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RINGS OF QUOTIENTS OF Φ-ALGEBRAS

D. G. JOHNSON

Let $\mathscr{X}$ be a completely regular (Hausdorff) space. Fine, Gillman, and Lambek have studied the (generalized) rings of quotients of $C(\mathscr{X}) = C(\mathscr{X}; \mathbf{R})$, with particular emphasis on the maximal ring of quotients, $Q(\mathscr{X})$. In this note, we start with a characterization of $Q(\mathscr{X})$ that differs only slightly from one of theirs. This characterization is easily altered to fit more general circumstances, and so serves to obtain some results on non-maximal rings of quotients of $C(\mathscr{X})$, and to generalize these results to the class of Φ-algebras.

We consider only commutative rings with unit. Let A be one such, and recall that the (unitary) over-ring B of A is called a *rational extension* or *ring of quotients* of A if it satisfies the following condition: given $b \in B$, for every $0 \neq b' \in B$ there is $a \in A$ with $ba \in A$ and $b'a \neq 0$. A ring without proper rational extensions is said to be *rationally complete*. For the rings to be considered here (all are semi-prime), the condition above can be replaced by the simpler condition: for $0 \neq b \in B$, there exists $a \in A$ such that $0 \neq ba \in A$ ([1], p. 5). Accordingly, we make the following

DEFINITION. If B is an over-ring of A and $0 \neq b \in B$, say that b is *rational over* A if there is $a \in A$ with $0 \neq ba \in A$.

Let $m\beta\mathscr{X}$ denote the minimal projective extension of $\beta\mathscr{X}$ and $\tau: m\beta\mathscr{X} \to \beta\mathscr{X}$ the minimal perfect map ([2]). In [1], it is shown that $Q(\mathscr{X})$ is a dense, point-separating subalgebra of $D(m\beta\mathscr{X})$, the set of all continuous maps from $m\beta\mathscr{X}$ into the two-point compactification of the real line which are real-valued on a dense subset of $m\beta\mathscr{X}$ (see, also, [3]). Since $Q(\mathscr{X})$ contains every ring of quotients of $C(\mathscr{X})$, this leads to

PROPOSITION 1. *If B is any ring of quotients of $C(\mathscr{X})$, then there exist a compact (Hausdorff) space $\mathscr{Y}$ and minimal perfect maps α and γ such that B is a point-separating subalgebra of $D(\mathscr{Y})$ and the following diagram commutes*:

$$\begin{array}{ccc} m\beta\mathscr{X} & \xrightarrow{\alpha} & \mathscr{Y} \\ & \searrow^{\tau} & \downarrow{\scriptstyle\gamma} \\ & & \beta\mathscr{X}\,. \end{array}$$

$\mathscr{Y}$ is the obvious identification space, and the proof consists of a routine argument to show that the quotient map α is closed, whence $\mathscr{Y}$ is Hausdorff. Since $C(\mathscr{X}) \subseteqq B$, the existence of γ follows immediately. (Note that, although $D(m\beta\mathscr{X})$ is an algebra, $D(\mathscr{Y})$ for other spaces $\mathscr{Y}$ is, in general, only a partial algebra.)

For our purposes, it is convenient to view $C(\mathscr{X})$ as a subalgebra of $D(\beta\mathscr{X})$. *This allows us to decree that all spaces are compact (Hausdorff).*

Let us say that any space $\mathscr{Y}$ that is situated in a commutative diagram of the form

$$\begin{array}{ccc} m\mathscr{X} & \xrightarrow{\ \alpha\ } & \mathscr{Y} \\ & \searrow_{\tau} & \downarrow{\scriptstyle \gamma} \\ & & \mathscr{X}\,, \end{array}$$

where all maps are minimal perfect, is *near to* $\mathscr{X}$. (Of course, the existence of γ automatically guarantees the existence of α.) Note that we have already adopted the convention of identifying $f \in D(\mathscr{X})$ with its image $f \circ \gamma$ in $D(\mathscr{Y})$ whenever convenient. With this convention, if A is a subalgebra of $D(\mathscr{Y})$ and $f \in D(\mathscr{Y})$ then we may consider f as an element of an over-ring of A—$D(m\mathscr{X})$—, even if there is no subalgebra of $D(\mathscr{Y})$ containing both A and f.

