]) \ll h^\beta \tag{1}$$

**for all intervals $[a, a+h]$ of length $h < 1$, and a fixed exponent $0 < \beta < 1$. (The notation $X \ll Y$ is a substitute for $X = O(Y)$.) Measures $\mu$ occur in the theory of sets of fractional Hausdorff dimension.**

**In the following statements $S$ is a subset of $(-\infty, \infty)$ of Lebesgue measure 0, depending only on $\mu$ and $\Lambda$.**

THEOREM 1. **For $r = 2, 4, 6, \cdots$ and $t \notin S$, there is a constant $B_r(t)$ so that**

$$\int_{-\infty}^{\infty} |\sum a_k \cos(\lambda_k tx + b_k)|^r \mu(dx) \leqq B_r(t)(\sum |a_k|^2)^{r/2} .$$

**Here $B_r(t)$ is independent of the sequences $(a_j)$ and $(b_k)$.**

THEOREM 2. **For $t \notin S$ the normalized sums**

$$(\tfrac{1}{2}N)^{-1/2} \sum_{k \leqq N} \cos(\lambda_k tx + b_k)$$

**tend in law (with respect to the probability $\mu$) to the normal law. Here the convergence is uniform for all sequences $(b_k)$.**

Theorem 1 is a random form of a fact apparently known from the advent of the study of lacunary series; Theorem 2 bears the same relation to the work of Salem and Zygmund [4]. Probability enters critically in the theorems because $\beta < 1$: for *any* increasing sequence $\Lambda$ there is a measure $\mu$ fulfilling (1) for every $\beta < 1$ and such that the $t$-set defined in Theorem 1 is of first category.

**1.** In this section and later we use the notations

$$e(y) \equiv e^{iy},\ \hat{\mu}(y) \equiv \int_{-\infty}^{\infty} e(yx)\mu(dx) ,$$

$-\infty < y < \infty$. In the following estimation $|y| > 1$.

$$I = \int_1^2 |\hat{\mu}(ty)|^2 dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_1^2 e(tyx_1 - tyx_2)dt \cdot \mu(dx_1)\mu(dx_2)$$

$$\leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \inf(1, 2|yx_1 - yx_2|^{-1})\mu(dx_1)\mu(dx_2) .$$

Let $r > 0$ be the integer defined by $2^{-r} < |y|^{-1} \leqq 2^{1-r}$; we sum the integrand over the sets

$$(|x_1 - x_2| > 1), (1 > |x_1 - x_2| \geqq \tfrac{1}{2}), \cdots, (2^{1-r} > |x_1 - x_2| > 2^{-r})$$

and finally over the set $(2^{-r} > |x_1 - x_2|)$. In each case the product measure can be estimated by (1) and Fubini's Theorem; summing up we obtain $I \ll |y|^{-\beta}$. A more convenient form is valid for all real $y$:

$$\int_1^2 |\hat{\mu}(ty)|dt \ll (1 + |y|)^{-1/2\beta} . \tag{2}$$

2. To prove Theorem 1 we require an elementary lemma.

LEMMA. *Let $(v_k)_1^\infty$ be a sequence of real numbers and $r$ a positive integer. Let $T$ be the sum of the moduli of all Fourier-Stieltjes coefficients*

$$\hat{\mu}(d_1 v_{k_1} + d_2 v_{k_2} + \cdots)$$

*where $1 \leqq k_1 < k_2 < \cdots, d_1, d_2, \cdots$ are integers $\neq 0$, and*

$$|d_1| + |d_2| + \cdots \leqq 2r ;$$

*the number of integers $d_1, d_2, \cdots$ varies between 1 and $2r$.*
*Then*

$$\int |\sum a_k e(v_k x)|^{2r} \mu(dx) \leqq (1 + T)(r!)^2 (\sum |a_k|^2)^r .$$

*Proof.* We first expand $(\sum a_k e(v_k x))^r$ by the multinomial formula, obtaining a sum of terms

$$r!(e_1! e_2! \cdots e_r!)^{-1} a_{k_1}^{e_1} \cdots a_{k_r}^{e_r} e(e_1 v_{k_1} x + \cdots + e_r v_{k_r} x) .$$

Of course $1 \leqq k_1 < \cdots < k_r$, and the $r$-tuple $(e_1, \cdots, e_r)$ is variable, subject to the equality $e_1 + \cdots + e_r = r$. Next to this expansion we place that of the conjugate, using exponents $f_1, \cdots, f_r$. Multiplying these expansions and integrating with respect to $\mu$, we collect the integrals in two steps.

First we consider terms in the product in which $(e_1, \cdots, e_r) = (f_1, \cdots, f_r)$. Making a term-by-term comparison with $(\sum |a_k|^2)^r$, we find a sum $\leqq r!\ (\sum |a_k|^2)^r$.

For the remaining terms we note the factor $\hat{\mu}(e_1 v_{k_1} - f_1 v'_{k_1} + \cdots)$ attached to the number $|a_{k_1}|^{e_1 + f_1} \cdots$, and note that the former number is counted in $T$. Thus the sum here is $\leqq (r!)^2 \max |a_k|^{2r}$, and the proof is complete.