Now let A be a Φ-algebra that is closed under bounded inversion; i.e., an archimedean lattice ordered algebra with a multiplicative identity that is a weak order unit, in which $1/a \in A$ whenever $1 \leqq a \in A$. Let $\mathscr{X} = \mathscr{M}(A)$, the space of maximal ideals of A with the hull-kernel topology. It is shown in [4] that A is (isomorphic with) a point-separating subalgebra of $D(\mathscr{X})$. If $\mathscr{Y}$ is any space that is near to $\mathscr{X}$, let $A_{\mathscr{Y}} = \{f \in D(\mathscr{Y})$: for each nonempty open set $\mathscr{U}$ in $\mathscr{Y}$, there are a nonempty open set $\mathscr{V} \subseteqq \mathscr{U}$ and $g \in A$ such that $f|_{\mathscr{V}} = g|_{\mathscr{V}}\}$. Note that $A_{\mathscr{Y}}$ is always a lattice. However, it need not be an algebra:

EXAMPLE. Let $\mathscr{X} = \mathscr{Y}$, the one-point compactification of the countable discrete space, and let $A = C(\mathscr{X})$. Then $A_{\mathscr{Y}} = D(\mathscr{Y})$, which is not an algebra.

REMARK. One readily shows that the open sets $\mathscr{V}$ appearing in the definition of $A_{\mathscr{Y}}$ can always be shown to have the form $\gamma^{\leftarrow}[\mathscr{V}_1]$, where $\mathscr{V}_1$ is open in $\mathscr{X}$. It follows that

$$A_{\mathscr{Y}} = \{f \in D(\mathscr{Y}) \colon f \circ \alpha \in A_{m\mathscr{X}}\}\ .$$

PROPOSITION 2. (i) *Every element of* $A_{\mathscr{Y}}$ *is rational over* A^*

(and, hence, over A).

(ii) *$A_{\mathscr{Y}}$ contains every rational extension of A and A^* in $D(\mathscr{Y})$.*

Proof. (i) Let $0 \neq f \in A_{\mathscr{Y}}$, and let $\mathscr{U}$ be a nonempty open set contained in $\operatorname{coz} f$. Since $f \in A_{\mathscr{Y}}$, there exist a nonempty open set $\mathscr{V} = \gamma^{\leftarrow}[\mathscr{V}_1] \subseteqq U$, where $\mathscr{V}_1$ is open in $\mathscr{X}$, and $h \in A^*$ such that $f|_{\mathscr{V}} = h|_{\mathscr{V}}$. Choose $0 \neq g \in A^*$ with $\overline{\operatorname{coz} g} \subseteqq \mathscr{V}_1$. Then $0 \neq fg = hg \in A^*$.

(ii) Let $f \in D(\mathscr{Y}) \backslash A_{\mathscr{Y}}$. Then, there is a nonempty open set $\mathscr{U}$ such that f agrees with no member of A on any nonempty open subset of $\mathscr{U}$. Choose $g \in A^*$ with $\phi \neq \overline{\operatorname{coz} g} \subseteqq \mathscr{U}$.

There is no $h \in A$ with $hg \neq 0$ while $fh \in A$. For, such h would agree with a unit h_1 of A on some nonempty open subset $\mathscr{V}$ of $\mathscr{U}$ (since A is closed under bounded inversion), whence

$$f|_{\mathscr{V}} = (h/h_1)f|_{\mathscr{V}} ,$$

while $(1/h_1)hf \in A$, a contradiction. Thus, f is contained in no rational extension of A.

Although $A_{\mathscr{Y}}$ may contain many different rational extensions of A, it is not true that it is the union of such extensions, as is seen in the example preceding Proposition 2. However, in those spaces $\mathscr{Y}$ for which $A_{\mathscr{Y}}$ is an algebra, $A_{\mathscr{Y}}$ is a Φ-algebra and is the largest ring of quotients of A that "lives on" $\mathscr{Y}$. In particular, this happens when $D(\mathscr{Y})$ is an algebra (e.g., when $\mathscr{Y}$ is basically disconnected or an F-space). Hence, $A_{m\mathscr{X}}$ is a Φ-algebra, since $m\mathscr{X}$ is extremally disconnected, and we obtain the following generalizations of results in [1].

THEOREM 1. *$A_{m\mathscr{X}}$ is rationally complete; thus, $A_{m\mathscr{X}} = \mathscr{Q}(A)$, the maximal ring of quotients of A.*

THEOREM 2. *$A_{m\mathscr{X}}$ is uniformly dense in $D(m\mathscr{X})$.*

THEOREM 3 ([1]). *$D(m\mathscr{X})$ is rationally complete.*

The proofs of Theorems 1 and 3 are virtually identical, and are related to one found on p. 30 of [1]; we prove 1. To do so, we will employ the following characterization of rational completeness (see [1], p. 7).

The commutative ring B is rationally complete if and only if it satisfies: for any dense ideal I of B, every element of $\operatorname{Hom}_B(I, B)$ *is a multiplication by an element of B. (In the present setting, an ideal I of $A_{m\mathscr{X}}$ is dense if and only if $\cup\{\operatorname{coz} f : f \in I\}$ is dense in $m\mathscr{X}$.)*

Proof of Theorem 1. Let I be a dense ideal in A, and let

$\phi \in \mathrm{Hom}_{A_{m\mathscr{X}}}(I, A_{m\mathscr{X}})$. By Zorn's lemma, choose a family $\{\mathscr{U}_\kappa : \kappa \in K\}$ of open sets in $m\mathscr{X}$ satisfying:

(i) $\mathscr{U} = \bigcup \mathscr{U}_\kappa$ is dense in $m\mathscr{X}$;

(ii) the $\mathscr{U}_\kappa$ are pairwise disjoint;

(iii) for each κ, there is $f_\kappa \in I$ such that f_κ is bounded away from zero on $\mathscr{U}_\kappa$ and both f_κ and $\phi(f_\kappa)$ agree with members of A on $\mathscr{U}_\kappa$.

Let $f \in D(m\mathscr{X})$ satisfy

$$f\Big|_{\mathscr{U}_\kappa} = \frac{\phi(f_\kappa)}{f_\kappa}\Big|_{\mathscr{U}_\kappa}$$

for each $\kappa \in K$. This is possible, since $m\mathscr{X}$ is extremally disconnected, so $m\mathscr{X} = \beta\mathscr{U}$.

If $g \in I$ and $x \in \mathscr{U}_\kappa$, then

$$f(x)g(x) = \frac{\phi(f_\kappa)(x)}{f_\kappa(x)}g(x) = \frac{g\phi(f_\kappa)}{f_\kappa}\Big|_{\mathscr{U}_\kappa}(x) = \frac{f_\kappa\phi(g)}{f_\kappa}\Big|_{\mathscr{U}_\kappa}(x) = \phi(g)(x) .$$

It follows that ϕ is multiplication by f. Clearly, $f \in A_{m\mathscr{X}}$, and the proof is complete.

Proof of Theorem 2. Let $f \in D(m\mathscr{X})$, $\varepsilon > 0$. By Zorn's lemma, choose a family $\{\mathscr{U}_\kappa : \kappa \in K\}$ of open sets in $m\mathscr{X}$ which satisfies:

(i) $\mathscr{U} = \bigcup \mathscr{U}_\kappa$ is dense in $m\mathscr{X}$;

(ii) the $\mathscr{U}_\kappa$ are pairwise disjoint;

(iii) for $x, y \in \mathscr{U}_\kappa$, $|f(x) - f(y)| < \varepsilon$ (in particular, f is real-valued on $\mathscr{U}_\kappa$).

For each $\kappa \in K$, choose $x_\kappa \in \mathscr{U}_\kappa$, and define $g: \mathscr{U} \to \mathbf{R}$ by

$$g(y) = f(x_\kappa) \quad \text{if} \quad y \in \mathscr{U}_\kappa .$$

Since $m\mathscr{X} = \beta\mathscr{U}$, g can be extended to $\hat{g} \in D(m\mathscr{X})$. Clearly, $\hat{g} \in A_{m\mathscr{X}}$, and

$$|f - \hat{g}| \leqq \varepsilon .$$

Now the analogue of Proposition 1 for Φ-algebras is routinely obtained.

In case $\mathscr{Y} = m\mathscr{X}$ and $A = C(\mathscr{X})$ one readily translates the definition of $A_{\mathscr{Y}}$ (using the fact that $m\mathscr{X}$ is extremally disconnected, and hence that every dense subspace is C^*-embedded) as follows:

$$A_{m\mathscr{X}} = \varinjlim \{C(\mathscr{S}) : \mathscr{S} \text{ is a dense open subset of } \mathscr{X}\} .$$

Thus, the Fine-Gillman-Lambek result that this direct limit is $Q(\mathscr{X})$ follows from Theorem 1.

It is easily seen that any Φ-algebra A is a rational extension of its bounded subring A^*, and hence that $(A^*)_{\mathscr{Y}} = A_{\mathscr{Y}}$ for any space $\mathscr{Y}$ near to $\mathscr{M}(A)$. Thus, if A is closed under uniform convergence, then $\mathscr{Q}(A) = \mathscr{Q}(A^*) = Q(\mathscr{M}(A))$, since $A^* = C(\mathscr{M}(A))$. In the general case, this may fail to hold. (So, more generally, $A_{\mathscr{Y}} \neq C(\mathscr{M}(A))_{\mathscr{Y}}$ even when $A \subsetneqq C(\mathscr{M}(A))$.)