To prove Theorem 1 it will be enough to give a proof for sequences $\Lambda$ with a gap $q \geqq 2r$, for in any case $\Lambda$ is a union of $1+[\log q/\log 2r]$ sequences with gaps of this size. According to the lemma, it is sufficient to show that for almost all $t$, the sum $T$ is finite, where $T$ is calculated for the sequence $v_k \equiv t\lambda_k$. Thus $T$ is a sum of numbers

$$|\hat{\mu}(td_1\lambda_{k_1} + \cdots + td_s\lambda_{k_s})| ,$$

where $d_1 \neq 0, \cdots, d_s \neq 0$, $|d_1| + \cdots + |d_s| \leqq 2r$. Because $q \geqq r$ and $|d_1| + \cdots + |d_{s-1}| \leqq 2r - 1$,

$$|d_1\lambda_{k_1} + \cdots + d_s\lambda_{k_s}| \geqq \frac{1}{r}\lambda_{k_s} ,$$

whence

$$\int_1^2 |\hat{\mu}(td_1\lambda_{k_1} + \cdots + td_s\lambda_{k_s})|dt \ll \lambda_{k_s}^{-1/2\beta} .$$

But the number of forms $d_1\lambda_{k_1} + \cdots + d_s\lambda_{k_s}$ having a certain $k = k_s$ is $\ll k^{2r}$. Thus $\int_1^2 Tdt < \infty$ because $\sum_1^\infty k^{2r}\lambda_k^{-1/2\beta} < \infty$. This proves Theorem 1 for the interval $1 < t < 2$ and the same argument is plainly valid for $(-\infty, \infty)$.

3. In the proof of Theorem 2 it is again necessary to estimate sums like $T$, but it is no longer possible to make such sums converge. Instead, we must estimate their rate of increase.

LEMMA. *Let $d_1 \neq 0, \cdots, d_s \neq 0$ be integers and*

$$p = |d_1| + \cdots + |d_s| .$$

*The number of s-tuples $1 \leqq k_1 < \cdots < k_s \leqq N$ for which*

$$|d_1\lambda_{k_1} + \cdots + d_s\lambda_{k_s} - \lambda| \leqq 2^j \qquad (j = 1, 2, 3, \cdots) \tag{3}$$

*is bounded as follows for all real $\lambda$ and $N \geqq 1$:*

(a) $\leqq B(p, q)j^p$ *if $p = 1$ or $p = 2$.*

(b) $\leqq B(p, q)j^p N^{1/2(p-1)}$ *if $p > 2$.*

*Proof.* The argument for $s = 1$ is very simple and is contained implicitly in that now given for $s = 2$, $p \geqq 2$. Here we distinguish two cases, according as $|d_1\lambda_{k_1}| \leqq q^{-1}|d_2\lambda_{k_2}|$, or not. In the first case we can write

$$d_1\lambda_{k_1} + d_2\lambda_{k_2} = (1 + \theta)d_2\lambda_{k_2} , \qquad |\theta| \leqq q^{-1} < 1 .$$

Let $k < k^*$ be two values of $k_2$ occurring in this case. Then

$$|\lambda_k(1 + \theta) - \lambda_{k^*}(1 + \theta^*)| \leqq 2^{j+1}$$

$$\lambda_{k^*} \leqq (\lambda_k + 2^{j+1})(1 - q^{-1})^{-2} .$$

From this it follows that $k^* - k \ll j$, so that $k_2$ is restricted to $\ll j$ values. Once $k_2$ is chosen, $k_1$ is similarly confined, and so the first case distinguished before gives a contribution $\ll j^2$. Moreover this case always obtains when $|d_1| \leqq |d_2|$, and in particular when $s = 2$, $p = 2$; thus (a) is proved. Again, if $|d_1\lambda_{k_1}| > q^{-1}d_2\lambda_{k_2}$ then

$$k_1 < k_2 \leqq k_1 + \log |d_1|/\log q$$

and $(k_1, k_2)$ is restricted to $\ll N$ values. Because $p > 2$, this is consistent with (b).

When $s \geqq 3$ we choose an integer $A = A_{q,s}$ so that $2A^{-q}p \leqq 1$ and first estimate the number of solutions of (3) wherein $k_{s-1} + A < k_s$. Then

$$d_1\lambda_{k_1} + \cdots + d_s\lambda_{k_s} = (1 + \theta)d_s\lambda_{k_s} , \qquad |\theta| \leqq \tfrac{1}{2} .$$

We find as above that $k_s$ can assume $\ll j$ different values, and once $k_s$ is fixed we find by induction (on $p$ or on $s$) that the remaining choices are $\ll j^{p-1}N^{1/2(p-2)}$ in number. Finally, if $k_{s-1} < k_s \leqq k_{s-1} + A$, then $(k_1, k_2)$ has at most $AN$ values, and for each one of these the number of choices is $\ll j^{p-2}N^{1/2(p-3)}$. This proves the lemma.

Much more precise estimates are given by Erdös and Gál, but these don't seem to be applicable [1].

4. In the proof of Theorem 2 we use the multinomial expansion of $(\sum_{k\leqq N} \cos(t\lambda_k x + b_k))^r$ into a finite combination of sums (with coefficients to be considered later)

$$\sum_{1\leqq k_1<\cdots<k_s\leqq N}\sum \cos^{e_1}(t\lambda_{k_1}x + b_{k_1}) \cdots \cos^{e_s}(t\lambda_{k_s}x + b_{k_s}) .$$

Here $e_1 \geqq 1, \cdots, e_s \geqq 1$, and $e_1 + \cdots + e_s = r$. This sum is $\leqq N^s$ in modulus, and so it can be neglected if $s < \frac{1}{2}r$. When $r$ is even, say $r = 2v$, there occurs a *dominant* contribution determined by the choice $s = v$, $e = \cdots = e_v = 2$. This requires closer argument and we exclude it for the moment; in every $s$-tuple $(e_1, \cdots, e_s)$ remaining at least one component must be *odd*.

To exploit the last remark we expand

$$\cos^{e_1}(t\lambda_{k_1}x + b_{k_1}) \cdots \cos^{e_s}(t\lambda_{k_s}x + b_{k_s})$$

into a linear combination of exponentials $e((tx)(d_1\lambda_{k_1} + \cdots + dr\lambda_{k_r}))$, wherein $1 \leqq |d_1| + \cdots + |d_s| \leqq r$.

We can handle the dominant term in almost the same way, using the identity $2\ \cos^2 u = 1 + \cos 2u$. In the multinomial formula there occurs the factor $r!\,2^{-v}(v = \frac{1}{2}r)$. Hence the dominant term contains the constant 1 with a coefficient

$$2^{-v}\cdot r!2^{-v}\cdot\binom{N}{v} = 2^{-r}r!(v!)^{-1}N^{v} + 0(N^{v-1}) .$$

Now the $r^{\text{th}}$ moment

$$m_r = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^r e^{-1/2u^2}du = 2^{-v}r!(v!)^{-1} .$$

Thus the constant term is $2^{-v}N^v m_r + 0(N^{v-1})$, and this is correct because the 'norming' constant is $(\frac{1}{2}N)^{-1/2}$.

In the dominant term there occur other exponentials, but each of them is of the type considered above. It remains now to be proved that the random error, say $R_N$, encountered in the moment of

$$\sum_{k\leqq N}\cos\,(t\lambda_k x + b_k)$$

is almost surely $o(N^v)$ as $N\to+\infty$. But in fact these errors are Fourier-Stieltjes coefficients

$$|\hat{\mu}(td_1\lambda_{k_1} + \cdots + td_s\lambda_{k_s})|$$

where $1\leqq k_1 < \cdots < k_s \leqq N$ and $1\leqq |d_1| + \cdots + |d_s| \leqq r$. From the previous lemma and from the estimation (2), we find that

$$\int_1^2 R_N dt \ll N^{v-1/2}$$

and therefore, by Chebyshev's inequality, $R_{N^3} = o(N^{3v})$ almost surely. Because $(N+1)^3 = N^3 + o(N^3)$ this completes the proof.

It is not difficult to formulate and prove a similar theorem for the *union* of sequences $t\Lambda \cup s\Lambda$, where $(t, s)$ is a point in the plane. When $\mu$ is absolutely continuous, however, we can suppress one of the variables and obtain a central-limit theorem for sums

$$\sum_{k\leqq N}\cos\,(\lambda_k x + b_k) + \sum_{k\leqq N}\cos\,(\lambda_k tx + b_k') .$$

The central-limit phenomenon here is false for certain sequences $\Lambda$ and certain values of $t$: $\lambda_k = 2^k$ and $t = 2$. The existence of even one $t > 1$ rendering the central-limit theorem false is presumably a strong restriction on a lacunary sequence.

5. We conclude by stating a theorem and a conjecture related to it. As before $S$ is a set of measure 0 in $(-\infty, \infty)$ depending only on $\Lambda$ and $\mu$.

THEOREM 3. *For each $t \notin S$, each closed set $E$, and each $\varepsilon > 0$,* *there is an integer* $N = N(t, \varepsilon, E)$ *such that*

$$\left| \int_E \left| \sum_{k \geqq N} a_k e(\lambda_k tx) \right|^2 \mu(dx) - \mu(E) \sum_{k \geqq N} |a_k|^2 \right| \leqq \varepsilon \sum_{k \geqq N} |a_k{}^2 .$$

The proof is very similar to that of Theorem 1, and to some extent depends upon Theorem 1; however, it is necessary here to use the estimate (a) of the lemma in § 3.

COROLLARY. *If* $\sum |a_k|^2 = +\infty$, *then* $\sum_1^\infty a_k e(\lambda_k tx)$ *diverges almost everywhere with respect to* $\mu$.

It is natural to conjecture that $\sum_1^\infty a_k e(\lambda_k tx)$ converges almost everywhere, provided $\sum |a_k|^2 < \infty$.

*Added in proof.* This follows from theorems on orthogonal series.

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UNIVERSITY OF ILLINOIS