EXAMPLE. Let $A = Q(\mathbf{R})$. Then (see [1], p. 34),

$$A = \mathscr{Q}(A^*) \neq D(m\mathbf{R}) = D(M(A^*)) = Q(M(A^*)) .$$

For any Φ-algebra A and any space $\mathscr{Y}$ near to $\mathscr{X} = \mathscr{M}(A)$, every subalgebra of $A_{\mathscr{Y}}$ that contains A is a ring of quotients of A. Of interest are those that separate points of $\mathscr{Y}$; prime candidates are the maximal subalgebras of $A_{\mathscr{Y}}$ containing A, which are easily seen to exist.

The results that follow are obtained using ideas and methods employed by Nanzetta in [6] (see his 2.1, 2.3, 4.1). Conversion of his arguments to the present setting is largely an exercise in careful bookkeeping, and the details are omitted.

THEOREM 4. *If B is a maximal subalgebra of $A_{\mathscr{Y}}$, then B is a lattice* (hence, a Φ-algebra).

We will use the term "maximal subalgebra of $A_{\mathscr{Y}}$" to denote only those that contain A.

DEFINITION. Let B be a subalgebra of $D(\mathscr{Y})$. A function $f \in D(\mathscr{Y})$ is said to be *locally in B* if each point of $\mathscr{Y}$ has a neighborhood on which f coincides with some member of B. The subalgebra B is said to be *local* (in $D(\mathscr{Y})$) if each member of $D(\mathscr{Y})$ that is locally in B is a member of B.

THEOREM 5. *Every maximal subalgebra of $A_{\mathscr{Y}}$ is local.*

As in [6], this fact yields the following result.

THEOREM 6. *Let B be a maximal subalgebra of $A_{\mathscr{Y}}$, and let $\mathscr{S}$ be a stationary set of B. If $|\mathscr{S}| > 1$, then*

(i) *$\mathscr{S}$ is closed;*

(ii) *$\mathscr{S}$ is nowhere dense;*

(iii) *$\mathscr{S}$ is connected.*

COROLLARY. *If $\mathscr{Y}$ is totally disconnected, then every maximal subalgebra of $A_{\mathscr{Y}}$ separates points of $\mathscr{Y}$.* (Note that this may occur even when $A_{\mathscr{Y}}$ is not an algebra: see the example preceding Proposition 2.)

It is not known whether every space $\mathscr{Y}$ near to $\mathscr{X}$ supports (i.e., is the structure space of) a ring of quotients of $C(\mathscr{X})$. Apparently, an answer to this question awaits a more systematic description of the collection of spaces near to $\mathscr{X}$.

Note that $(A_{\mathscr{Y}})^*$, the set of bounded elements of $A_{\mathscr{Y}}$, is always a Φ-algebra. Hence, it is always a ring of quotients of A^*—the largest bounded ring of quotients of A^* in $D(\mathscr{Y})$. As mentioned above, it is not known whether $(A_{\mathscr{Y}})^*$ always separates points of $\mathscr{Y}$; it clearly does so if and only if $A_{\mathscr{Y}}$ does. However, the example that follows shows that $A_{\mathscr{Y}}$ may separate points in $\mathscr{Y}$ even though $\mathscr{Y}$ supports no ring of quotients of A.

EXAMPLE. Let $\mathscr{S} = \{(x, \sin(1/x)); x \in (0, 1]\}$, let $\mathscr{X}$ denote the one-point compactification of $\mathscr{S}$, and let $\mathscr{Y} = \mathscr{S} \cup (\{0\} \times [-1, 1])$. Let A denote the Φ-algebra of all functions $f \in D(\mathscr{X})$ that satisfy the following condition:

There is a real number x_0, $0 < x_0 < 1$, and a real polynomial p such that

$$f\left(x, \sin\frac{1}{x}\right) = p\left(\frac{1}{x}\right) \quad \text{for} \quad 0 < x < x_0$$

(cf. [4], 3.6). Then $(A_{\mathscr{Y}})^* = C(\mathscr{Y})$, whereas no subalgebra of $D(\mathscr{Y})$ containing A separates points in $\mathscr{Y}$ ([6], Theorem 4.6).

In passing, it should be noted that the development here has proceeded independently of [1]. The only results from that work that have been employed in an essential way came from Chapter 1 of [1], which consists of standard facts about rings of quotients of commutative rings (see, e.g., [5]). Thus, one can rapidly and efficiently reach the high points of the theory developed in [1] along the lines suggested by this note.

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NEW MEXICO STATE UNIVERSITY